可解量子多体系の相関関数や形状因子と非平衡量子ダイナミクス:
１次元ボース気体の量子ソリトン状態

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基研研究会
「非平衡系の物理—その普遍的理解を目指して—」
YITP, Kyoto, Aug 1-4, 2012 (Aug. 4)

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  XXZ鎖（量子可積分系）の相関関数のレビュー (1)

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  量子ソリトン状態の構築とそのダイナミクス (2), (3), (4)
  (ダークソリトン）

ポイント:
(1) 厳密解の方法の着実な発展の紹介
(2) 冷却原子系の実験との関連性
(3) 孤立量子系における非平衡「緩和」ダイナミクス
(4) １次元ボース気体での厳密な量子BEC状態
（凝縮率＜1＋量子多体効果　―＞　平均場を越える？）

Part I

- See PDF file: review-part.pdf

Part Ⅱ: Quantum dark soliton

- 1D Bose gas with repulsive interaction

  \+ experimental backgrounds

- Construction of the **quantum state** of a **dark soliton** in the weak coupling case (0<c<< 1)

- **Exact relaxation dynamics** (video shaw):

Solving the Bethe equations numerically, we derive exact time evolution for a dipped density profile collapsing into a flat profile

(Relaxation of quantum states (M. Rigol et al PRL(2007) ;

P. Reimann, PRL (2008); Cf. von Neumann's quantum ergodic theorem (1929)); H. Tasaki, PRL(1998)

1D bosons interacting through the δ-function potentials

The Lieb-Liniger Hamiltoninan is given by

$$\mathcal{H}_{LL} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{j,k=1}^{N} c\,\delta(x_j - x_k)\,.$$

We introduce field operators for the 1D bosons, $\psi(x)$, $\psi(x)^\dagger$ satisfying the commutation relations:

$$[\psi(x), \psi^\dagger(y)] = \delta(x-y)$$

In the second quantized form, we have for $\mathcal{H}_{LL}$

$$\mathcal{H} = \int_0^L \{\partial_x \psi^\dagger(x) \partial_x \psi(x) + c\,\psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x)\} dx$$

The field operators satisfy the nonlinear Schrödinger equation:

$$i\partial_t \psi = -\partial_x^2 \psi + 2c\psi^\dagger \psi^\dagger \psi$$

Dark soliton: a solution of classical NLS equation
n: density, n=N/L where N the particle number
$\beta = 1 - v^2/(2\mu)$

We assume that $\psi(x) \to n$ for $x = \pm\infty$.

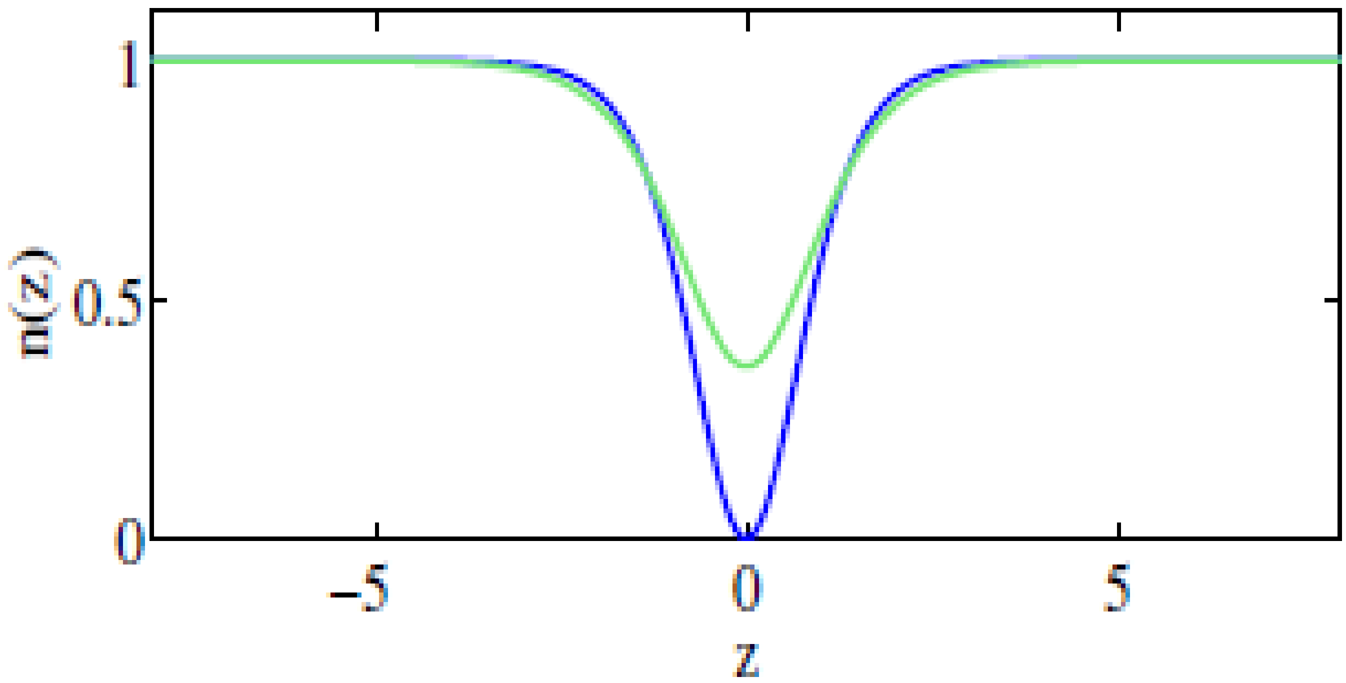
$$i\partial_t \hat{\psi} = -\partial_x^2 \hat{\psi} + 2c\hat{\psi}^\dagger \hat{\psi}\hat{\psi} - \mu\hat{\psi}$$

$$\psi(x) = \sqrt{n}\left(\sqrt{1-\beta} + i\sqrt{\beta}\tanh^2(x\sqrt{\beta n c})\right)$$

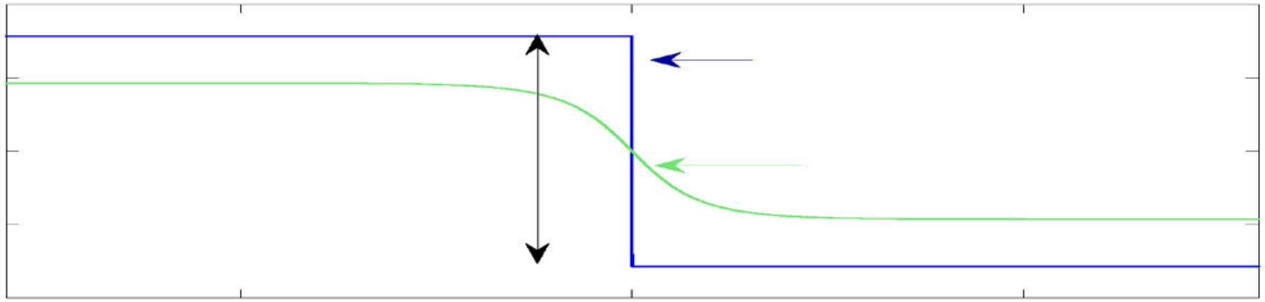
Here we have $\mu = n^2 c$.

Dark soliton (a solution of classical NLSE)

- Amplitude profile



Phase profile of a dark soliton



Brief review: experiments of cold atoms

- Through Interference of two plane-wave laser beams, one can construct one-dimensional trap of cold atoms.

Cf. T. Kinoshita et al., Nature **440**, 900 (2006); Science **305,** 1125(2004)

A dark soliton can be realized by the phase printing method
Cf. Becker et al., Nature Physics vol. 41 (2008)

