Fate of homogeneous and isotropic solutions in massive gravity

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AEG, Chunshan Lin, Shinji Mukohyama, JCAP **11** (2011) 030 [arXiv:1109.3845]
AEG, Chunshan Lin, Shinji Mukohyama, JCAP **03** (2012) 006 [arXiv:1111.4107]
Antonio de Felice, AEG, Shinji Mukohyama [arXiv:1206.2080]

Nonlinear massive gravity theory and its observational test
YITP, August 2, 2012

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AEG, Chunshan Lin, Shinji Mukohyama, JCAP **11** (2011) 030 [arXiv:1109.3845] → Background
AEG, Chunshan Lin, Shinji Mukohyama, JCAP **03** (2012) 006 [arXiv:1111.4107] → Linear Perturbations
Antonio de Felice, AEG, Shinji Mukohyama [arXiv:1206.2080] → Nonlinear instability

Nonlinear massive gravity theory and its observational test
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Motivations

- We now have a general massive gravity theory with 5 degrees of freedom.
- Addressing the dark energy problem (decoupling gravity from vacuum energy; self-acceleration) has been among the motivations of NLMG.
- Can we get a cosmology with self-acceleration?
- Look for simplest solutions in the simplest version of the theory.
  $\Longrightarrow$Does it work?
  (continuity with GR, stability, description of thermal history...)
  - yes $\Rightarrow$ predictions of observables to constrain the theory
  - no $\Rightarrow$ relax the solution and/or theory

Why do we get the ghost degree?
Counting the physical degrees of freedom

Classify perturbations with respect to 3d rotational symmetries:

- DOF in metric $\delta g_{\mu\nu}$:

+4 scalars	+4 vectors	+2 tensors

- $\delta g_{0\mu}$ components are nondynamical:

-2 scalars	-2 vectors	-0 tensors

[
- In GR, general coordinate invariance $x^\mu \to x^\mu + \xi^\mu$:

-2 scalars	-2 vectors	-0 tensors

$\Longrightarrow$ GR has only 2 tensors (gravity waves).
]

- In a generic massive theory, no gauge invariance:

+2 scalars	+2 vectors	+2 tensors

- However, we expect massive spin–2 particle to have 5 d.o.f. (1 s, 2 v, 2 t). The extra scalar is the BD ghost.

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Gauge invariant theory

- Introduce four scalar fields (à la Stückelberg), one for each broken gauge degree: ϕ^a $(a = 0, 1, 2, 3)$
- Requiring Poincaré symmetry in the field space. Invariant "line element":

$$ds^2_\phi = \eta_{ab}\, d\phi^a\, d\phi^b$$

- Mass term depends only on $g_{\mu\nu}$ and the *fiducial metric*

$$f_{\mu\nu} = \eta_{ab}\, \partial_\mu \phi^a\, \partial_\nu \phi^b$$

- Requiring that the sixth degree (BD ghost) is canceled at any order, the most general action is:

$$S_m[g_{\mu\nu}, f_{\mu\nu}] = M_p^2 m_g^2 \int d^4x \sqrt{-g}\,(\mathcal{L}_2 + \alpha_3 \mathcal{L}_3 + \alpha_4 \mathcal{L}_4)$$

$$\left[\begin{array}{l} \mathcal{L}_2 = \frac{\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\rho\sigma}}{2} K^\alpha{}_\mu K^\beta{}_\nu \\ \mathcal{L}_3 = \frac{\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\sigma}}{3!} K^\alpha{}_\mu K^\beta{}_\nu K^\gamma{}_\rho \quad \text{and} \quad K^\mu{}_\nu \equiv \delta^\mu{}_\nu - \left(\sqrt{g^{-1}f}\right)^\mu{}_\nu \\ \mathcal{L}_4 = \frac{\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}}{4!} K^\alpha{}_\mu K^\beta{}_\nu K^\gamma{}_\rho K^\delta{}_\sigma \end{array}\right]$$

de Rham, Gabadadze, Tolley '10

Which theory?

Massive gravity zoology in 3+1

1. Drop Poincaré symmetry in the field space
$$f_{\mu\nu} = \bar{\eta}_{ab}\,\partial_\mu\phi^a\,\partial_\nu\phi^b\,,$$
with generic $\bar{\eta}$. `Hassan, Rosen, Schmidt-May '11`
2. Ghost-free bigravity: introduce dynamics for the fiducial metric `Hassan, Rosen '11`
3. Ghost-free trigravity, multigravity etc... `Khosravi et al '11` `Nomura, Soda '12`
4. Quasi-dilaton, varying mass, ... `d'Amico et al '12` `Huang, Piao, Zhou '12`

The list is still growing...

