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1. Introduction

Various Instabilities in Secular Timescale

CFS instability

- Fluid modes (f, p, g-modes) may become unstable due to gravitational radiation
- Instability occurs in dissipative timescale

(Chandrasekhar 70, Friedman & Schutz 78)

r-mode instability

- Fluid elements oscillate due to Coriolis force
- Instability occurs due to gravitational radiation

(Andersson 98, Friedman & Morsink 98)

g-mode instability

- Fluid elements oscillate due to restoring force of buoyancy
- Instability occurs in nonadiabatic evolution or in convective unstable cases

Kelvin-Helmholtz instability

- Instability occurs when the deviation of the velocity between the different fluid layers exceeds some critical value

\[ e^{i(m \varphi - \omega t)} \]

Rotating frame

- \( J_\rightarrow <0 \)
- \( J_\rightarrow >0 \)

Inertial frame

- Occurs when \( m \Omega > \omega \)
- \( J_\rightarrow >0 \)
- \( J_\rightarrow >0 \)
Dynamics of r-mode instabilities

Saturation amplitude of r-mode instability

3D simulation
- Saturation amplitude of $o(1)$
- Imposing large amplitude of radiation reaction potential in the system to control secular timescale with dynamics (Lindblom et al. 00)

1D evolution with partially included 3 wave interaction
- Saturation amplitude of $\sim o(0.001)$, which depends on interaction term

Final fate of r-mode instability

3D simulation
- Evolution starting from the amplitude $o(1)$
- Imposing large amplitude of radiation reaction potential
- Energy dissipation of r-mode catastrophically decays to differentially rotating configuration in dynamical timescale

1D evolution including mode couplings network
- After reaching the saturation amplitude $\sim o(0.001)$, Kolmogorov-type cascade occurs
- Destruction timescale is secular

(Schenk et al. 2001)

(Gressman et al. 02, Lin & Suen 06)
Dynamics of r-mode instabilities

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3D simulation

- Evolution
- Imposing large amplitude of radiation reaction potential
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Alternative approaches

- From linear regime to nonlinear regime
- From dynamical timescale to secular timescale are necessary!

(Schenk et al. 2001)
Amplitude of r-mode instability

- Isolated neutron star in the supernova remnant Cassiopeia A
- Compelling evident that the central compact object is neutron star
- Restriction to the amplitude of the r-mode instability by not detecting gravitational waves

\[ \alpha \approx 0.14 - 0.005 \]

Possibility of gravitational wave source

- Possibility of parametric resonance by nonlinear mode-mode interaction
- Amplification to \( \alpha \sim 1 \)

Necessary to obtain a common knowledge for the basic properties of r-mode instability!
2. Dynamics beyond acoustic timescale

- Timescale which cannot be reached by GR hydrodynamics
  - Need to separate the hydrodynamics and the radiation term
- Instability driven by gravitational radiation
  - Need to impose gravitational waves

“Newton gravity + gravitational radiation reaction” are at least necessary

Gravitational radiation reaction

(Blanchet, Damour, Schafer 90)

Quadrupole radiation metric

(includes 2.5PN term)

\[ h_{ij} = \frac{4G}{5c^5} \frac{d^3 I_{ij}^{TT}}{dt^3} \]

amplification factor to control the radiation reaction timescale

Gravitational radiation reaction potential

\[ \Phi^{(RR)} = \frac{1}{2} \left( -\psi + h_{ij} x^j \nabla^i \Phi \right) \]

\[ \triangle \psi = 4\pi h_{ij} x^j \nabla^i \rho \]
Dynamics beyond the acoustic timescale

- Shortest timescale in the system restricts the maximum timestep for evolution
  - Acoustic timescale in Newtonian gravity (control acoustic timescale)
- Relax the restriction from the rotation of the background star
  - Introduce rotating reference frame

Anelastic approximation

Kill the degree of freedom of the sound wave propagation

\[ v^j \nabla_j h + (\Gamma - 1) h \nabla_j v^j = 0 \]

\[ \nabla_j (\rho v^j) = 0 \]

No shocks

Linear regime

\[ \nabla_j (\rho_{eq} v^j) = 0 \]

\[ \nabla_j (\rho v^j) = 0 \]

Propagation of the sound wave

\[ \frac{\partial \rho}{\partial t} = \frac{1}{c_s^2} \frac{\partial P}{\partial t} = -\nabla_j (\rho v^j) \]

\[ \frac{\partial}{\partial t} \nabla_j (\rho v^j) + \Delta P = S \]

Imposing the anelastic approximation changes the structure of the pressure equation

\[ \left( \triangle - \frac{1}{c_s^2} \frac{\partial}{\partial t} \right) P = S \]
Basic equations in rotating reference frame (Lie derivative)