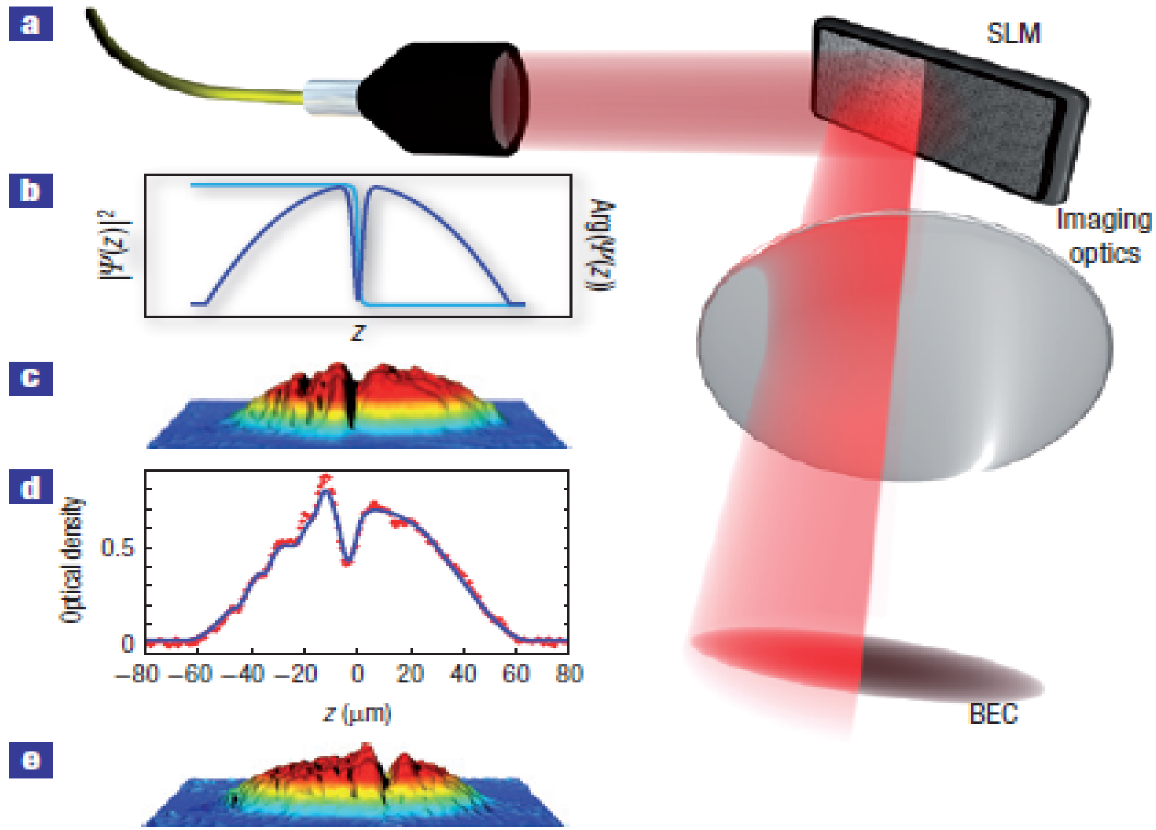


Figure 1 Principle of dark-soliton generation. a, Optical set-up. A spatial light modulator (SLM) is used to imprint a phase step by exposing part of the condensate to a far-detuned laser beam. **b**, Theoretical curve of a dark soliton's density $|\Psi|^2$

Note: dark solitons observed in cold atomic experiments should be related to a quantum state

- Bose-Einstein condensates of cold atomic systems should be described by a quantum state (pure state).

More than 30 years ago, it was conjectured that type II excitations should be related to a dark soliton

JOURNAL OF THE PHYSICAL SOCIETY OF JAPAN, Vol. 49, No. 4, OCTOBER, 1980

Solitons in a One-Dimensional Bose System with the Repulsive Delta-Function Interaction

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(Received May 12, 1980)

Lieb's type II excitation, derived exactly by means of a Bethe ansatz, in a one-dimensional Bose system with the repulsive delta-function interaction is identified with the soliton mode, which is a solution of the Pitaevskii-Gross (or nonlinear Schrödinger) equation for the corresponding Bose field.

However, it was not clear how to construct a quantum state which leads to a dark soliton in the weak coupling limit

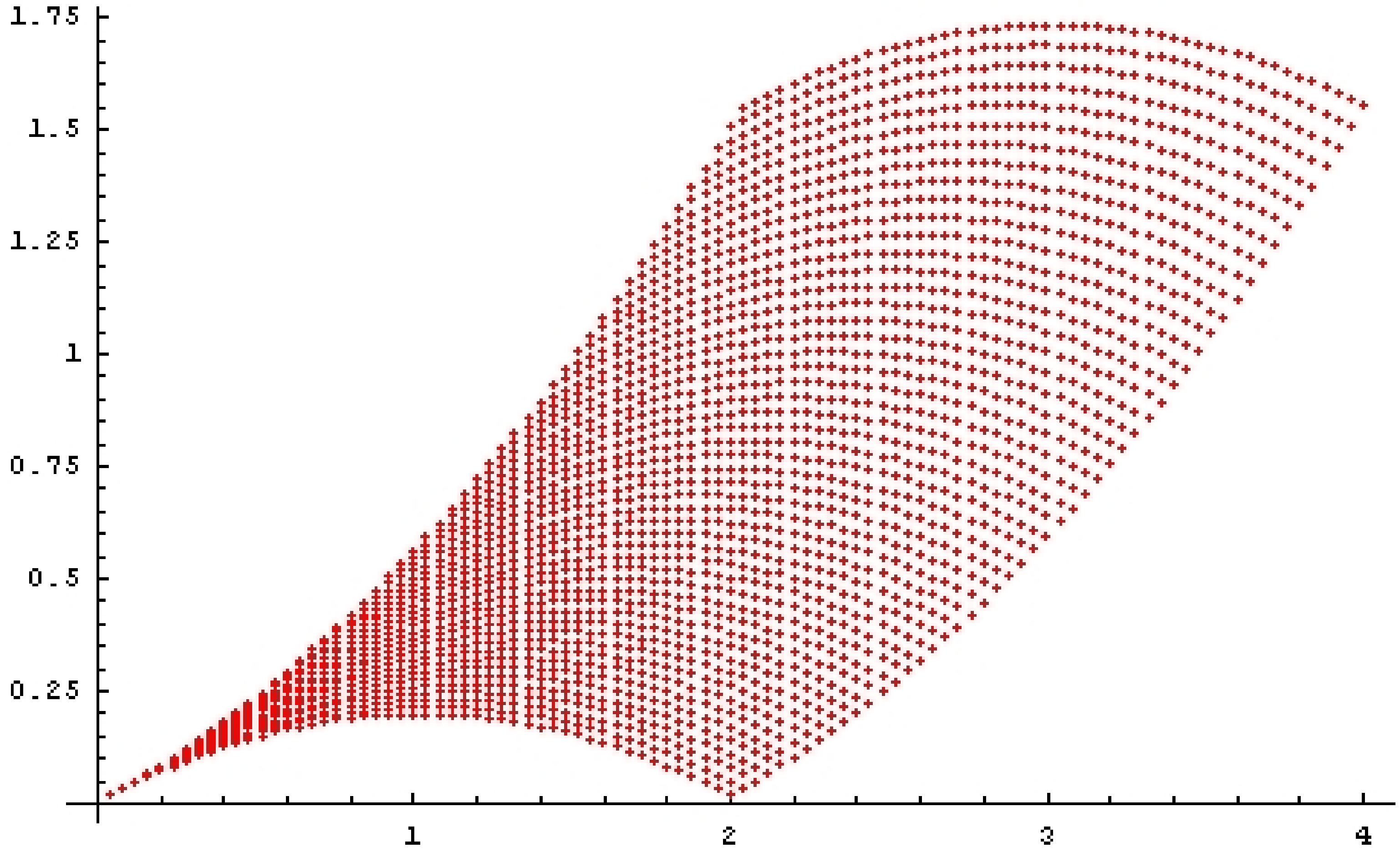
The argument of Ishikawa - Takayama was based on the similarity of the dispersion relations.

- Lieb's Type II excited state

  = Bethe eigenstate with one hole

- The density profile of a type II eigenstate is flat or constant.
- Remark: Every Bethe state is translationally invariant.

Spectrum of 1D Bose Gas (c=100)



Construction of the quantum state of a dark soliton (J. Sato et al., arXiv:1204.3960)

- We take superposition of excited states with on hole:

$$|X, N\rangle = \Sigma_{q=0}^{N-1} \exp(2\pi\ i\ pq)\ |p\rangle$$

Here $|p\rangle$ denotes the Bethe eigenstate with one hole corresponding to momentum $2\pi\ p/L$

``Delta function'' becomes a dark soliton

Application of form-factor formulas for the 1D Bose gas (Slavnov)

- If we solve the Bethe equations numerically very high precision, one can evaluate form factors numerically exactly.

- Errors are $O(10^{-16})$ (Jun Sato)

The density profile

- $|X, N\rangle$: The quantum state of a dark soliton

- $\langle X,N|\ \psi(x)^{\dagger}\ \psi(x)\ |X,N\rangle$:

  the density profile as a function of x

- We observe that the density profile overlaps that of the dark soliton with $v=v_c/2$, completely for small c
- Hereafter we put N=L and n=N/L=1

