超流動安定性と分岐理論

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[1] Y. Kato and S. Watabe, J. Low Temp. Phys. **158**, 92 (2010)
Generalized Criterion for Stability of Superflow Past an Obstacle

[2] Y. Kato and S. Watabe, Phys. Rev. Lett. **105**, 035302 (2010)
Dynamical density fluctuations of superfluids near the critical velocity

[3]加藤雄介、渡部昌平、「超流動安定性の新たな判定法」
物性研究2010年12月号

Introduction 1/3

- 超流動　障害物が存在する状況で実現するエネルギー散逸のないマクロな流れ

- 超流動の安定性をもたらすもの
  ボース・アインシュタイン凝縮(BEC)（=巨視的波動関数Ψの存在）＋ $\boxed{?}$

$\boxed{?}$ ＝有限の圧縮率　（Bloch et al. Rev. Mod. Phys. 2008, Appendix）

理想ボースガスのBECは圧縮率が無限大

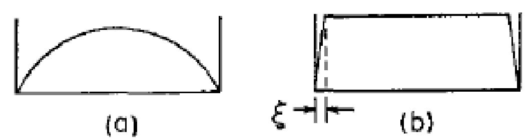


FIG. 3. The form of the condensed mode: (a) for the ideal gas; (b) for the interacting gas; ξ is a coherence length.

Hohenberg-Martin (1965) Annals of Phys.

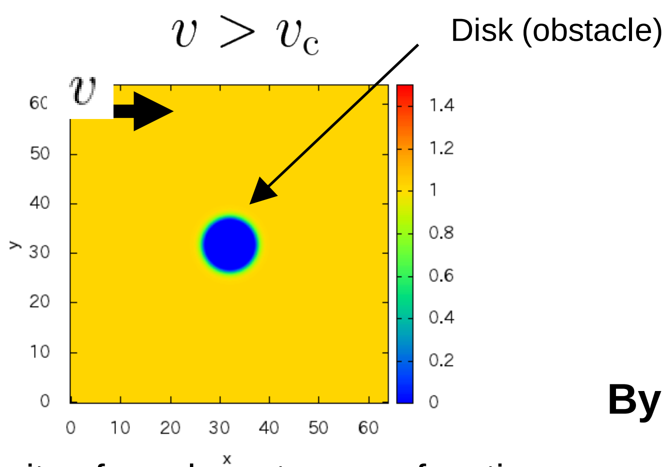
「柔らかすぎない」BEC⇒超流動性

以上は超流動速度が十分小さいときの話

Introduction 2/3

障害物が存在する状況で、超流動速度が臨界速度(Vc)を超えると定常流解が存在しない。

numerical cal. of time-dependent-Gross-Pitaevskii equation(=NLS) in 2D



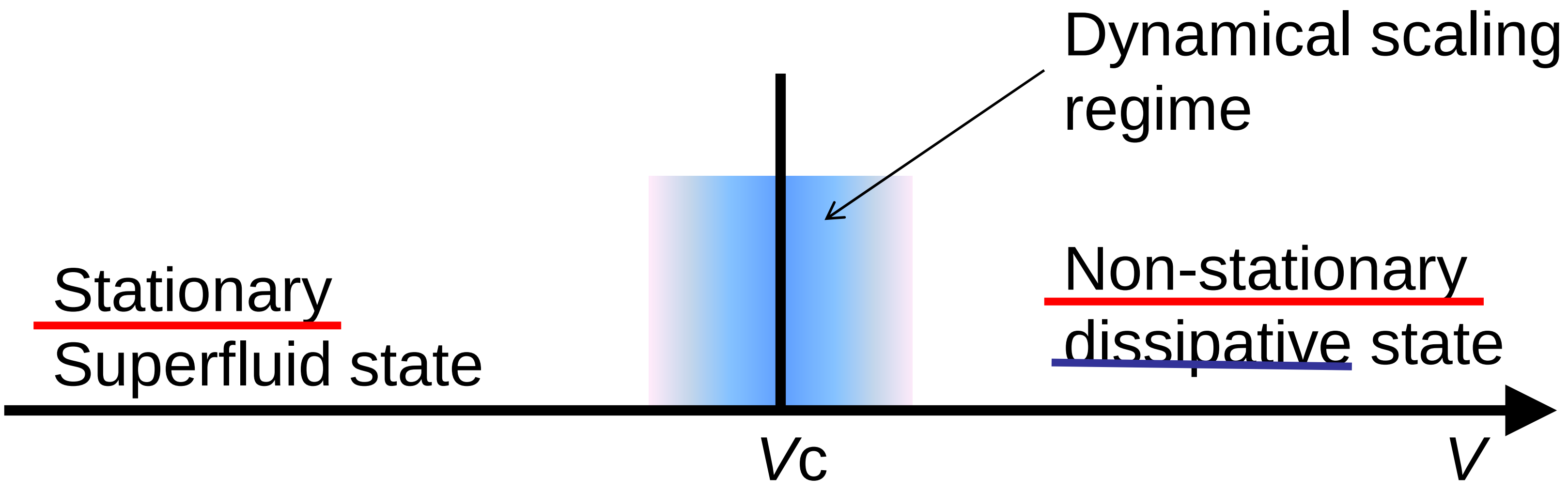
Density of condensate wave function

By M. Kunimi

臨界速度以下特に臨界速度近傍で、超流動の安定性をもたらすもの
ボース・アインシュタイン凝縮(BEC)(=巨視的波動関数Ψの存在)＋有限の圧縮率＋ $\boxed{??}$

研究目的: $\boxed{??}$ の同定 ⇒ 結論: $\boxed{??}$ = 抑制された動的(局所)密度ゆらぎ

Introduction 3/3(解題): bifurcation theory



Conventional theory of Equilibrium phase transition does not apply to the transition from V<Vc to V>Vc.

But bifurcation theory is useful to describe the transition from a stationary state to a non-stationary state

Model (1/2): Gross-Pitaevskii theory (1961) weakly-interacting, zero temperature

We start with the following (dimensionless) Hamiltonian:

$$\hat{H} = \int \mathrm{d}\boldsymbol{r} \left[\frac{1}{2}\nabla\hat{\Psi}^\dagger(\boldsymbol{r})\cdot\nabla\hat{\Psi}(\boldsymbol{r}) + \left(U(\boldsymbol{r})-\mu\right)\hat{\Psi}^\dagger(\boldsymbol{r})\hat{\Psi}(\boldsymbol{r})\right] + \frac{1}{2}\int \mathrm{d}\boldsymbol{r}\,\hat{\Psi}^\dagger(\boldsymbol{r})\hat{\Psi}^\dagger(\boldsymbol{r})\hat{\Psi}(\boldsymbol{r})\hat{\Psi}(\boldsymbol{r})$$

Kinetic term One-body potential, chemical potential Two-body (contact type) repulsion

where $\hat{\Psi}(\boldsymbol{r})$ is a Bosonic field operator.