In this talk, I will only allow extensions of the type 1.

Which cosmology?

Goal

- Homogeneous and isotropic universe solution, which can accommodate the history of the universe.
- Preserved homogeneity/isotropy for linear perturbations

- FRW ansatz for the both physical and fiducial metrics

$$ds^2 = -N(t)^2\,dt^2 + a(t)^2\,\Omega_{ij}\,dx^i\,dx^j$$

$$ds_\phi^2 = -n(\phi^0)^2\,(d\phi^0)^2 + \alpha(\phi^0)^2\,\Omega_{ij}\,d\phi^i\,d\phi^j$$

$$\left[\begin{array}{c} \Omega_{ij} = \delta_{ij} + \frac{K\,\delta_{il}\delta_{jm}x^l\,x^m}{1-K\,\delta_{lm}x^l x^m} \\ \langle\phi^a\rangle = \delta^a_\mu\,x^\mu \end{array}\right]$$

Is this form for $f_{\mu\nu}$ the only choice?

- Case with different $f_{\mu\nu} \longrightarrow$ FRW d'Amico et al '11; Koyama et al '11; Volkov '11,'12; Kobayashi et al '12
- Although background dynamics homogeneous+isotropic, there *is* a broken FRW symmetry in the Stückelberg sector, which *can* be probed by perturbations.

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Cosmological solutions for Minkowski fiducial metric

AEG, Lin, Mukohyama '11a

- No flat FRW, for Minkowski fiducial. `d'Amico et al '11`
- But open FRW solutions exist

$$ds^2 = -N^2\, dt^2 + a^2\, \Omega_{ij}^{(K<0)}\, dx^i\, dx^j$$

$$ds_\phi^2 = -n^2\, dt^2 + \alpha^2\, \Omega_{ij}^{(K<0)}\, dx^i\, dx^j$$

$$\left[n = \dot{\alpha}/\sqrt{|K|}\right] \Longleftarrow \text{Minkowski in open chart}$$

Minkowski in open coordinates

- Minkowski metric $\quad ds_\phi^2 = -[d\tilde{\phi}^0]^2 + \delta_{ij} d\tilde{\phi}^i d\tilde{\phi}^j$
- After coordinate transformation

$$\tilde{\phi}^0 = \frac{\alpha(\phi^0)}{\sqrt{|K|}} \sqrt{1 + |K| \delta_{ij}\phi^i\phi^j}\,, \qquad \tilde{\phi}^i = \alpha(\phi^0)\phi^i\,.$$

becomes:

$$ds_\phi^2 = -\frac{[\alpha'(\phi^0)]^2}{|K|}[d\phi^0]^2 + [\alpha(\phi^0)]^2\, \Omega_{ij}(\{\phi^i\})\, d\phi^i d\phi^j$$

- No closed FRW chart of Minkowski $\Longrightarrow$ no closed solution

Cosmological solutions for Minkowski fiducial metric

AEG, Lin, Mukohyama '11a

Equation of motion for $\phi^0 \Longrightarrow$ 3 branches of solutions:

$$\left(\frac{\dot{a}}{N} - \sqrt{|K|}\right) J_\phi\left(\frac{\alpha}{a}\right) = 0$$

- Branch I $\Longrightarrow \dot{a} = \sqrt{|K|}N \Longrightarrow g_{\mu\nu}$ is also Minkowski (open chart)

  $\Longrightarrow$ *No cosmological expansion!*

- Branch II$_\pm$ $\Longrightarrow J_\phi(\alpha/a) = 0$

  $$\left[J_\phi(X) \equiv 3 + 3\,\alpha_3 + \alpha_4 - 2\,(1 + 2\,\alpha_3 + \alpha_4)\,X + (\alpha_3 + \alpha_4)\,X^2\right]$$

$$\alpha = a\,X_\pm, \quad \text{with } X_\pm \equiv \frac{1 + 2\alpha_3 + \alpha_4 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4}}{\alpha_3 + \alpha_4} = \text{constant}$$

For $K = 0$, this branch not present. Only Branch I remains.

- Open universe?