Slavnov's formula (1990) for form factors

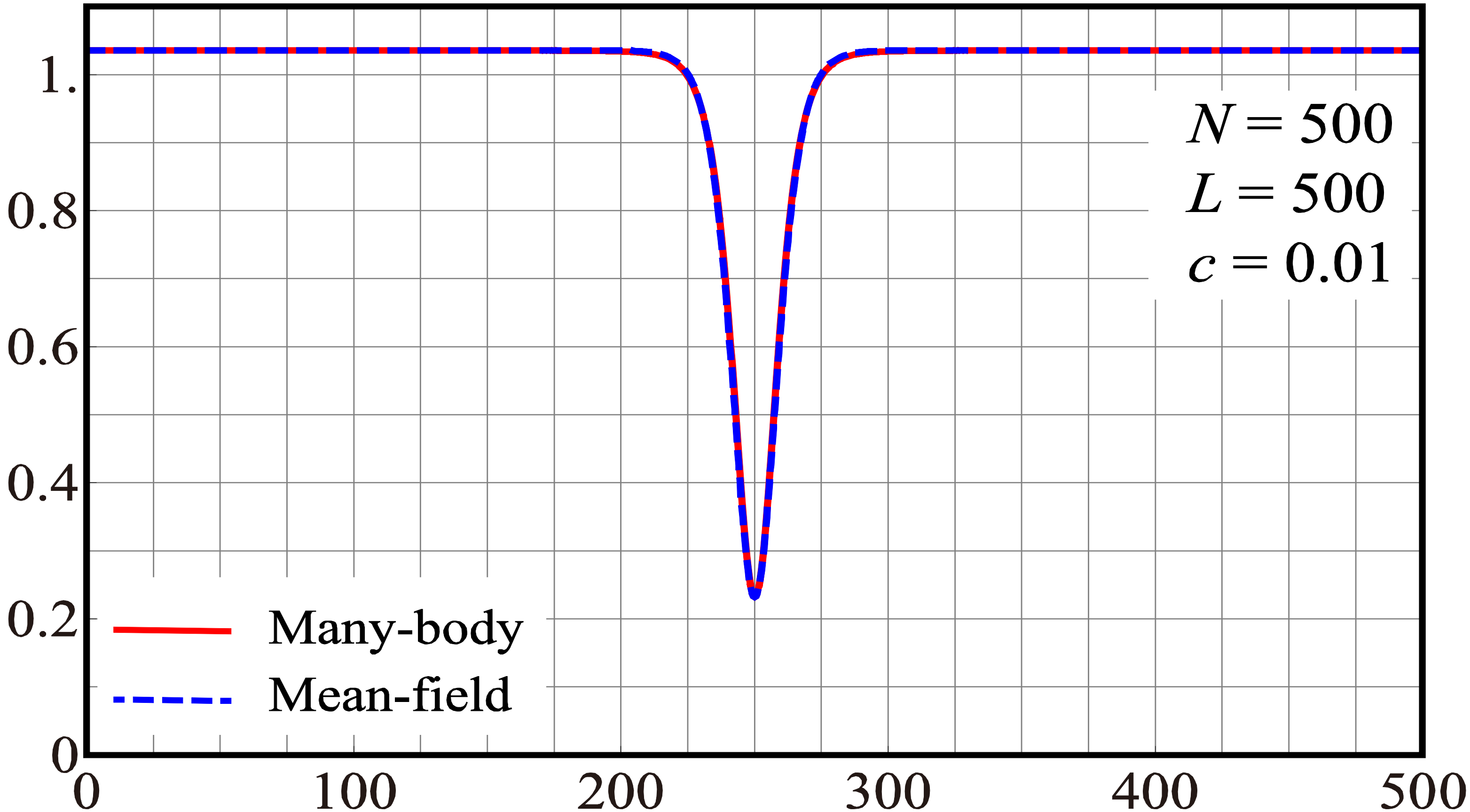
- <p'| $\psi(x)^\dagger \psi(x) |p\rangle$

$$= \frac{i^N}{(p-p')} \text{``Cauchy det''} * \det U$$

Here |p> and |p'> are two given Bethe eigenstates with momentum p and p'

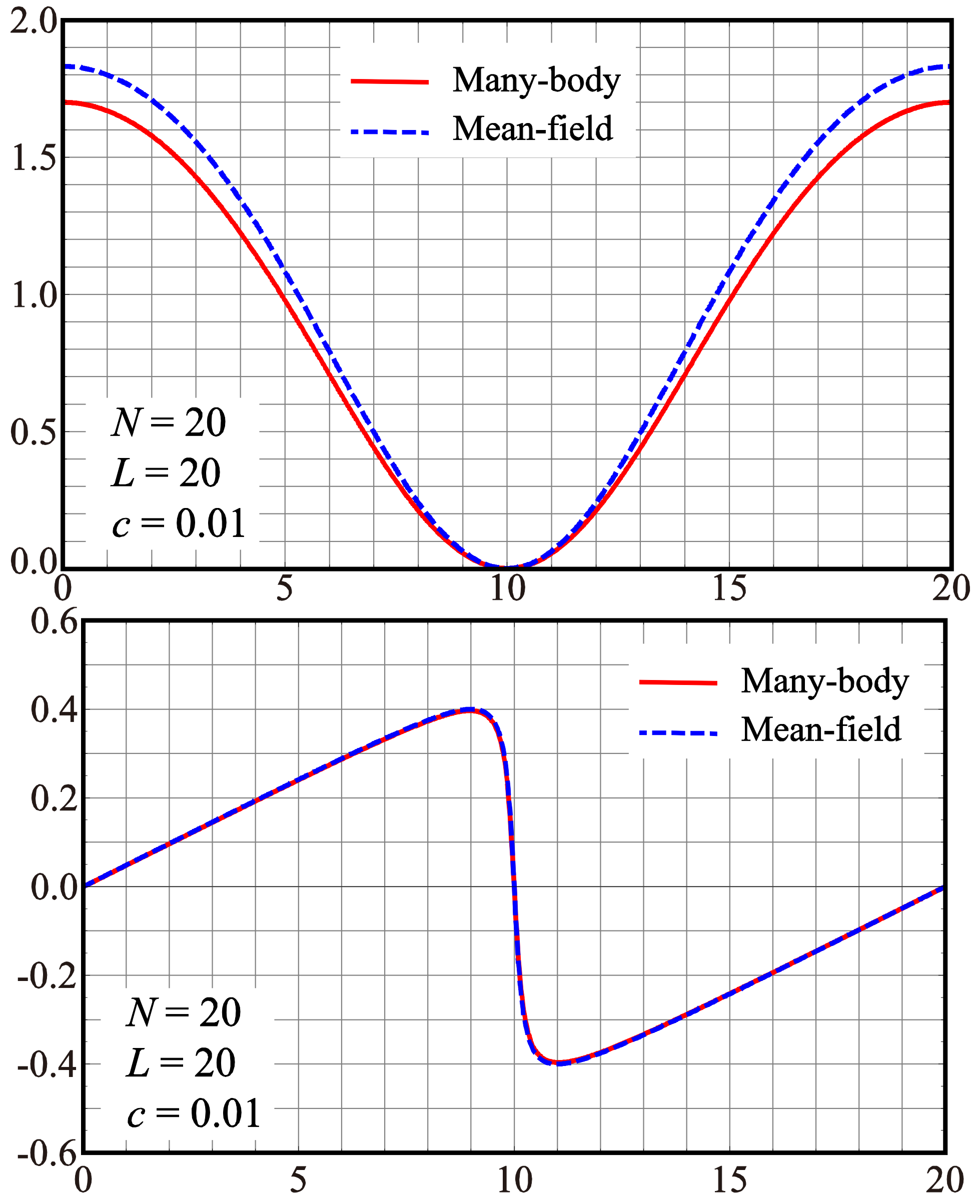
Quantum state of a dark soliton:

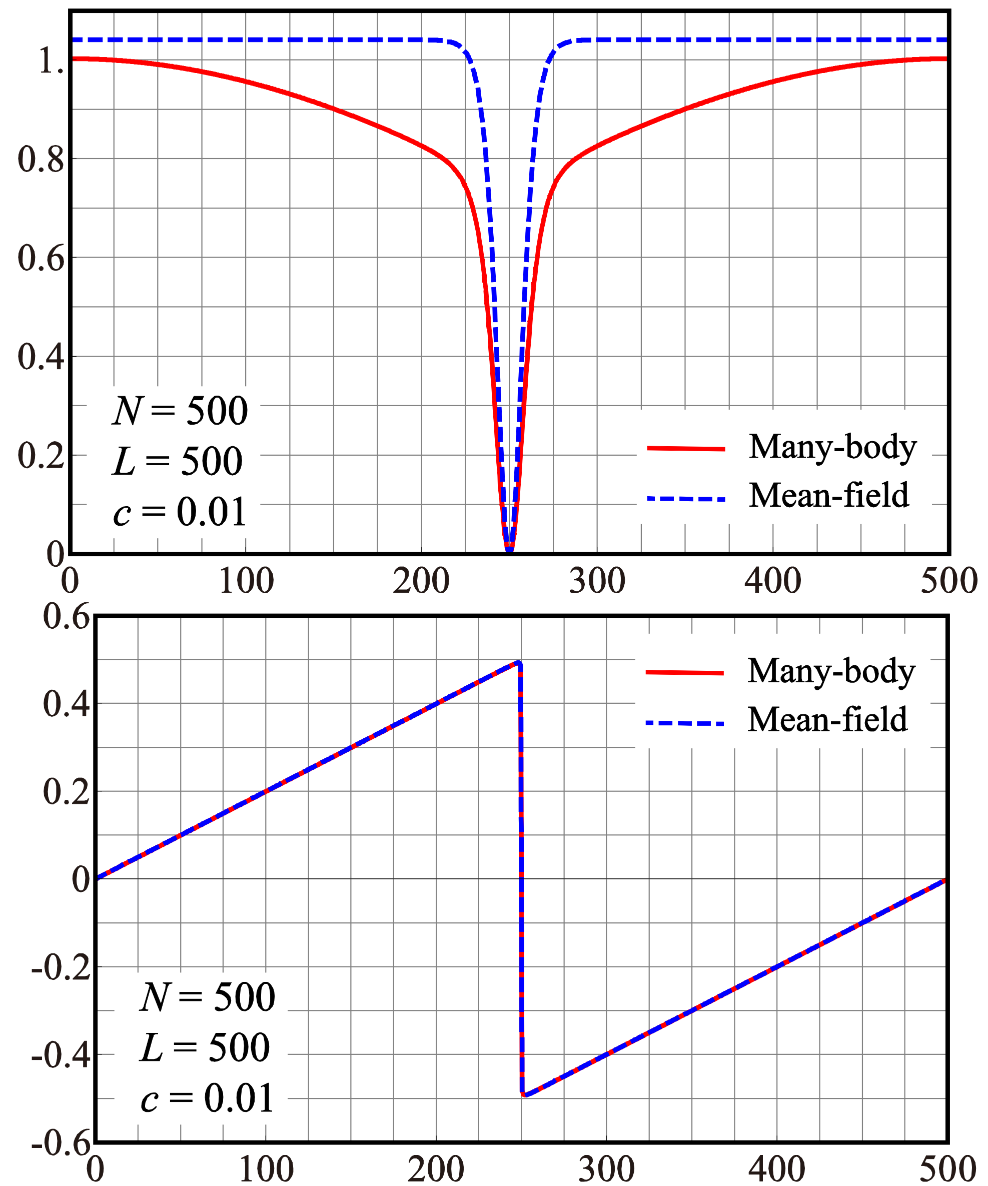
1. density profile overlaps the classical dark soliton, completely for small c. $v=v_c/2$