Heisenberg Equation of Motion for $\hat{\Psi}(\boldsymbol{r},t)$ is given by

$$\begin{aligned} \mathrm{i}\frac{\partial\hat{\Psi}(\boldsymbol{r},t)}{\partial t} &= \left[\hat{\Psi}(\boldsymbol{r},t),\hat{H}\right] \\ &= -\frac{1}{2}\boldsymbol{\nabla}^2\hat{\Psi}(\boldsymbol{r},t) + \left(U(\boldsymbol{r})-\mu\right)\hat{\Psi}(\boldsymbol{r},t) + \hat{\Psi}^\dagger(\boldsymbol{r},t)\hat{\Psi}(\boldsymbol{r},t)\hat{\Psi}(\boldsymbol{r},t) \end{aligned}$$

Most particles are assumed to condense. Field operator can then be treated as a c-number.

$$\hat{\Psi}(\boldsymbol{r},t) \longrightarrow \Psi(\boldsymbol{r},t)$$ (Gross-Pitaevskii approximation).

Model (2/2): Gross-Pitaevskii theory (1961) in conventional dimensional form

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = -\frac{\hbar^2\nabla^2}{2m}\Psi(\mathbf{r},t) + (U(\mathbf{r})-\mu)\Psi(\mathbf{r},t) + g|\Psi(\mathbf{r},t)|^2\Psi(\mathbf{r},t)$$

Kinetic term

One-body potential,chemical potential

Two-body potential

It describes weakly interacting Bosons in condensed phase at zero temperature.

$$\Psi(\boldsymbol{r},t) = \sqrt{\rho(\boldsymbol{r},t)}e^{i\varphi(\boldsymbol{r},t)}$$: Condensate wave function

$$\boldsymbol{v}(\boldsymbol{r},t) = \frac{\hbar}{m}\nabla\varphi(\boldsymbol{r},t)$$: Gradient of phase ⇔ Superfluid velocity

⇓

$$\oint \mathrm{d}\boldsymbol{r}\cdot\boldsymbol{v}(\boldsymbol{r}) = \frac{h}{m}\times\text{integer}$$

Quantization of vorticies(Onsager-Feynman)

Breakdown of SF by vortex-emission

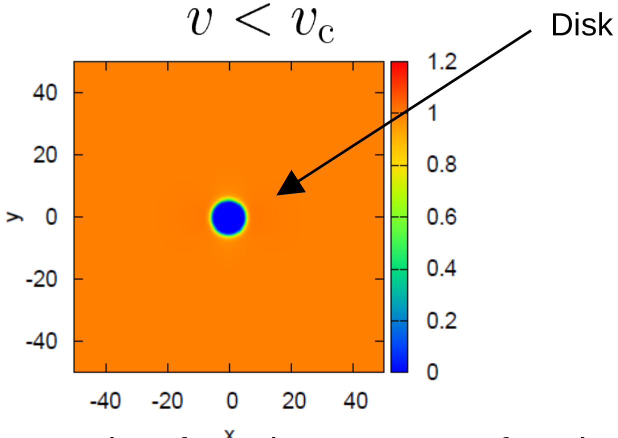
Confirmed by numerical cal. of time-dependent-Gross-Pitaevskii equation