**Time evolution**

\[
\frac{\partial \rho}{\partial t} = 0
\]

\[
\frac{\partial}{\partial t}(\rho u_i) + \nabla_j(\rho u_i v^j) = -\nabla_i p - \rho \nabla_i (\Phi + \Phi^{RR}) - \rho (u_i^{(eq)} + v^j) \nabla_j u_i \quad \text{(eq)} + \rho u_j \nabla_i v^j \quad \text{(eq)}
\]

up to 1st order of $\epsilon$

**Spatial component of the momentum velocity**

\[
u_i^{(eq)} + u_i = \tilde{\gamma}_{ij} (v^j_{(eq)} + v^j)
\]

**Spatial metric**

\[
\tilde{\gamma}_{ij} = \delta_{ij} + h_{ij}
\]

**Pressure poisson equation**

\[
\Delta p = S_p
\]

**Anelastic approximation (constraint)**

\[
\nabla_j (\rho v^j) = 0
\]

Need a special technique to satisfy constraints throughout the evolution

**Boundary condition:**

P=0 at the stellar surface
Procedure

Similar procedure to SMAC method, which is used to solve Navie-Stokes incompressible fluid (McKee et al. 08)

1. **Time update the linear momentum**
   \[
   (\rho \Delta u_i)^{(n+1)} = (\rho \Delta u_i)^{(n)} - \Delta t [\nabla_i p + \cdots]
   \]
   Note that the velocity does not automatically satisfy anelastic condition

2. **Introduce an auxiliary function** \( \phi \) and solve the following Poisson’s equation
   \[
   (\Delta \phi)^{(n+1)} = \partial_j (\rho \Delta v^j)^{(n+1)}
   \]
   **Boundary Condition:** \( \phi = 0 \) at the stellar surface

3. **Adjust the 3-velocity in order to satisfy the anelastic condition**
   \[
   (\rho \Delta v^i)^{(n+1)} = (\rho \Delta v^i)^{(n)} - (\partial^i \phi)^{(n+1)}
   \]

4. **Introduce another auxiliary function** \( \psi \) and solve the following Poisson’s equation
   \[
   \Delta \psi = \delta^{ij} [\partial_j (\rho \Delta u_i)^{(n+1)} - \partial_j (\rho \Delta u_i)^{(n)}]
   \]
   **Boundary Condition:** \( \psi = 0 \) at the stellar surface

5. **Time update the pressure**
   \[
   p^{(n+1)} = p^{(n)} + \frac{\psi^{(n+1)}}{\Delta t}
   \]
3. Nonlinear r-mode instability

Equilibrium configuration of the star

- Rapidly rotating neutron star
- Uniformly rotating, n=1 polytropic equation of state

Eigenfunction and eigenvector of r-mode in incompressible star

Eigenfunction of the velocity
\[ \delta u = \alpha \Omega R \left( \frac{r}{R} \right)^l Y^{(B)}_{ll} \]
\[ \alpha = 1 \times 10^{-4} \]

Impose eigenfunction type perturbation on the equilibrium velocity to trigger r-mode instability

Eigenfrequency (rotating reference frame)
\[ \omega = \frac{2m}{l(l+1)} \Omega \]

Check the excitation of the eigenfrequency

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Due to
• slow rotation approximation
• anelastic approximation
the eigenfrequency does not perfectly agree

Eigenfrequency of the r-mode from slow rotation approximation
(Yoshida & Lee 01)

Our excitation frequency

g-mode?
Gravitational Waveform

Saturation amplitude is around $\alpha \approx 10^{-3}$
No velocity profile appears in the equatorial plane in linear and slow rotation regime of r-mode instability.

Effect of rapid rotation and nonlinearity

Shock wave seems to form at the surface as the times goes on

"Destruction" of r-mode instability
• Lindblom et al. shows in their paper that the catastrophic decay is due to the shocks and the breaking waves at the surface

• Anelastic approximation kills the dominant contribution of the density fluctuation

• Computation with small amplitude of velocity perturbation with Newtonian hydrodynamics may answer the question

(Might be very difficult)
4. Summary

We investigate the r-mode instability of uniformly rotating stars by means of three dimensional hydrodynamical simulations in Newtonian gravity with radiation reaction

- We have succeeded in constructing a nonlinear anelastic approximation in the rotating reference frame, which kills the propagation of sound speed, in order to evolve the system beyond the dynamical timescale.

- When the current multipole contribution is dominant to the r-mode instability (density fluctuation effect is negligible), the instability seems to last for at least hundreds of rotation periods.

- Studies of no anelastic approximation with small amplitude of velocity perturbation may help us for a better understanding.