Matrix element of the field operator almost coincides with the classical scalar field of dark soliton for small c

- <X,N-1| $\phi(\mathrm{x})$ | X,N>: Matrix element of field operator between dark soliton states with N and N−1 particles.
- It gives the order parameter of BEC. (according to the Onsager−Penrose criterion)
- The amplitude profile and the phase profile coincide with those of the classical dark soliton with v=2 π/L , respectively.


2.0
1.5
1.0
0.5
0.0
0
5
10
15
20
Many-body
Mean-field
$N = 20$
$L = 20$
$c = 0.01$
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0
5
10
15
20
Many-body
Mean-field
$N = 20$
$L = 20$
$c = 0.01$



1.
0.8
0.6
0.4
0.2
0
0
100
200
300
400
500
$N = 500$
$L = 500$
$c = 0.01$
Many-body
Mean-field
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
100
200
300
400
500
$N = 500$
$L = 500$
$c = 0.01$
Many-body
Mean-field

Onsager-Penrose criterion of BEC

The one-particle reduced density matrix

- $\rho_1(x,y) = \langle X,N | \psi(x)^\dagger \psi(y) | X,N \rangle$

- If the largest eigenvalue of $\rho_1(x,y)$ is much larger than all the other eigenvalues, the system has BEC.

Depletion rate < 1

- $\langle X,N|\ \psi(x)^\dagger \psi(x)\ |X,N\rangle$

$= |\langle X,N-1|\ \psi(x)\ |X,N\rangle|^2 + \Sigma_n\ |\langle n|\ \psi(x)\ |X,N\rangle|^2$

ここで　nは|X, N-1> とは異なる全ての状態

右辺の第二項の大きさはBEC凝縮率が１より小さい程度(depletion rate) を示す

Remark 1: Various applications of Slavnov's form-factor formula (1990) for 1D Bose gas

- We can exactly follow the time evolution of the quantum many-body system both in real-time and real-space.

To each eigenstate |p> we derive a set of numerical solutions of the Bethe equations numerically with quite high precision.

- Some similarity with Quench dynamics（P. Calabrese and J.S. Caux, 2007; F. Essler (transverse Ising chain))

Remark 2: Historical comments

- For attractive case, quantum state of a bright soliton was constructed in 1980's.
- By superposing Bethe eigenstates of an N-string, bright soliton states were constructed.
- In optical fibers, bright soliton states are studied in several experiments.

Bright soliton is derived from a quantum state through a different limit

Journal of the Physical Society of Japan
Vol. 53, No. 6, June, 1984, pp. 1933–1938

Classical Soliton as a Limit of the Quantum Field Theory

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(Received February 8, 1984)

In the Nonlinear Schrödinger model, the correspondence between classical soliton and its quantum field theory is investigated. It is shown explicitly that moving soliton as well as soliton at rest arise as the matrix elements of a field operator in the limit $n \to \infty$, where n is the particle number making the bound state.

Exact non-equilibrium dynamics of a quantum dark soliton

(1) Relaxation for N=1000

(2) Recurrence for N=20

By Jun Sato (JSPS Fellow, Ochanomizu Univ.)

J. Sato et al, PRL vol. 108, 110401 (2012)

Time evolution of the density profile

The density operator is defined by

$$\rho(x,t) = \psi^{\dagger}(x,t)\psi(x,t)$$

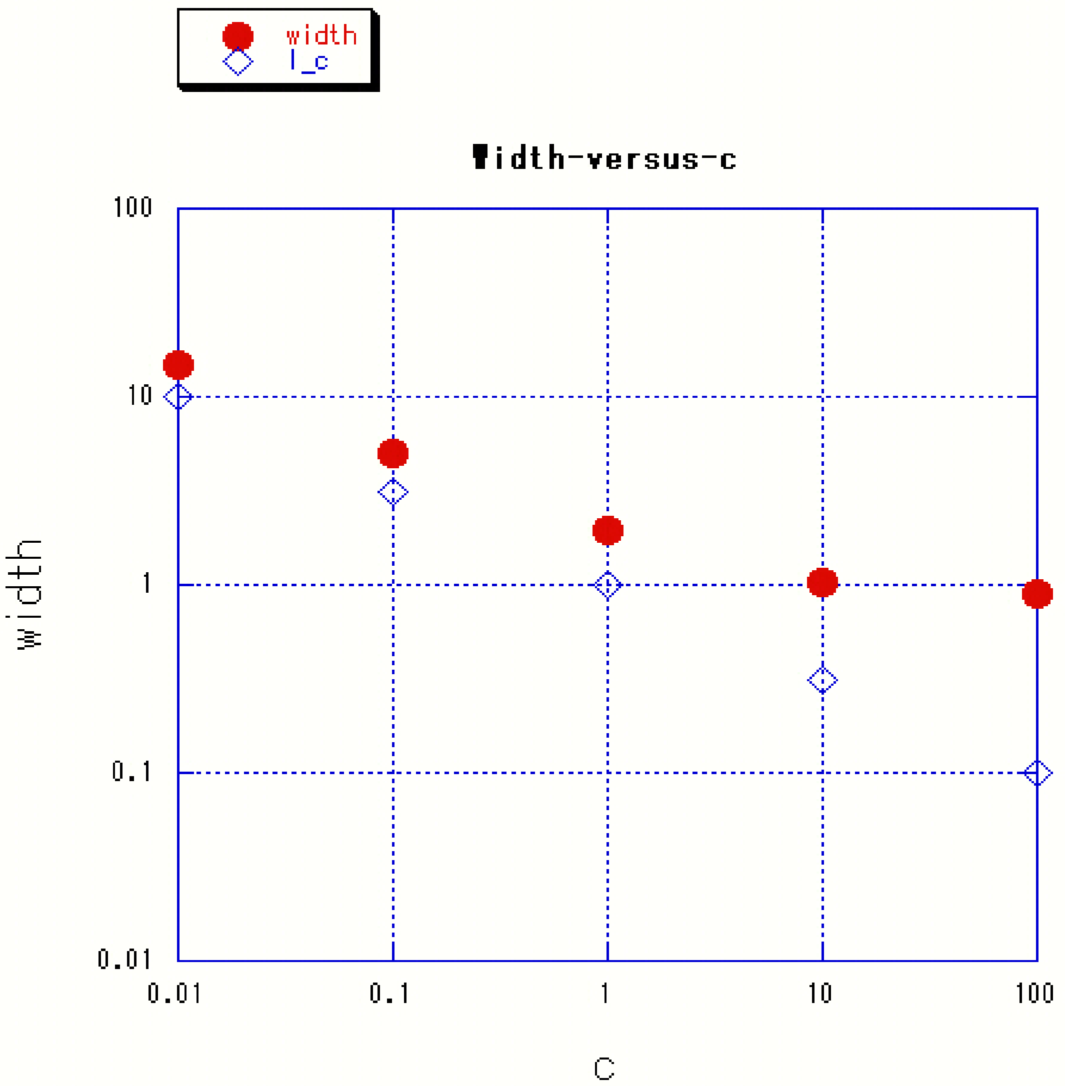
The density profile at time t is calculated by

$$\langle X(t)|\rho(x)|X(t)\rangle = \sum_{p,p'=0}^{N-1} e^{2\pi i(p-p')q/N} \times e^{i(P-P')x - i(E_p - E_{p'})t}\langle P'|\rho(0)|P\rangle,$$