  $\Longrightarrow$ observations: (curvature contribution)$_0 \lesssim 1\%$

Extension to generic fiducial metric

AEG, Lin, Mukohyama '11b

- Extending the field space metric, the line elements are

$$ds^2 = -N^2\, dt^2 + a^2\, \Omega_{ij} dx^i\, dx^j$$

$$ds_\phi^2 = -n^2\, dt^2 + \alpha^2\, \Omega_{ij} dx^i\, dx^j$$

Generically, we can have spatial curvature with either sign

e.g. at the cost of introducing a new scale (H_f), de Sitter fiducial can be brought into flat, open and closed FRW form.

Equations of motion for $\phi^0 \Rightarrow$ 3 branches of solutions

$$(a\,H - \alpha H_f)\; J_\phi \left(\frac{\alpha}{a}\right) = 0$$

- Branch I : $a\,H = \alpha\, H_f$ $\left[H_f \equiv \frac{\dot{\alpha}}{\alpha\, n}\right]$
- Branch II$_\pm$: 2 cosmological branches

$$\alpha(t) = X_\pm\, a(t)$$

$\Longrightarrow$ same solution as in Minkowski fiducial

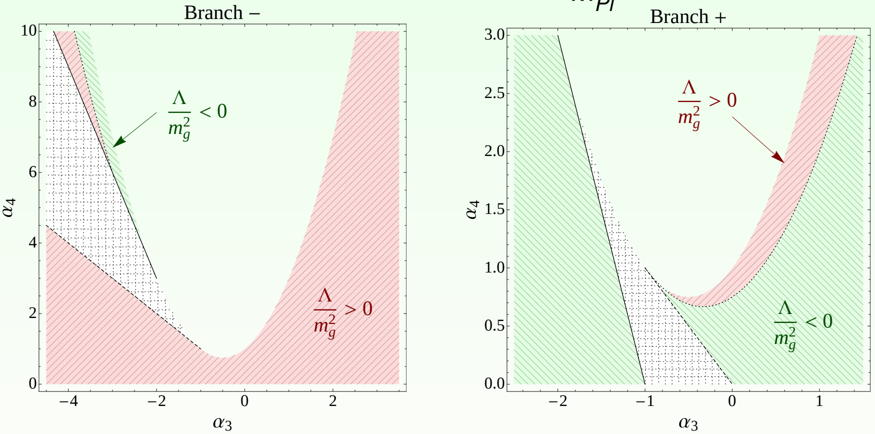
- Expansion in Branch I can be determined by the matter content $\Rightarrow$ in principle, can have cosmology.

Branch $\mathrm{II}_{\pm}$: Self-acceleration

- Evolution of Branch $\mathrm{II}_{\pm}$, with generic (conserved) matter

$$\left[H \equiv \frac{\dot{a}}{a\,N}\right] \longrightarrow 3\,H^2 + \frac{3\,K}{a^2} = \Lambda_{\pm} + \frac{1}{M_{Pl}^2}\,\rho$$

→ *independent of H_f*



$$\Lambda_{\pm} \equiv -\frac{m_g^2}{(\alpha_3+\alpha_4)^2}\left[(1+\alpha_3)\left(2+\alpha_3+2\,\alpha_3^2-3\,\alpha_4\right) \pm 2\,\left(1+\alpha_3+\alpha_3^2-\alpha_4\right)^{3/2}\right]$$

Perturbing the solution

- Lack of BD ghost does not guarantee stability. Higuchi '87
- Branch II is disconnected from Branch I. Do we still have 5 dof? Scalar sector may include additional couplings, giving rise to potential conflict with observations. Does Vainshtein mechanism still work?
- Can we distinguish massive gravity from other models of dark energy/modified gravity?

Perturbations and gauge invariant variables
AEG, Lin, Mukohyama '11b

- Perturbations in the metric, Stückelberg fields and matter fields:

$$g_{00} = -N^2(t)\,[1+2\phi]\,, \quad g_{0i} = N(t)a(t)\beta_i\,, \quad g_{ij} = a^2(t)\left[\Omega_{ij}(x^k) + h_{ij}\right]$$

$$\varphi^a = x^a + \pi^a + \frac{1}{2}\pi^b\partial_b\pi^a + O(\epsilon^3)\,, \qquad \sigma_I = \sigma_I^{(0)} + \delta\sigma_I$$