Frisch et al. 1992

2D problem; Vc for flow around a disk

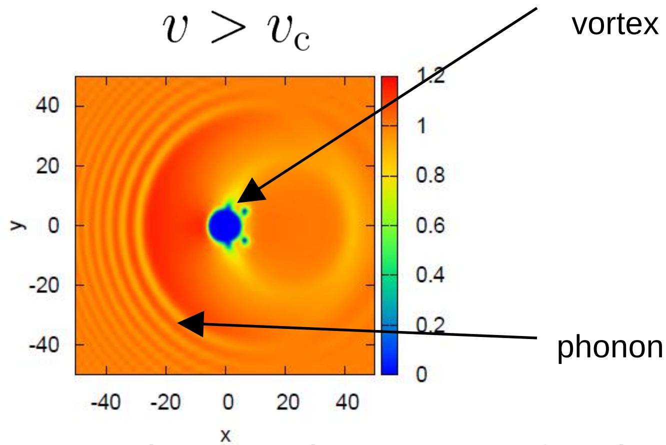
Direction of superflow v

$v < v_c$

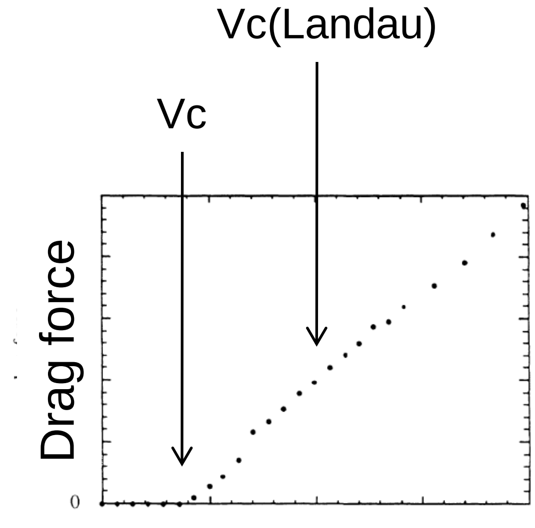


Density of condensate wave function

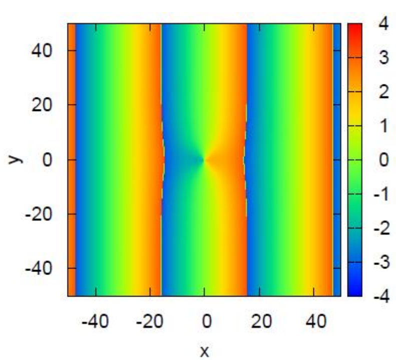
$v > v_c$



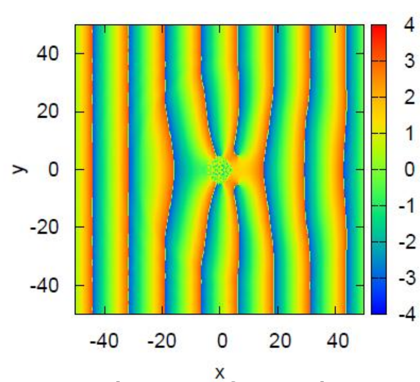
Density of condensate wave function



Velocity of fluid



Phase of condensate wave function



Phase of condensate wave function

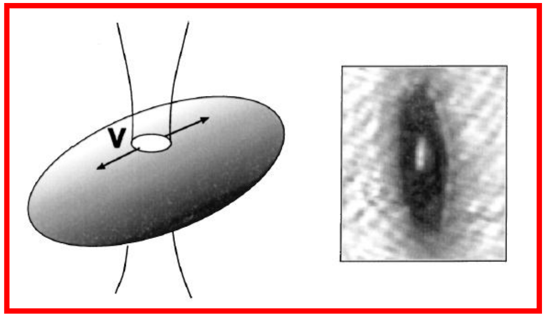
By M. Kunimi

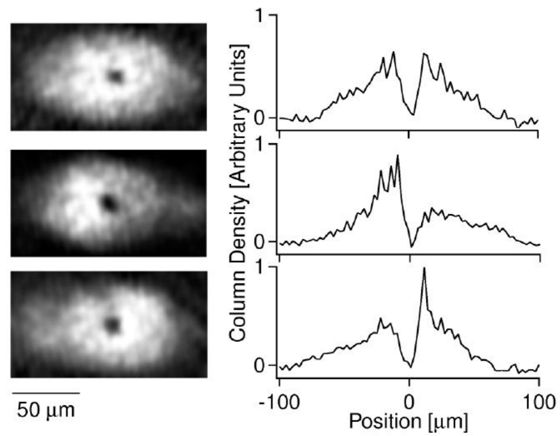
Critical velocity in Cold atoms

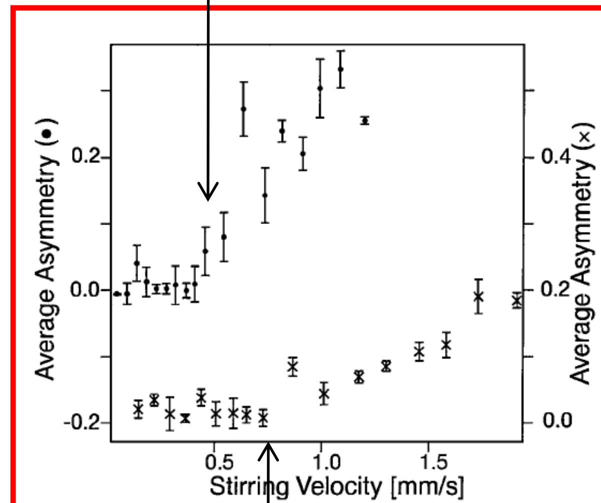
In reality, $v_c < v_{c,\mathrm{Landau}}$; vortex creation in Cold Atoms

Vc（Vc,Landau=4.8mm/s）

stirring condensate by laser beam







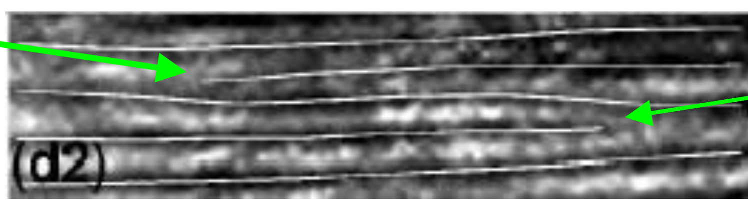
Raman et al. 1999

Onofrio et al. 2000

Vc (VcLandau=7.0mm/s)

Phase of condensate wave function

vortex

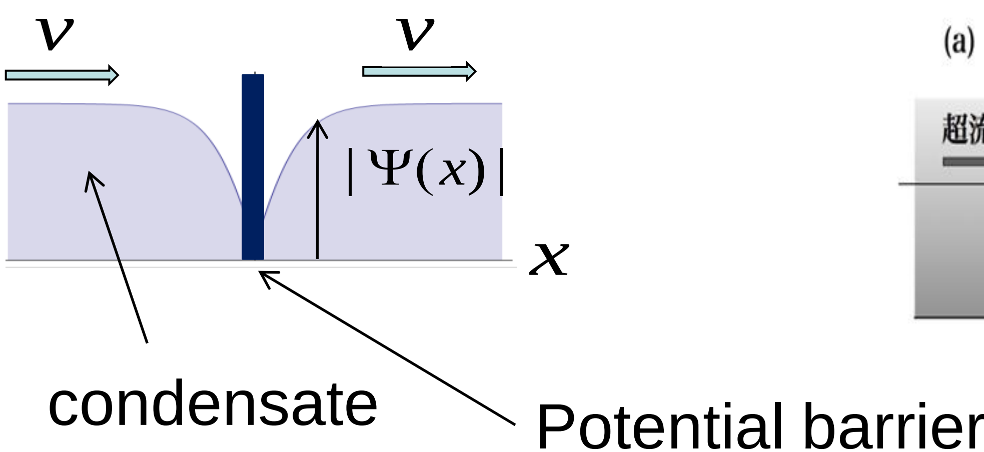


vortex

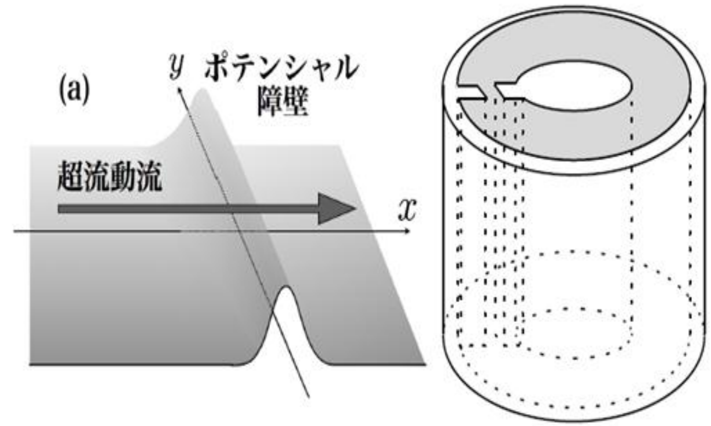
(interference fringe)Inouye et al. (2001)

Simpler Analogue of vortex-emission instability
one-dimensional superflow in the presence of potential barrier

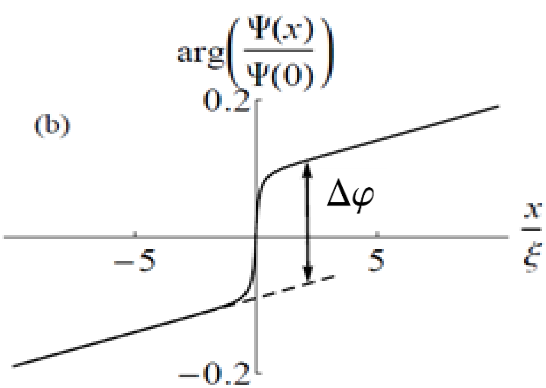
(1D Gross-Pitaevskii eq.)



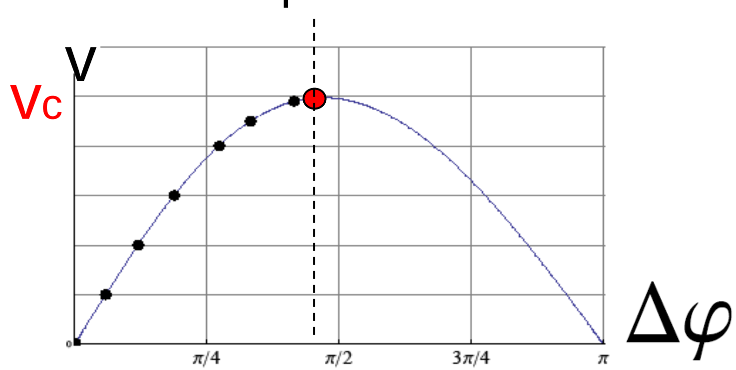
(2D Gross-Pitaevskii eq.)