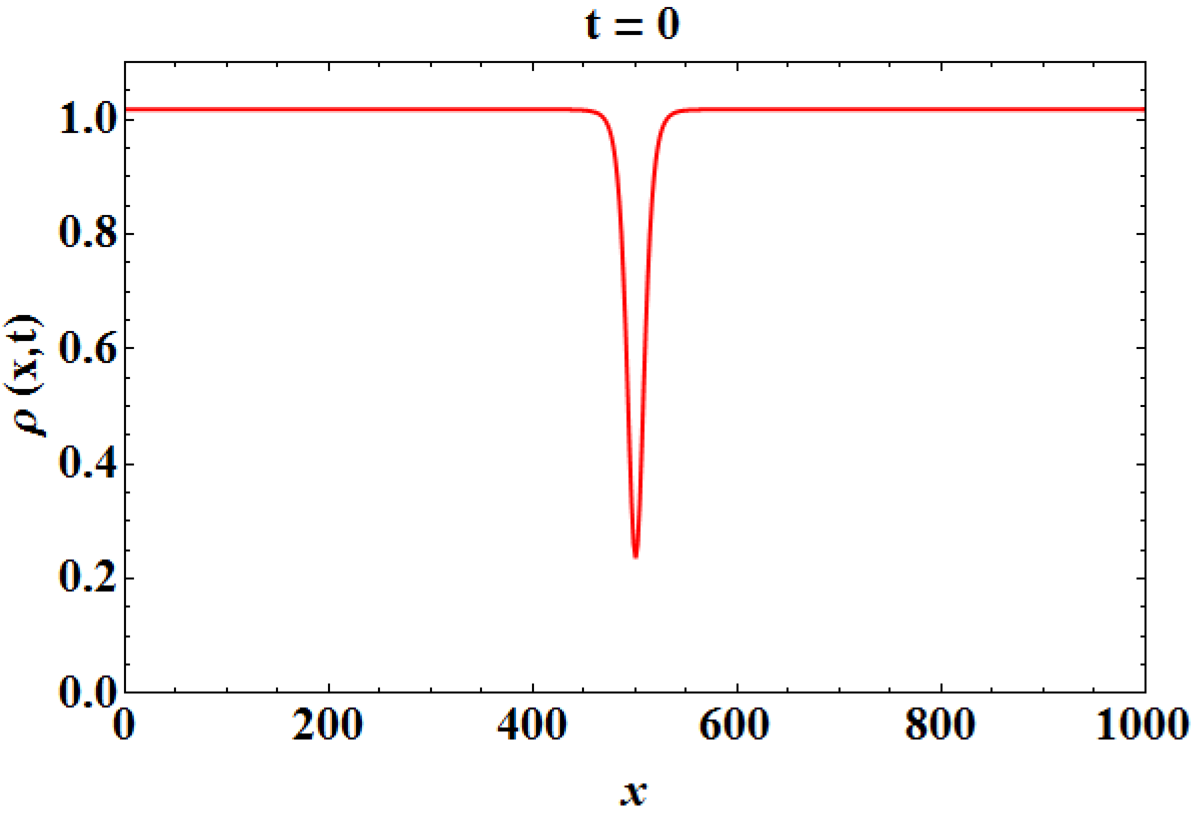
where $P = 2\pi p/L$ and $P' = 2\pi p'/L$ denote the total momentum of the Bethe eigenstates $|P\rangle$ and $|P'\rangle$, respectively.

We evaluate the form factor $\langle P'|\rho(0)|P\rangle$ by Slavnov's formula. It is expressed in terms of a determinant.