← *matter sector*

- Scalar-vector-tensor decomposition:

$$\beta_i = D_i\beta + S_i\,, \qquad \pi_i = D_i\pi + \pi_i^T\,,$$

$$h_{ij} = 2\psi\Omega_{ij} + \left(D_iD_j - \tfrac{1}{3}\Omega_{ij}\triangle\right)E + \tfrac{1}{2}(D_iF_j + D_jF_i) + \gamma_{ij}$$

$$D_i \leftarrow \Omega_{ij}\,, \quad \triangle \equiv \Omega^{ij}D_iD_j$$
$$D^iS_i = D^i\pi_i^T = D^iF_i = 0$$
$$D^i\gamma_{ij} = \gamma^i_i = 0$$

- Gauge invariant variables without Stückelberg fields:

Originate from $g_{\mu\nu}$ and matter fields $\delta\sigma_I$ →

$$Q_I \equiv \delta\sigma_I - \mathcal{L}_Z\sigma_I^{(0)}\,,$$
$$\Phi \equiv \phi - \tfrac{1}{N}\partial_t(NZ^0)\,,$$
$$\Psi \equiv \psi - \tfrac{\dot a}{a}Z^0 - \tfrac{1}{6}\triangle E\,,$$
$$B_i \equiv S_i - \tfrac{a}{2N}\dot F_i\,,$$

$$\left(\begin{array}{c} Z^0 \equiv -\frac{a}{N}\beta + \frac{a^2}{2N^2}\dot E \\ Z^i \equiv \frac{1}{2}\Omega^{ij}(D_jE + F_j) \\ \text{Under } x^\mu \to x^\mu + \xi^\mu : \\ Z^\mu \to Z^\mu + \xi^\mu \end{array}\right)$$

- However, we have 4 more degrees of freedom:

Associated with Stückelberg fields

$$\psi^\pi \equiv \psi - \frac{1}{3}\triangle\pi - \frac{\dot a}{a}\pi^0\,, \qquad E^\pi \equiv E - 2\pi\,, \qquad F_i^\pi \equiv F_i - 2\pi_i^T$$

Quadratic action

- After using background constraint for Stückelberg fields:

$$S^{(2)} = \underbrace{S^{(2)}_{\rm EH} + S^{(2)}_{\rm matter} + S^{(2)}_{\Lambda_\pm}}_{\text{depend only on } Q_I,\,\Phi,\,\Psi,\,B_i,\,\gamma_{ij}} + \underbrace{\tilde{S}^{(2)}_{\rm mass}}_{\tilde{S}^{(2)}_{\rm mass} = S^{(2)}_{\rm mass} - S^{(2)}_{\Lambda_\pm}}$$

- The first part is equivalent to GR + $\Lambda_\pm$+ Matter fields σ_I.
- The additional term:

$$M_{GW}^2 \equiv m_g^2 \left(1 - \frac{a\,n}{\alpha\,N}\right) \frac{\alpha^2}{a^2} \times \left[(1 + 2\,\alpha_3 + \alpha_4) - \frac{\alpha}{a}(\alpha_3 + \alpha_4)\right]$$

$$\tilde{S}^{(2)}_{\rm mass} = M_p^2 \int d^4x\, N\, a^3 \sqrt{\Omega}\, M_{GW}^2 \times \left[3(\psi^\pi)^2 - \frac{1}{12} E^\pi \triangle(\triangle + 3K) E^\pi + \frac{1}{16} F^i_\pi (\triangle + 2K) F^\pi_i - \frac{1}{8}\gamma^{ij}\gamma_{ij}\right]$$

- The only common variable is γ_{ij}.
- E^π, ψ^π, F^π_i have no kinetic term!

Cancellation of kinetic terms

Other examples of cancellation

- Self-accelerating solutions in the decoupling limit
  de Rham, Gabadadze, Heisenberg, Pirtskhalava '10
- Inhomogeneous de Sitter solutions
  Koyama, Niz, Tasinato '11
- dS and Schwarschild dS solutions in the decoupling limit
  Berezhiani, Chkareuli, de Rham, Gabadadze, Tolley '11
- A branch of self-accelerating solutions in bimetric gravity
  Crisostomi, Comelli, Pilo '12
- Self-accelerating spherically symmetric, isotropic solutions
  Gratia, Hu, Wyman '12
- Branch of self-accelerating solutions in quasi-dilaton massive gravity
  d'Amico, Gabadadze, Hui, Pirtskhalava '12

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- Inhomogeneous de Sitter solutions
  Koyama, Niz, Tasinato '11
- dS a…
- A br…
- Self…
- Bra…

What is the fate of these degrees?