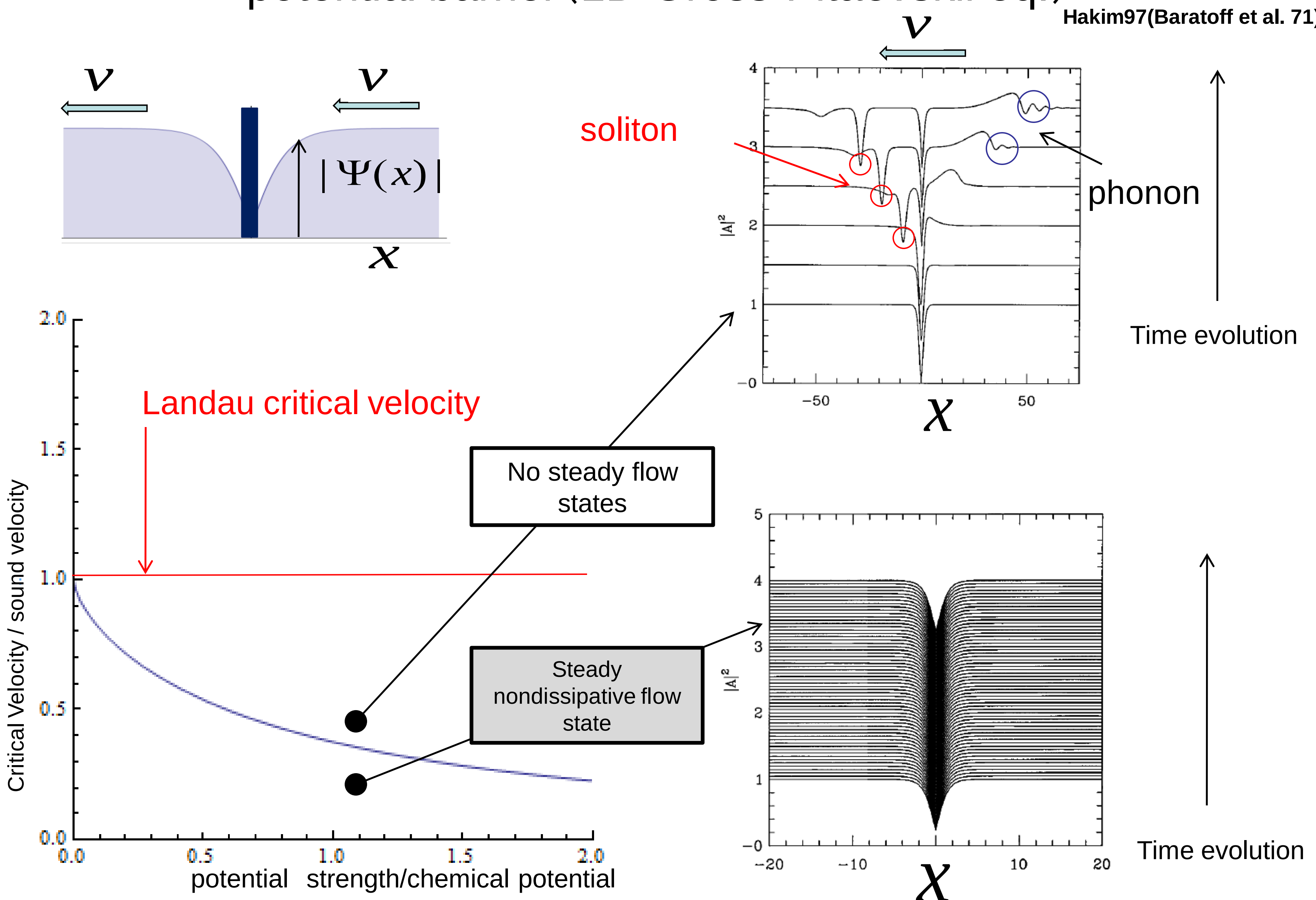
Josephson-like system



current-phase relation

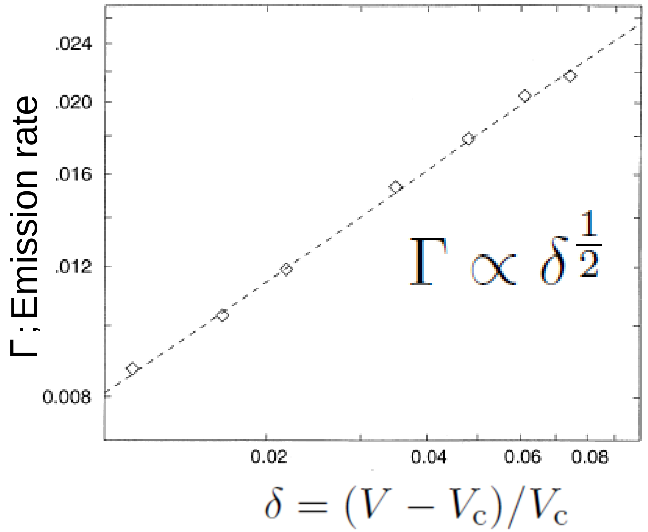


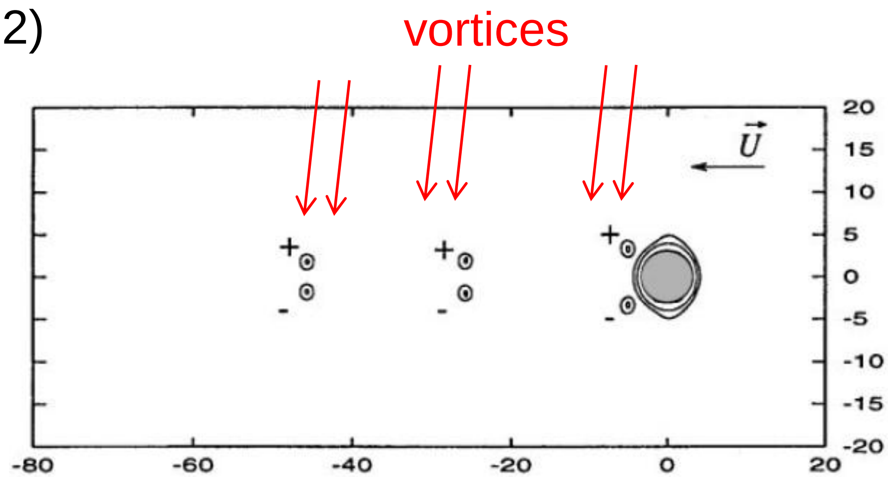
Stability of one-dimensional superflow in the presence of potential barrier（1D Gross-Pitaevskii eq.）