Width of quantum state is proportional to healing length $lc=1/(cn)^{1/2}$

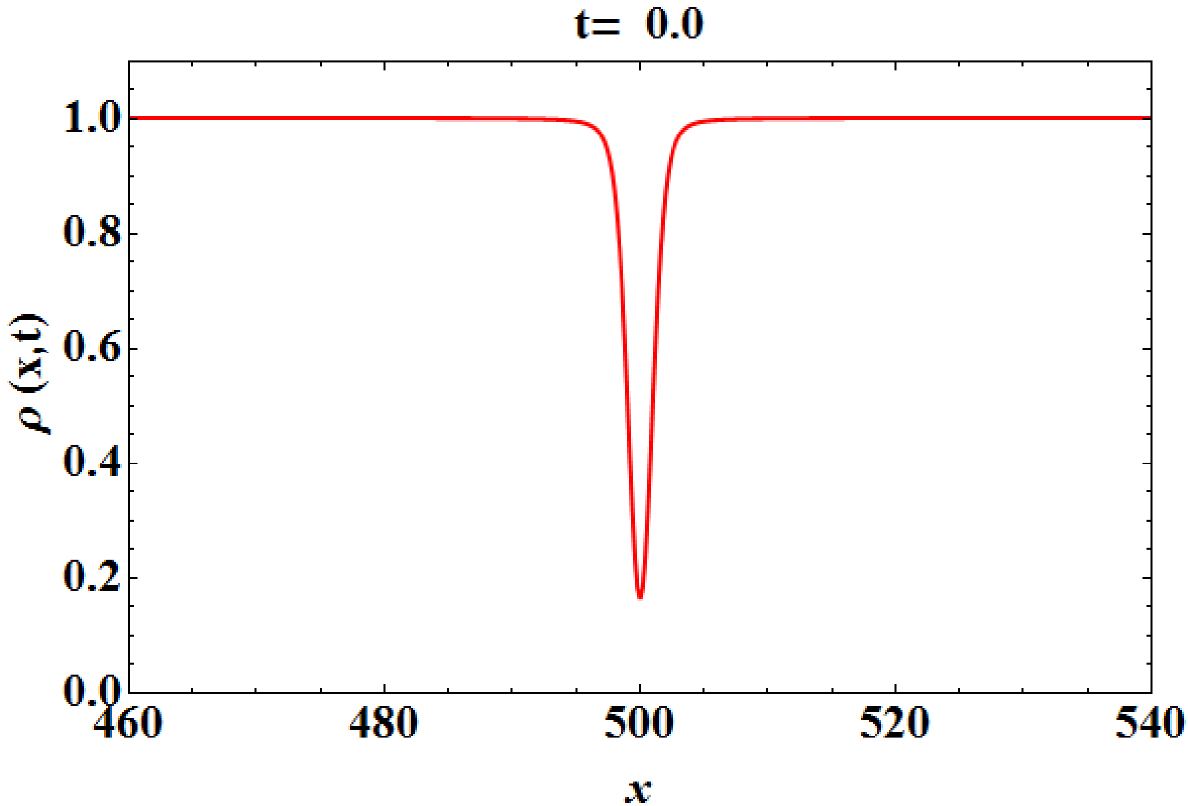


$N = 1000$



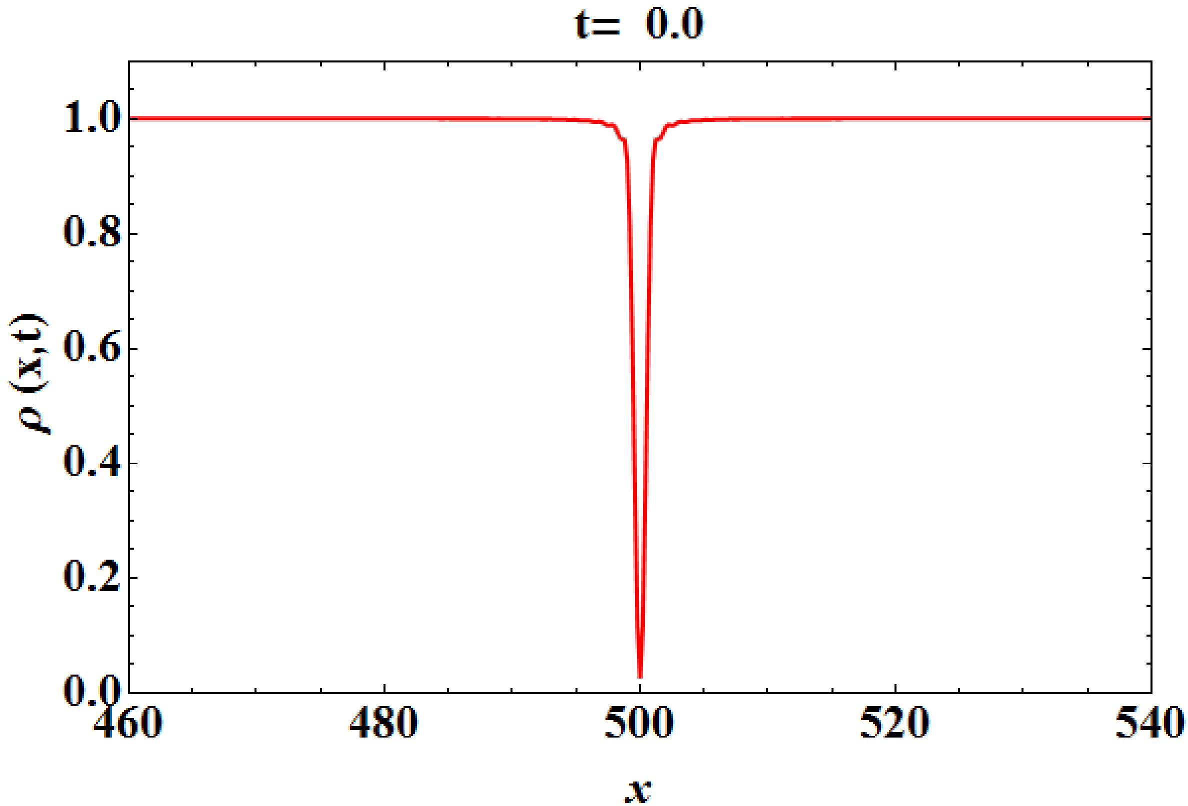
$c = 0.01$

$N = 1000$



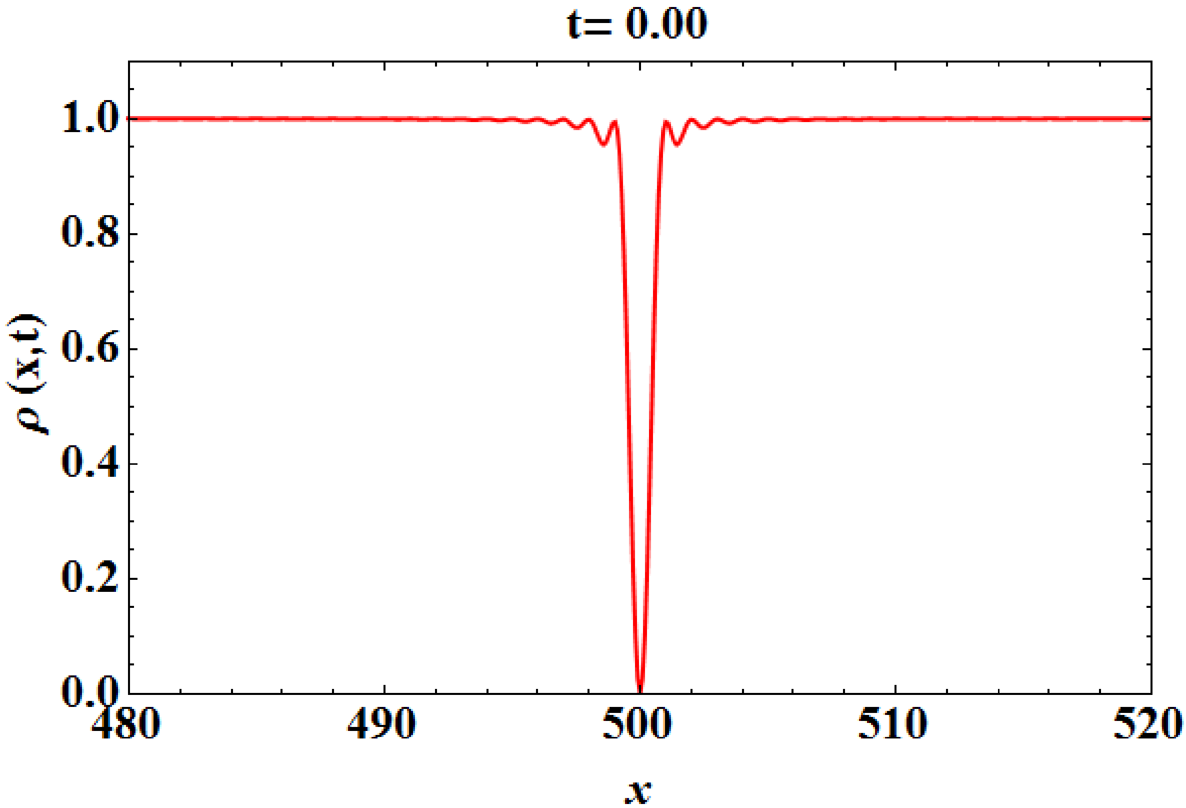
$c = 1$

$N = 1000$



$c = 10$

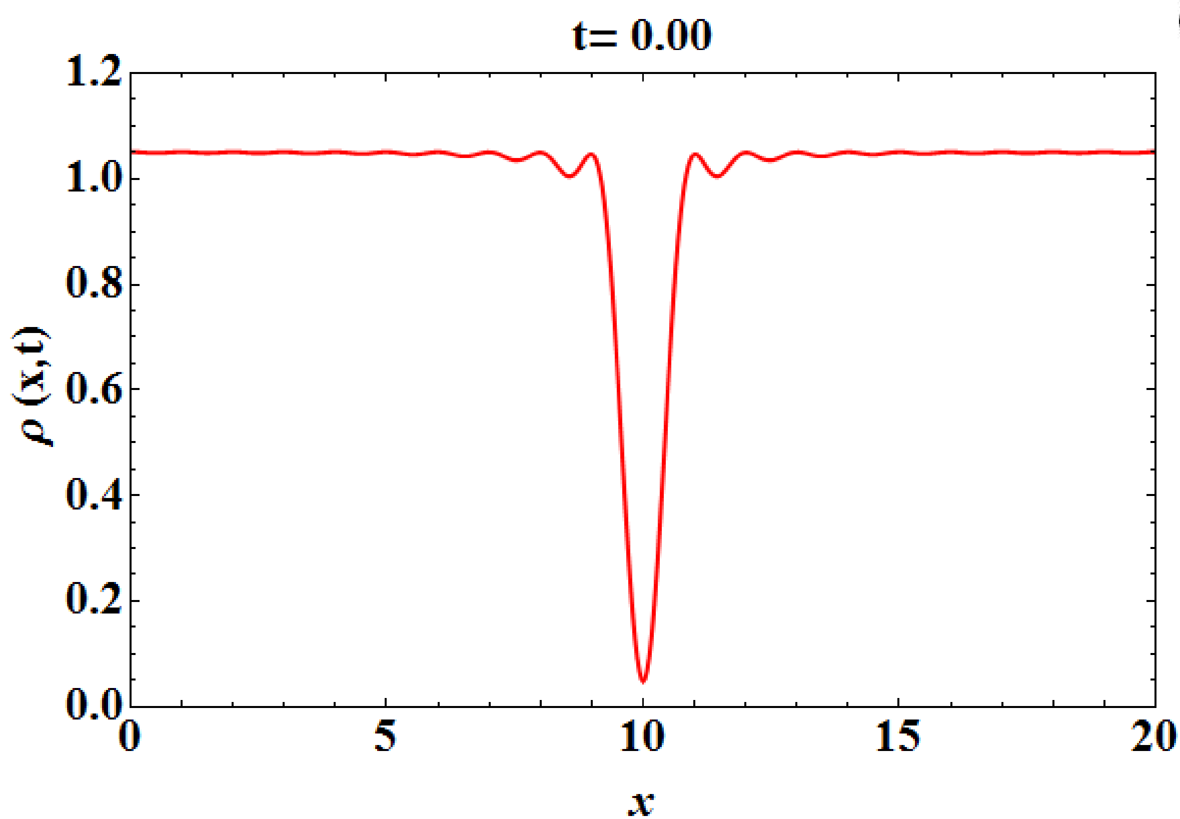
$N = 1000$