1. Infinitely strong coupling?
2. Infinitely heavy degrees? Then, they can be integrated out
   $\Longrightarrow$ same d.o.f. as in GR, Higuchi bound (or its analogue) irrelevant, no need for Vainshtein mechanism. The only signature imprinted in the perturbations is the GW signal.
   [*Discussed in Norihiro Tanahashi's talk yesterday*]
3. Sign of (high-order) kinetic term? $\Rightarrow$ Ghost

Need to go beyond linear order to determine which case is realized

Probing the nonlinear action with linear tools

- The cancellation seems to be a consequence of the symmetry of the background.
- Instead of computing the high order action, we slightly break the isotropy and compute the quadratic terms.
- The broken anisotropy allows us to obtain information on the high order terms in the exact FRW case.
- The deviation from isotropy in the background is interpreted as $k = 0$ perturbation in the isotropic solution.

Introducing anisotropy
de Felice, AEG, Mukohyama '12

- The simplest anisotropic extension of flat FRW is the degenerate Bianchi type–I metric

$$ds^2 = -N^2 dt^2 + a^2\, dx^2 + b^2\, \left(dy^2 + dz^2\right)$$

- Different fiducial metric $\Leftrightarrow$ different theory. In order to have continuity with the FRW solutions, we keep $f_{\mu\nu}$ isotropic:

$$ds_\phi^2 = -n^2\, dt^2 + \alpha^2\, (dx^2 + dy^2 + dz^2)$$

No spatial curvature $\Longrightarrow$ Non-Minkowski fiducial.
However, our analysis is valid for generic fiducial metrics, and can be generalized to non-zero curvature spaces.

Decomposition of perturbations

- Perturbations are decomposed with respect to the 2d rotational symmetry around the x axis

$$
\delta g_{\mu\nu} = \begin{array}{c} \\ 0 \\ 1 \\ i \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & j \end{array} \\ \begin{pmatrix} -2\,N^2\,\Phi & a\,N\,\partial_x\chi & b\,N\,(\partial_j B + v_j) \\ & a^2\,\psi & a\,b\,\partial_x\,(\partial_j\beta + \lambda_j) \\ & & b^2\left[\tau\,\delta_{ij} + 2\,E_{,ij} + h_{(i,j)}\right] \end{pmatrix} \end{array}
\qquad
\left[\begin{array}{c} i,j=2,3 \\ \partial^i v_i=0 \\ \partial^i \lambda_i=0 \\ \partial^i h_i=0 \\ \partial^i \pi_i=0 \end{array}\right]
$$

$$
\delta\phi^\mu = \begin{pmatrix} \pi^0 & \partial_x\pi^1 & \partial^i\pi+\pi^i \end{pmatrix}
$$

- Advantage of the axisymmetry: 2d scalars and 2d vectors decouple at linear level.
- Physical degrees in 2d scalar sector (*even modes*) (10 total) - (3 nondynamical) - (3 gauge) - (1 BD ghost) = 3
- Physical degrees in 2d vector sector (*odd modes*) (4 total) - (1 nondynamical) - (1 gauge) = 2
- We are interested in the stability of the gravity sector, so we do not include any matter fields. Only bare Λ.

Gauge invariant variables : Anisotropic case

GI constructed only out of $\delta g_{\mu\nu}$

$$\hat{\Phi} = \Phi - \frac{1}{2N}\partial_t\left(\frac{\tau}{H_b}\right)$$

$$\hat{\chi} = \chi + \frac{1}{2\,a\,H_b}\tau - \frac{a}{N}\partial_t\left[\frac{b}{a}\left(\beta - \frac{b}{2\,a}E\right)\right]$$

$$\hat{B} = B + \frac{1}{2\,b\,H_b}\tau - \frac{b}{2N}\partial_t E$$

$$\hat{\psi} = \psi - \frac{H_a}{H_b}\tau - \frac{b}{a}\partial_x^2\left(2\,\beta - \frac{b}{a}E\right)$$

$$\hat{v}_i = v_i - \frac{b}{2N}\partial_t h_i$$

$$\hat{\lambda}_i = \lambda_i - \frac{b}{2\,a}h_i$$

GI referring to $\delta\phi^a$

$$\hat{\tau}_\pi = \pi^0 - \frac{\tau}{2\,N\,H_b}$$

$$\hat{\beta}_\pi = \pi^1 - \frac{b}{a}\left(\beta - \frac{b}{2\,a}E\right)$$

$$\hat{E}_\pi = \pi - \frac{1}{2}E$$

$$\hat{h}_{\pi\,i} = \pi_i - \frac{1}{2}h_i$$

Strategy

- Use gauge invariant variables to keep track of the new massive graviton degrees. This removes the pure gauge combinations.
- Integrate out non-dynamical degrees
  (4 in the 2d scalar sector, 1 in the 2d vector sector)
- Expand around FRW solution for small anisotropy
- Diagonalize the Lagrangian: Bring the action to the canonical form by rescaling and rotating the fields.