Dynamical scaling above critical velocity

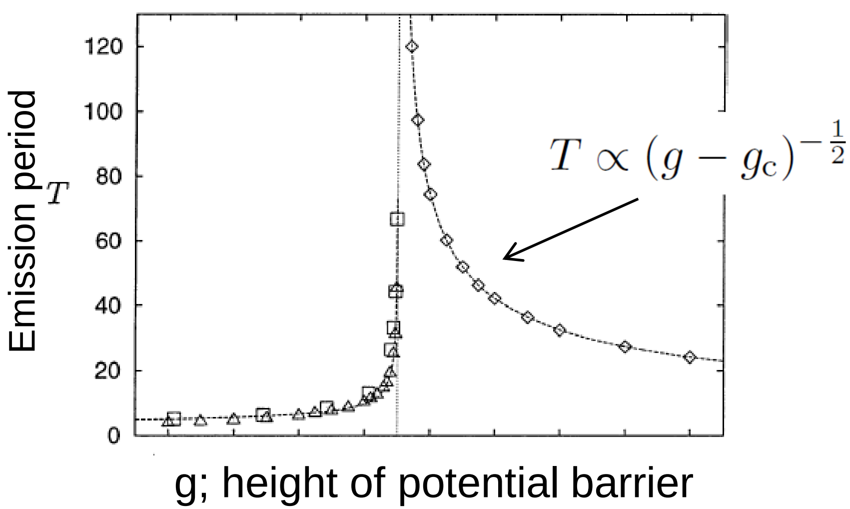
Vortex-emission rate(Huepe-Brachet 2002)

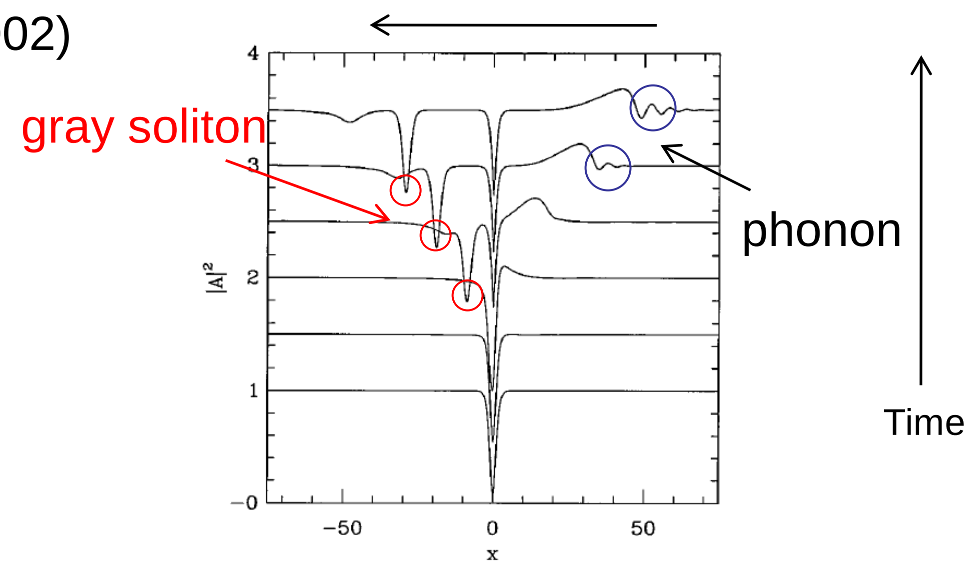




(Huepe-Brachet 2002)

Soliton-emission rate(Pham-Brachet 2002)

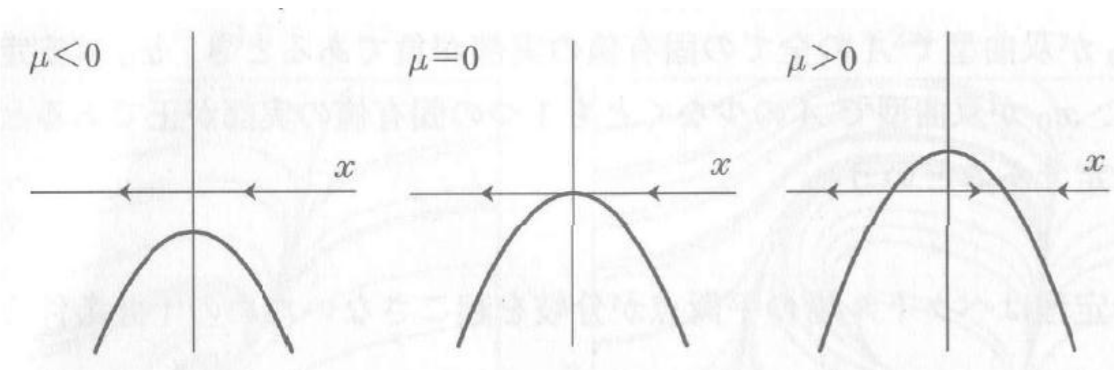




Bifurcation theory

=theory of parameter dependence of existence and stability of stationary solutions of dynamical nonlinear systems (equations)