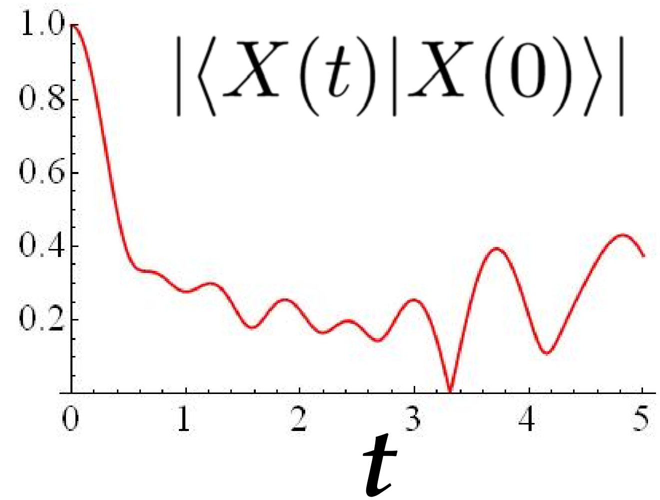


$c = 100$

$N = 20$, $c = 100$

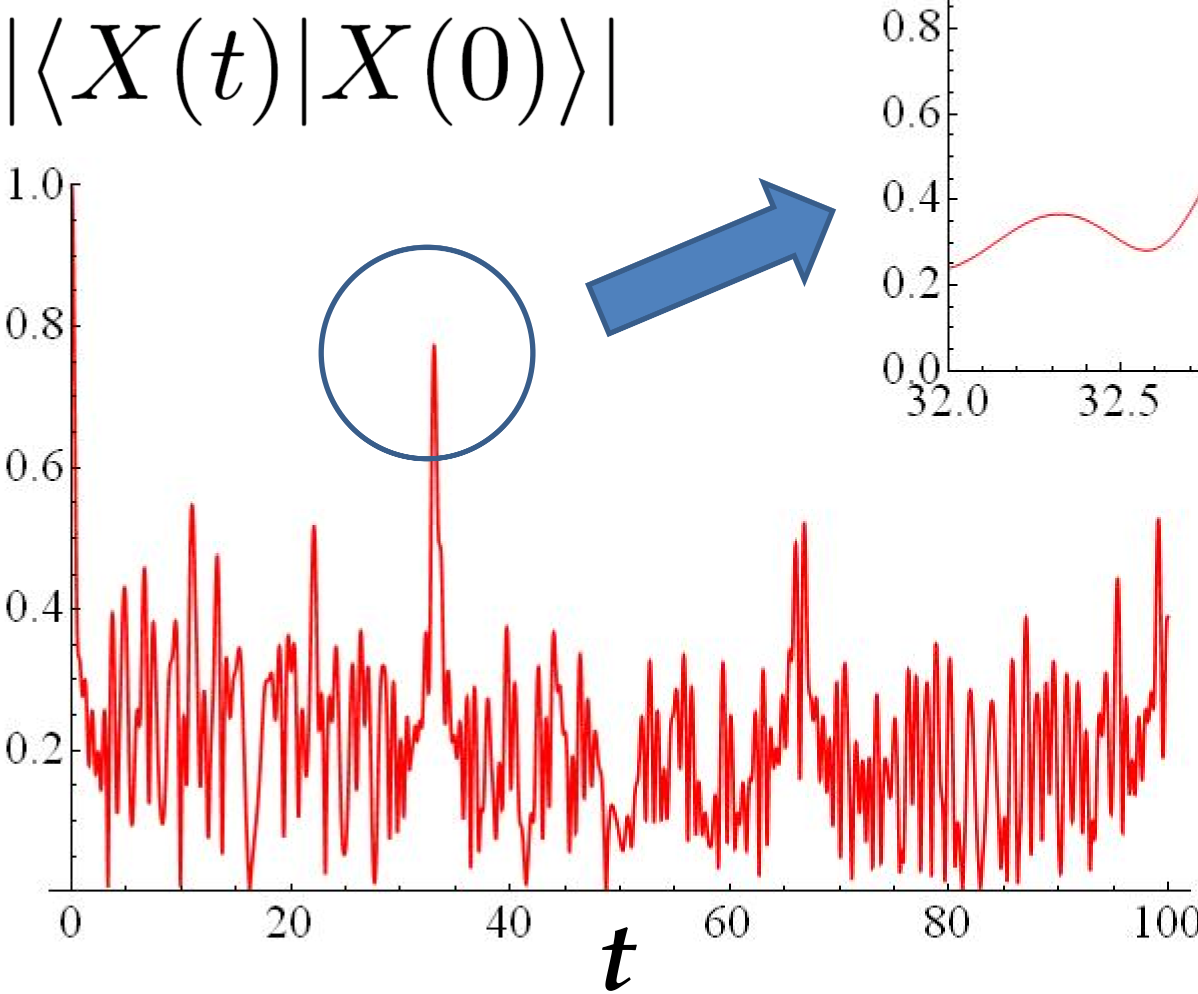
Relaxation process
in short time

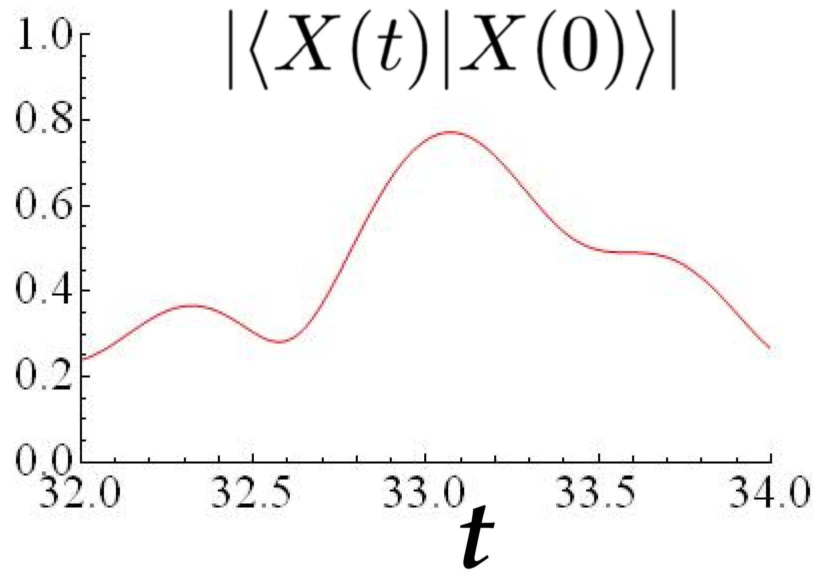




$N = 20, c = 100$

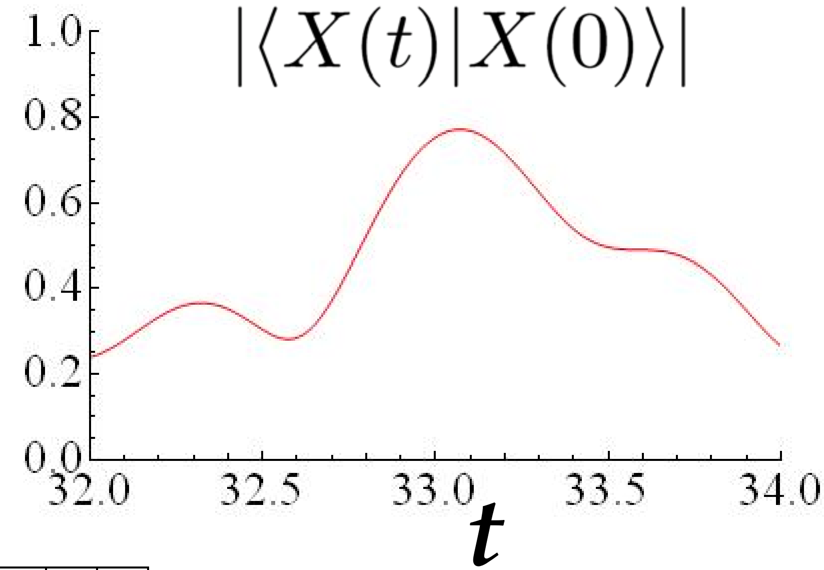
Recurrent at $t \sim 33$



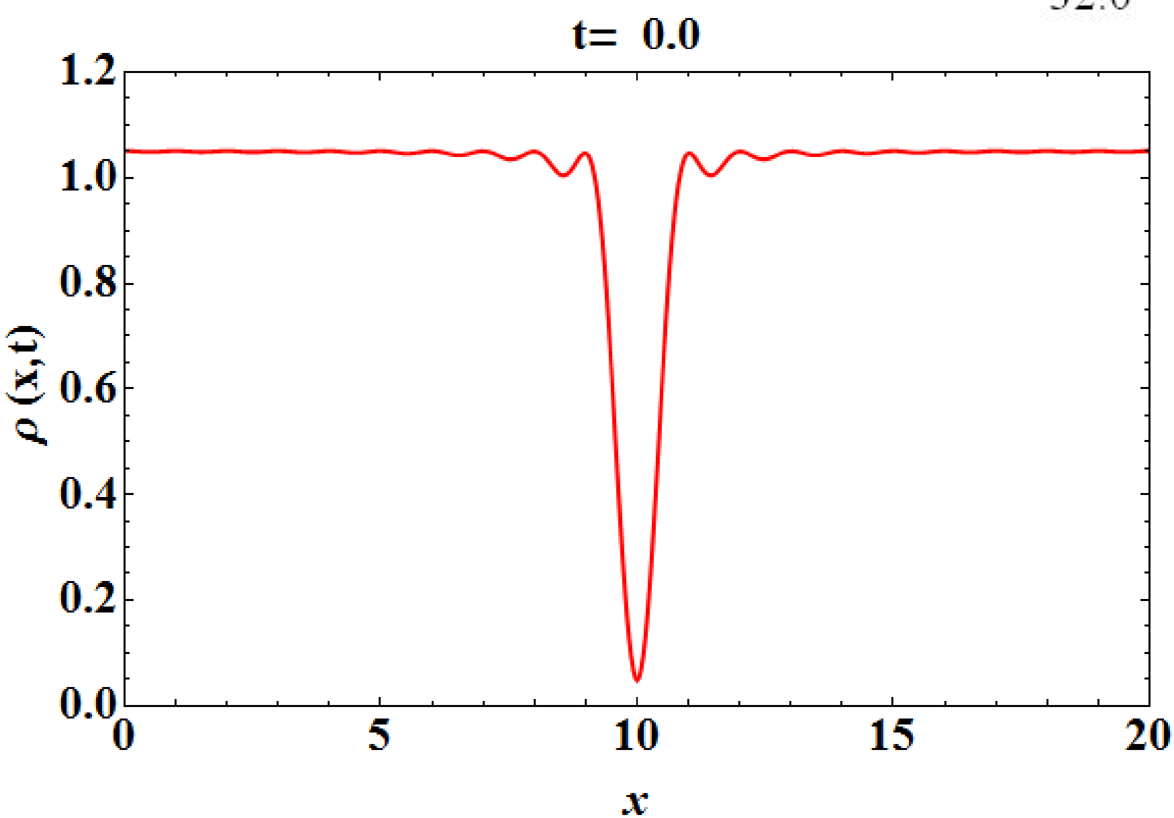


$t \sim 33$ recurrence ?