$\Longrightarrow$ **Obtain dispersion relations**

Small anisotropy expansion

- Equation of motion for $\phi^0 \Longrightarrow$ No factorization

$$\left(H_a - \frac{\alpha}{a} H_f\right) J_\phi^{\parallel}\left(\frac{\alpha}{b}\right) + 2\left(H_b - \frac{\alpha}{b} H_f\right) J_\phi^{\perp}\left(\frac{\alpha}{a}, \frac{\alpha}{b}\right) = 0 \qquad \left[\begin{array}{l} H_a \equiv \frac{\dot{a}}{a\,N} \\ H_b \equiv \frac{\dot{b}}{b\,N} \\ H_f \equiv \frac{\dot{\alpha}}{\alpha\,n} \end{array}\right]$$

- Small anisotropy around average scale factor $\bar{a} \equiv (a\,b^2)^{1/3}$

$$a = \bar{a}\left[1 + 2\,\sigma + \mathcal{O}(\sigma^2)\right] \qquad \left[|\sigma| \ll 1\right] \qquad b = \bar{a}\left[1 - \sigma + \mathcal{O}(\sigma^2)\right]$$

- Using FRW branch II solutions $\Longrightarrow \dfrac{\alpha}{\bar{a}} = X_\pm + \mathcal{O}(\sigma^2)$.

Odd sector – 2d vectors

- The action, after small anisotropy expansion, takes the form:

$$S^{(2)}_{\rm odd} \simeq \frac{M_{\rm Pl}^2}{2}\int N\, dt\, dk_L d^2 k_T \bar{a}^3 \left[K_{11}\frac{|\dot{\mathcal{Q}}_1|^2}{N^2} - \Omega_{11}^2\, |\mathcal{Q}_1|^2 + K_{22}\frac{|\dot{\mathcal{Q}}_2|^2}{N^2} - \Omega_{22}^2\, |\mathcal{Q}_2|^2\right]$$

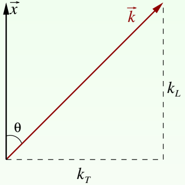
at leading order:

$$K_{11} = \frac{k_L^2\, k_T^4}{2\, k^2} \qquad K_{22} = \frac{\bar{a}^2\, k_T^2\, M_{GW}^2}{4\left(1 - \frac{\bar{a}^2\, n^2}{\alpha^2\, N^2}\right)}\,\sigma$$

$$\frac{\Omega_{11}^2}{K_{11}} = \frac{k^2}{\bar{a}^2} + M_{GW}^2 \qquad \frac{\Omega_{22}^2}{K_{22}} = \frac{1}{2\,\sigma}\left(1 - \frac{\bar{a}^2\, n^2}{\alpha^2\, N^2}\right)\frac{k^2}{\bar{a}^2}$$

1 GW in FRW

New degree



- condition for avoiding the ghost and gradient instability:

$$\left(1 - \frac{\bar{a}\, n}{\alpha\, N}\right)\sigma > 0$$

Even sector – 2d scalars

- The full quadratic action is formally (in terms of G.I. quantities)

$$S^{(2)}_{\rm even} = \frac{M_p^2}{2}\int N\,dt\,dk_L\,d^2k_T\,a\,b^2\,\mathcal{L}_{\rm even}$$

$$\mathcal{L}_{\rm even} = \frac{\dot{\mathcal{Y}}^\dagger}{N}\,K\,\frac{\dot{\mathcal{Y}}}{N} - \mathcal{Y}^\dagger\,\Omega^2\,\mathcal{Y} + \mathcal{Z}^\dagger\,\mathcal{A}\,\mathcal{Y} + \mathcal{Y}^\dagger\,\mathcal{A}^T\,\mathcal{Z} + \mathcal{Z}^\dagger\,\mathcal{B}\,\frac{\dot{\mathcal{Y}}}{N} + \frac{\dot{\mathcal{Y}}^\dagger}{N}\,\mathcal{B}^T\,\mathcal{Z} + \mathcal{Z}^\dagger\,\mathcal{C}\,\mathcal{Z}$$