Ex. $$\dot{x} = f(x,\mu) = \mu - x^2, \quad \mu, x \text{ real}$$

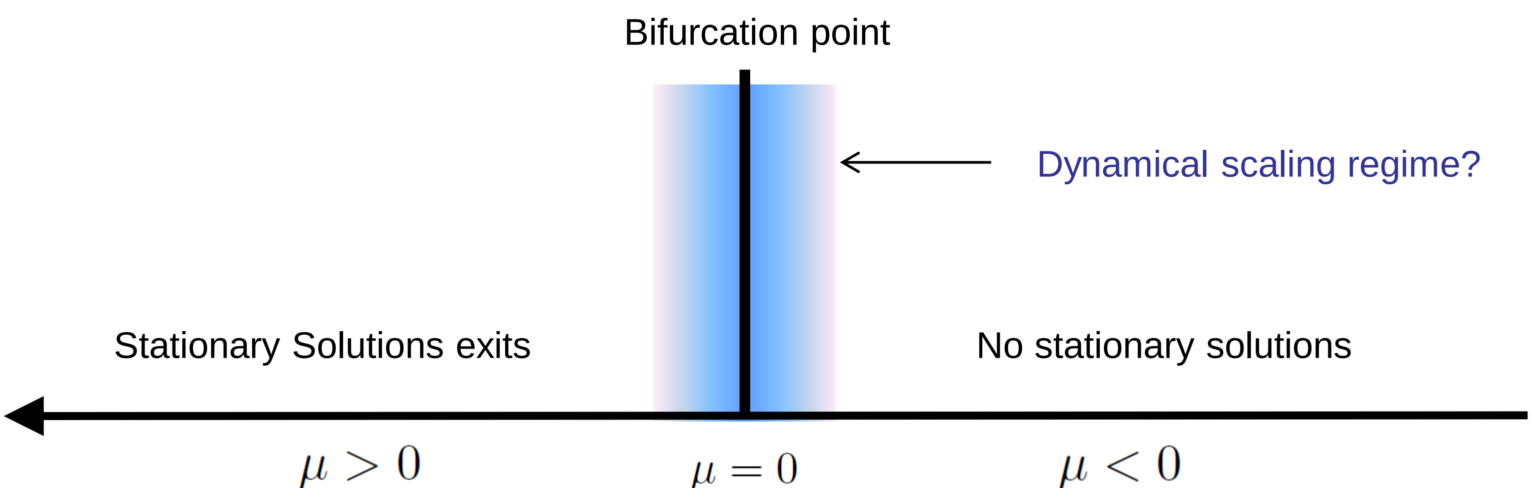


Stationary solutions

$\mu < 0$ No stationary solutions

$\mu = 0$ $$x(t) = 0$$

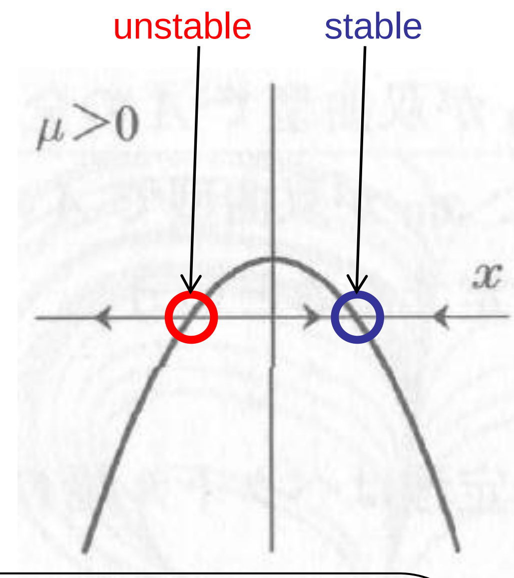
$\mu > 0$ $$x(t) = \pm\sqrt{\mu}$$



Stability analysis

$$\dot{x} = f(x,\mu) = \mu - x^2, \quad \mu, x \text{ real}$$

stationary solutions: $x(t) = \pm\sqrt{\mu} \qquad \mu > 0$



Linear stability analysis around stationary solutions.

$$\delta x(t) = \mp 2\sqrt{\mu}\,\delta x(t) + \mathcal{O}((\delta x(t))^2)$$

$$\delta x(t) = \exp(\mp 2\sqrt{\mu}t) \times \text{ constant}$$

Characteristic time (inverse of growth rate or converging rate) scales as

$$t^* \propto \mu^{-\frac{1}{2}}$$

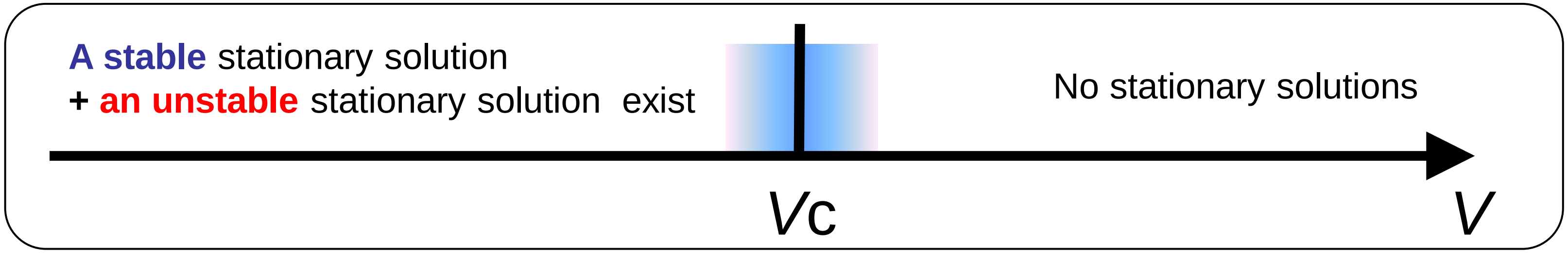
Dynamical scaling regime !

A stable stationary solution
+ **an unstable** stationary solution exist

No stationary solutions

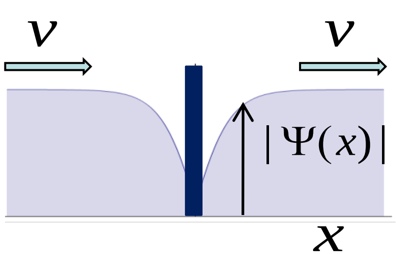
$\mu > 0$ $\mu = 0$ $\mu < 0$

Saddle-node bifurcation

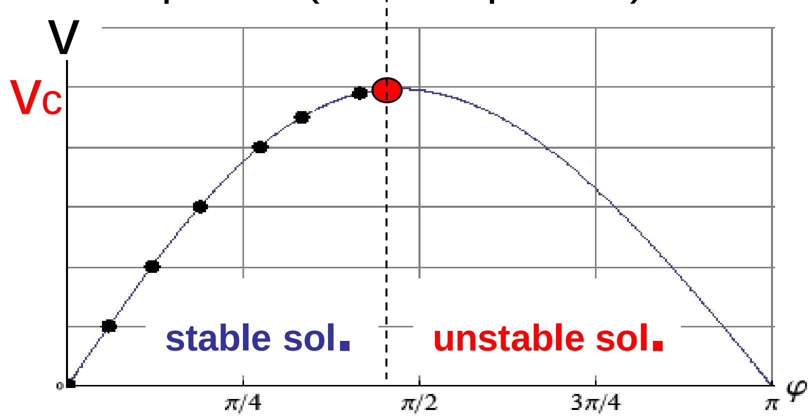


This type of bifurcation is called **saddle-node bifurcation**.

Soliton-emission instability = saddle-node bifurcation(Hakim 97)



Josephson (current-phase)relation



V<Vc a stable and an unstable solutions exist

V>Vc no stationary solution exists

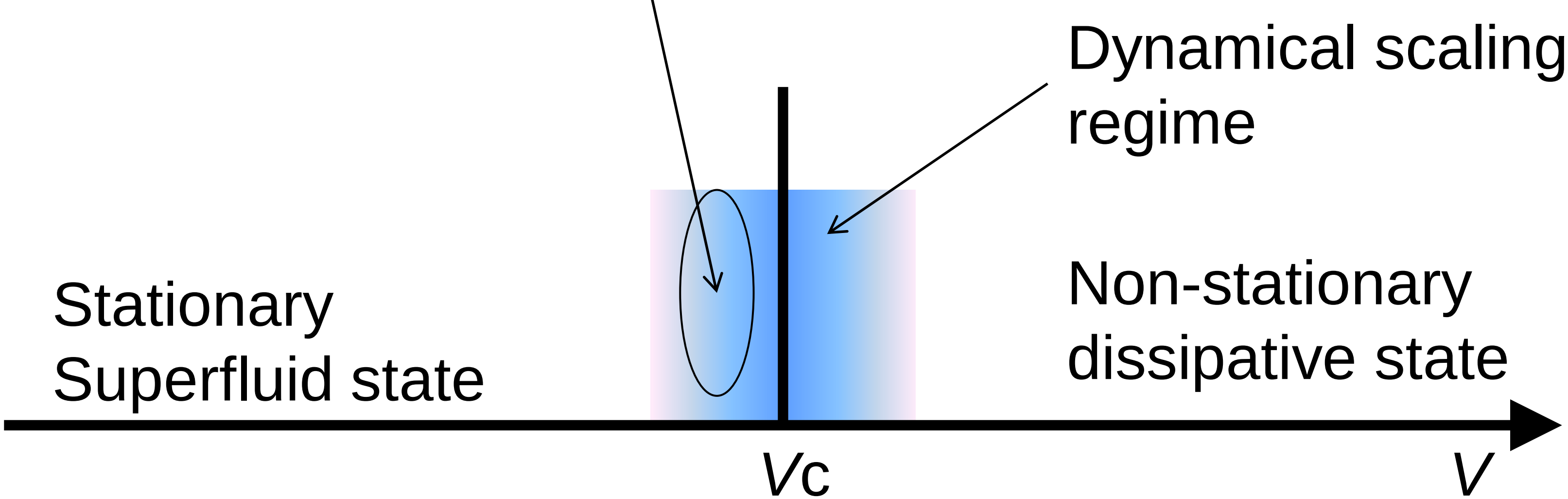
V=Vc, stable and unstable solutions are degenerate.