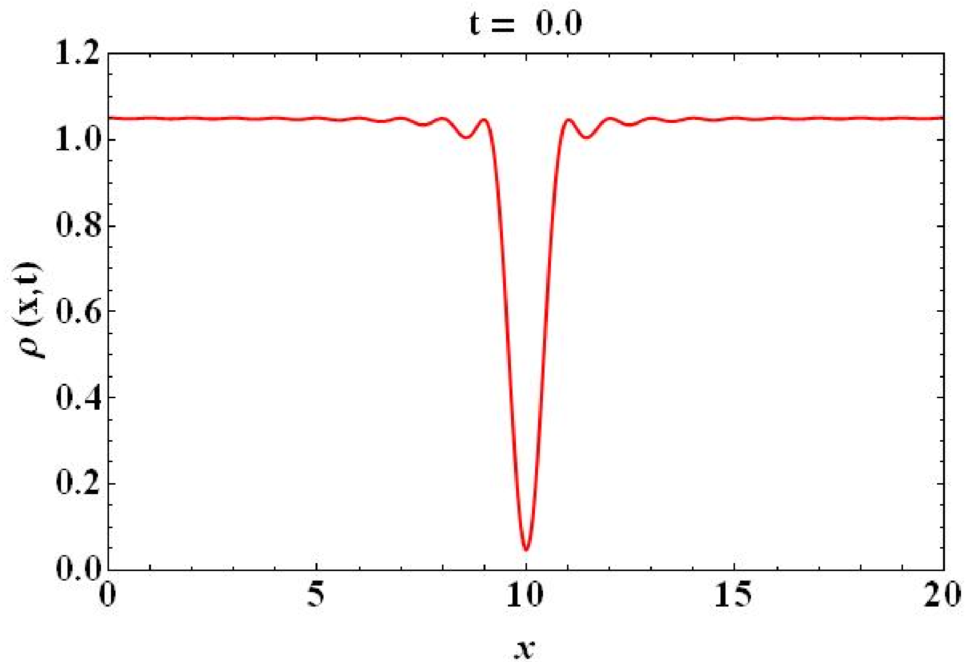
$N = 20, c = 100$

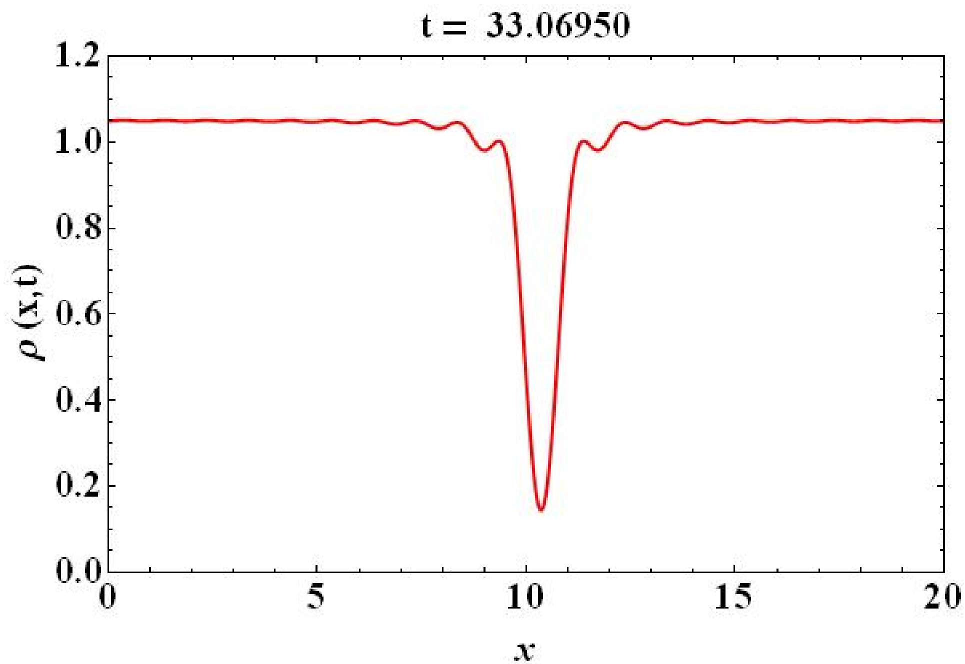


Observing it a bit longer

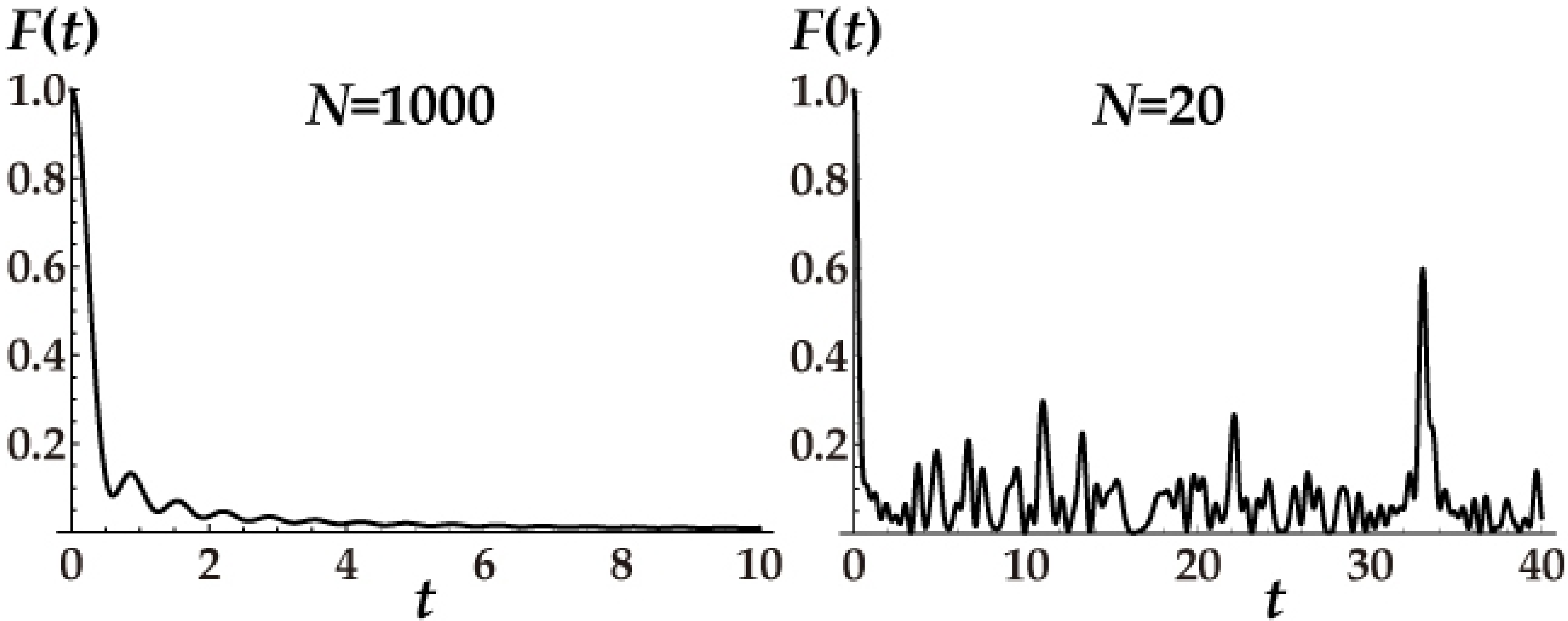


$t \sim 33$ recurrence ?


t = 0.0
ρ (x,t)
x


t = 33.06950
ρ (x,t)
x

Fidelity for large system (relaxation) and for small system (recurrence)



Collapse of a `localized' quantum state

- Two possible interpretations:
- (1) collapse due to nonlinear dispersion relation
- (2) Sum of many oscillations with different frequencies vanishes almost everywhere in time

- (2') quantum ergodic theorems (QET)

  (i) Reimann (2008)

  (ii) QET of von Neumann (1929)

Cf. M. Rigol et al.

Conclusions on quantum soliton state

- **We have constructed the quantum state of a dark soliton .** It should be useful for computing physical quantities.
- $\langle X,N-1 | \phi(x) | X,N \rangle$ gives the order parameter of BEC

- Solving the Bethe equation numerically and making use of Slavnov formula, we have calculated the time evolution for a very long time. (infinitely long time)

- For N=20, **recurrence phenomena may occur.**
- For N=1000, **relaxation** occurs for isolated quantum system

[Ref] J. Sato, R. Kanamoto, E. Kaminishi and T. D., Phys. Rev. Lett. 108, 110401 (2012); arXiv: 1204.3960.

Future problems and prospects

- (1) In XXZ chain and 1D Bose gas, we can study exact asymptotic behavior for various correlation functions.

  Physical applications should be interesting.

- (2) Quantum dynamics:

  For XXZ chain and 1D Bose gas, we can study time evolution of the quantum many-body systems over a very long period of time.

  Can we reconstruct any given Initial state

  in terms of Bethe states ? (Partially , yes)

(3) Quantum soliton states

Difference from the mean-field solution should be explored

Connections to experiments should be studied

Perturbative approaches

Can we construct a multi-soliton state ?