- $\mathcal{Y} \Rightarrow$ 3 dynamical degrees (in GR, 2 are gauge)
- $\mathcal{Z} \Rightarrow$ 4 non-dynamical degrees (including the BD ghost $\pi^0 - \frac{\tau}{2\,N\,H_b}$)
- E.O.M. for n.d. modes

$$\mathcal{Z} = -\mathcal{C}^{-1}\left(\mathcal{A}\,\mathcal{Y} + \mathcal{B}\,\frac{\dot{\mathcal{Y}}}{N}\right)$$

- Now all 3 d.o.f in the action are dynamical

$$\mathcal{L}_{\rm even} = \frac{\dot{\mathcal{Y}}^\dagger}{N}\,\bar{K}\,\frac{\dot{\mathcal{Y}}}{N} + \frac{\dot{\mathcal{Y}}^\dagger}{N}\,\bar{M}\,\mathcal{Y} + \mathcal{Y}^\dagger\,\bar{M}^T\,\frac{\dot{\mathcal{Y}}}{N} - \mathcal{Y}^\dagger\,\bar{\Omega}^2\,\mathcal{Y}$$

$$\left[\bar{K} = K - \mathcal{B}^T\,\mathcal{C}^{-1}\,\mathcal{B}\,,\qquad \bar{M} = -\mathcal{B}^T\,\mathcal{C}^{-1}\,\mathcal{A}\,,\qquad \bar{\Omega}^2 = \Omega^2 + \mathcal{A}^T\,\mathcal{C}^{-1}\,\mathcal{A}\right]$$

Even sector – 2d scalars

- The full quadratic action is formally (in terms of G.I. quantities)

$$S^{(2)}_{\rm even} = \frac{M_p^2}{2} \int N\,dt\,dk_L\,d^2k_T\,a\,b^2\,\mathcal{L}_{\rm even}$$

$$\mathcal{L}_{\rm even} = \frac{\dot{\mathcal{Y}}^\dagger}{N} K \frac{\dot{\mathcal{Y}}}{N} - \mathcal{Y}^\dagger \Omega^2 \mathcal{Y} + \mathcal{Z}^\dagger \mathcal{A} \mathcal{Y} + \mathcal{Y}^\dagger \mathcal{A}^T \mathcal{Z} + \mathcal{Z}^\dagger \mathcal{B} \frac{\dot{\mathcal{Y}}}{N} + \frac{\dot{\mathcal{Y}}^\dagger}{N} \mathcal{B}^T \mathcal{Z} + \mathcal{Z}^\dagger \mathcal{C} \mathcal{Z}$$

- $\mathcal{Y} \Rightarrow$ 3 dynamical degrees (in GR, 2 are gauge)
- $\mathcal{Z} \Rightarrow$ 4 non-dynamical degrees (including the BD ghost $\pi^0 - \frac{\tau}{2\,N\,H_b}$)
- E.O.M. for n.d. modes

$$\mathcal{Z} = -\mathcal{C}^{-1}\left(\mathcal{A}\,\mathcal{Y} + \mathcal{B}\,\frac{\dot{\mathcal{Y}}}{N}\right)$$

- Now all 3 d.o.f in the action are dynamical

$$\mathcal{L}_{\rm even} = \frac{\dot{\mathcal{Y}}^\dagger}{N} \bar{K} \frac{\dot{\mathcal{Y}}}{N} + \frac{\dot{\mathcal{Y}}^\dagger}{N} \bar{M} \mathcal{Y} + \mathcal{Y}^\dagger \bar{M}^T \frac{\dot{\mathcal{Y}}}{N} - \mathcal{Y}^\dagger \bar{\Omega}^2 \mathcal{Y}$$

$$\left[\bar{K} = K - \mathcal{B}^T \mathcal{C}^{-1} \mathcal{B}\,, \qquad \bar{M} = -\mathcal{B}^T \mathcal{C}^{-1} \mathcal{A}\,, \qquad \bar{\Omega}^2 = \Omega^2 + \mathcal{A}^T \mathcal{C}^{-1} \mathcal{A}\right]$$

- Use small anisotropy expansion and diagonalize $\bar{K}$ at leading order

1 GW in FRW ← $\kappa_1 = \frac{k_T^4}{8\,k^4}$

$$\kappa_2 = -\frac{2\,\bar{a}^2\,M_{GW}^2\,k_L^2}{\left(1 - \frac{\bar{a}^2\,n^2}{\alpha^2\,N^2}\right)}\,\sigma$$

$$\kappa_3 = - \frac{k_T^2}{2\,k_L^2}\kappa_2$$

wrong sign!