Further, Vortex-emission instability = saddle-node bifurcation (Pomeau 93,Rica 01)

⇨ Dynamical scaling of vortex-emission and soliton emission is due to saddle-node bifurcation

Dynamical density fluctuation

- Slightly below the critical velocity of soliton-emission instability, the dynamical density fluctuation is enhanced.



Model (1/4): Classical analogue of Bogoliubov equation

We express $\Psi(\boldsymbol{r},t)$ as the sum of the stationary solution of GP equation and the fluctuation around it.

$$\Psi(\boldsymbol{r},t) = \Psi_0(\boldsymbol{r}) + \delta\Psi(\boldsymbol{r},t)$$

Equation linearized w. r. t. $\delta\Psi(\boldsymbol{r},t)$ is given by

$$\mathrm{i}\frac{\partial\delta\Psi(\boldsymbol{r},t)}{\partial t} = K\delta\Psi(\boldsymbol{r},t) + \Psi_0^2(\boldsymbol{r})\delta\Psi^*(\boldsymbol{r},t) \quad \cdots\cdots\bigstar$$

with $K = -\frac{1}{2}\boldsymbol{\nabla}^2 + (U(\boldsymbol{r}) - \mu) + 2\Psi_0^*(\boldsymbol{r})\Psi_0(\boldsymbol{r})$

Normal mode is given in the form of $\delta\Psi(\boldsymbol{r},t) = u_j(\boldsymbol{r})\mathrm{e}^{-\mathrm{i}\epsilon_j t} - v_j^*(\boldsymbol{r})\mathrm{e}^{\mathrm{i}\epsilon_j t}$

Resultant two-component Equation for (u,v) is given by

$$\epsilon_j\begin{pmatrix} u_j(\boldsymbol{r}) \\ v_j(\boldsymbol{r}) \end{pmatrix} = \mathcal{L}\begin{pmatrix} u_j(\boldsymbol{r}) \\ v_j(\boldsymbol{r}) \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} K & -\Psi_0^2 \\ (\Psi_0^*)^2 & -K \end{pmatrix}$$

Bogoliubov equation

General solution to ★ is given by

j; index of normal mode

$$\delta\Psi(\boldsymbol{r},t) = \sum_{j \mathrm{s.t.} \mathrm{Re}\epsilon \geq 0} \left(u_j(\boldsymbol{r}) \quad \exp(-\mathrm{i}\epsilon_j t) - v_j^*(\boldsymbol{r}) \quad \exp(\mathrm{i}\epsilon_j^* t)\right)$$

Model (2/4): Bogoliubov theory (Bogoliubov1947) weakly interacting, zero temperature

Quantum fluctuation around a stationary condensate is taken into account as follows:

$\hat{\Psi}(\boldsymbol{r},t) = \Psi_0(\boldsymbol{r}) + \delta\hat{\Psi}(\boldsymbol{r},t)$ with

$$\delta\hat{\Psi}(\boldsymbol{r},t) = \sum_{j \mathrm{s.t.} \epsilon_j \geq 0} \Big(u_j(\boldsymbol{r})\hat{a}_j \exp(-\mathrm{i}\epsilon_j t) - v_j^*(\boldsymbol{r})\hat{a}_j^\dagger \exp(\mathrm{i}\epsilon_j t) \Big)$$

where $\hat{a}_j^\dagger$, $\hat{a}_j$ are bosonic creation and annihilation operators, respectively.

Orthonormal condition for (u,v) is given by

$$\int \mathrm{d}\boldsymbol{r} (u_k^* u_j - v_k^* v_j) = \delta_{kj} \qquad \int \mathrm{d}\boldsymbol{r} (u_k v_j - v_k u_j) = 0$$

Diagonal representation of Hamiltonian is given by

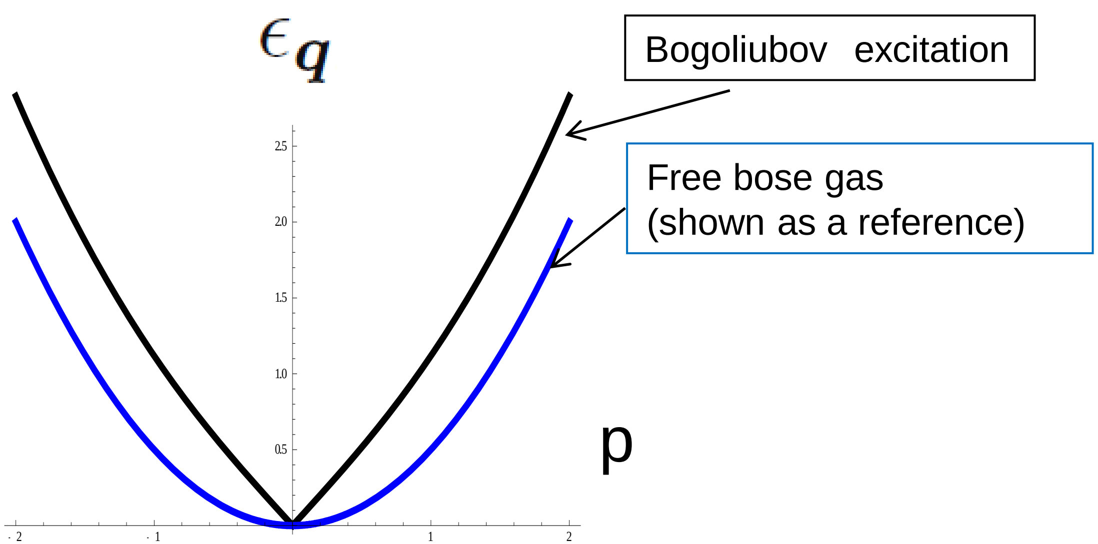
$$\hat{H} = E_\mathrm{g} + \sum_j \epsilon_j \hat{a}_j^\dagger \hat{a}_j + \mathcal{O}(\delta\hat{\Psi}^\dagger \delta\hat{\Psi}^2, (\delta\hat{\Psi}^\dagger)^2 \delta\hat{\Psi})$$