Dispersion relations

- If the ghost has a mass gap, it may still be heavy enough in FRW limit, to be integrated out from the low energy effective theory.
- It is still possible to diagonalize the system and obtain the eigenfrequencies at leading order in σ expansion

$$\omega_1^2 = \frac{k^2}{\bar{a}^2} + M_{GW}^2$$

$$\omega_2^2 = -\left(\frac{1 - \frac{\bar{a}^2\, n^2}{\alpha^2\, N^2}}{24\, \bar{a}^2\, \sigma}\right) \overbrace{\left[\sqrt{\left(10\, k_L^2 + 11\, k_T^2\right)^2 - 8\, k_L^2\, k_T^2} - \left(2\, k_L^2 + 5\, k_T^2\right)\right]}^{>8k^2 \qquad <10k^2}$$

$$\omega_3^2 = \left(\frac{1 - \frac{\bar{a}^2\, n^2}{\alpha^2\, N^2}}{24\, \bar{a}^2\, \sigma}\right) \left[\sqrt{\left(10\, k_L^2 + 11\, k_T^2\right)^2 - 8\, k_L^2\, k_T^2} + \left(2\, k_L^2 + 5\, k_T^2\right)\right]$$

- We assumed $\left(1 - \frac{\bar{a}\, n}{\alpha\, N}\right)\, \sigma > 0$, so mode 2 is the ghost. (For < 0, ghost is mode 3, but we have another ghost from the odd sector)
- $\omega^2 \propto k^2 \Rightarrow$ No mass gap!

Discussion

- Small background anisotropy $\Leftrightarrow$ perturbations in FRW.
- Quadratic kinetic term for $1 \gg |\sigma| \neq 0$
  $\Longleftrightarrow \phi_{k_1}\dot{\phi}_{k_2}\dot{\phi}_{k_3}$ type terms, with one $k_i = 0$.
- Homogeneous and isotropic solutions in massive gravity have ghost instability which arises from the cubic order action.
- This conclusion is valid for $\pm$ cosmological branch solutions of massive gravity with arbitrary fiducial metric.
- Nonlinear analysis indicate the kinetic term for the longitudinal degrees reappear at cubic order. d'Amico '12
- Similar solutions in variants of the theory (e.g. in bigravity, quasi-dilaton...) have the vanishing kinetic term behavior.
  $\Rightarrow$ Are they also unstable?

Discussion

Alternatives?

- Branch I solutions in bigravity, quasi-dilaton? Can the Higuchi/Vainshtein conflict be resolved by the dynamics of the fiducial metric? Fasiello, Tolley '12
- It is still possible to have a H&I physical metric, while either H or I is broken in Stückelberg sector.
- Inhomogeneous examples already exist, although d'Amico '12 showed that cancellation occurs in two such examples (d'Amico et al '11 and Koyama, Niz, Tasinato '11).
- In our analysis, anisotropy was introduced only as a technical tool. However, the kinetic terms of extra polarizations *are* second order. $\Rightarrow$ A universe with finite anisotropy, which looks isotropic at the background level may have a chance to evade the ghost.
  $\Longrightarrow$ Stay tuned for the talk by Chunshan Lin.

Bonus: Stückelberg equation of motion

- For FRW

$$\frac{\delta S}{\delta \phi^0} = 0 \longrightarrow (H - H_f\, X)\, J_\phi\,(X) = 0$$

with $X \equiv \alpha/a$ and

$$J_\phi(X) \equiv 3 + 3\,\alpha_3 + \alpha_4 - 2\,(1 + 2\,\alpha_3 + \alpha_4)\,X + (\alpha_3 + \alpha_4)\,X^2$$

- For axisymmetric Bianchi–I

$$(H_a - X_a\, H_f)\; J_\phi^{\|}\,(X_b) + 2\;(H_b - X_b\, H_f)\; J_\phi^{\perp}\,(X_a,\, X_b) = 0$$

with $X_a \equiv \alpha/a$, $X_b \equiv \alpha/b$, $J_\phi^{\|}(X_b) \equiv J_\phi(X_b)$ and

$$J_\phi^{\perp}(X_a, X_b) \equiv 3{+}3\,\alpha_3{+}\alpha_4{-}(1{+}2\,\alpha_3{+}\alpha_4)\,(X_a{+}X_b){+}(\alpha_3{+}\alpha_4)\,X_a X_b$$