Model (3/4) In the spatially uniform case, solution to Bogoliubov equation is given by

$$\begin{pmatrix} u_{\boldsymbol{q}}(\boldsymbol{r}) \\ v_{\boldsymbol{q}}(\boldsymbol{r}) \end{pmatrix} = \frac{\left(\varepsilon_q^2 - 1\right)^{-\frac{1}{4}}}{\sqrt{2\Omega}} \begin{pmatrix} \left[\varepsilon_q + \left(\varepsilon_q^2 - 1\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \\ \left[\varepsilon_q - \left(\varepsilon_q^2 - 1\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \end{pmatrix}$$

where Ω denotes volume and $\varepsilon_q \equiv \frac{q^2}{2} + 1$

Excitation energy is given by $\epsilon_{\boldsymbol{q}} = \sqrt{\varepsilon_q^2 - 1}$



Bogoliubov spectrum was found in experiments of cold atoms (Stamper-Kurn 1999) via probe detecting <u>density fluctuation S(q,ω),S(q)</u>

Gapless spectrum is a sign for Bogoliubov excitation being a Nambu-Goldstone mode(<u>phase fluctuation</u>?)

Model (4/4): Phase fluctuation, density fluctuation, and Bogoliubov excitation

$$\Psi(\boldsymbol{r},t)=\sqrt{\rho_0+\delta\rho(\boldsymbol{r},t)}\mathrm{e}^{\mathrm{i}\delta\theta(\boldsymbol{r},t)}$$

When $\mu=1,\ \rho_0=1$ and

$$\frac{\partial\delta\rho(\boldsymbol{r},t)}{\partial t}=-\nabla^2\delta\theta(\boldsymbol{r},t),\quad \frac{\partial\delta\theta(\boldsymbol{r},t)}{\partial t}=\left(-\frac{1}{4}\nabla^2+1\right)\delta\rho(\boldsymbol{r},t)$$

Solution is given by

$$\begin{pmatrix}\delta\rho(\boldsymbol{r},t)\\ \delta\theta(\boldsymbol{r},t)\end{pmatrix}=\exp(\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{r}-\omega t))\begin{pmatrix}q^2\\ -\mathrm{i}\omega\end{pmatrix}$$

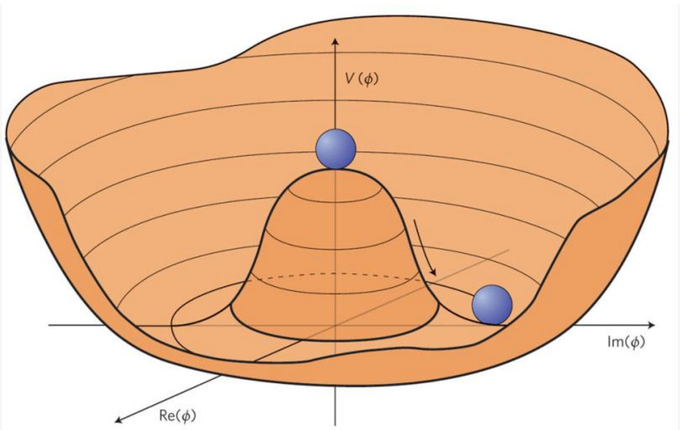
Density fluc. and phase fluc. couples

with $\omega=q\sqrt{1+\dfrac{q^2}{4}}$

$$\omega\to q,\quad q\to 0 \qquad \delta\rho\ll\delta\theta$$

In low energy, phase fluc. Dominates (∼NambuGoldstone mode)

$$\omega\to q^2/2,\quad q\to\infty \qquad \delta\rho\sim\delta\theta$$



How to measure density fluctuations?

Dynamical structure factor S(q,ω) is not useful in the spatially inhomogeneous systems.

We thus introduce the spectral function of local density:

$$\rho(\mathbf{r},\omega) = \sum_l |\langle l|\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})|\mathrm{g}\rangle|^2 \delta(\omega - E_l + E_\mathrm{g})$$

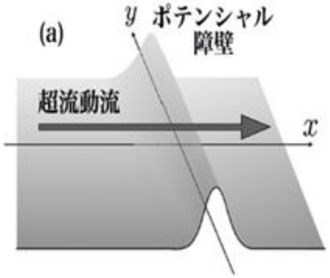
g; ground state (energy E_g) *l*; excited state（energy E_l）

⇕ Fourier Transform

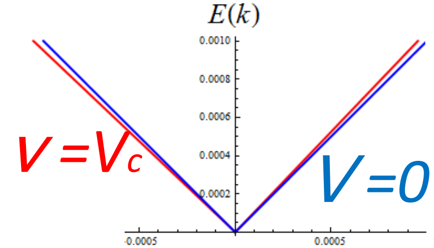
autocorrelation function of local density:

$$C(\mathbf{r},t) = \langle \hat{n}(\mathbf{r},t)\hat{n}(\mathbf{r},0) + \hat{n}(\mathbf{r},0)\hat{n}(\mathbf{r},t)\rangle/2 - \langle \hat{n}(\mathbf{r},0)\rangle^2$$

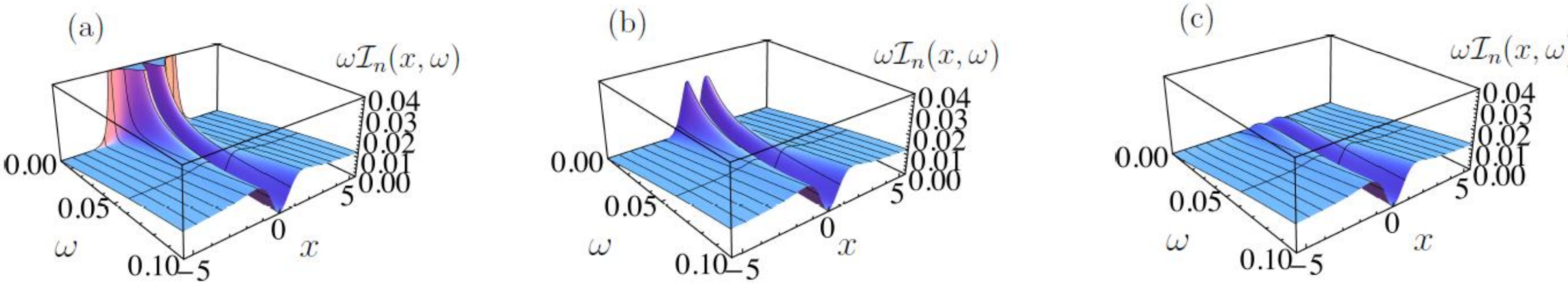
$$= \int_0^\infty \mathrm{d}\omega \rho(\mathbf{r},\omega)\cos\omega t$$



x,ω dependence of spectral function (U(x)=U δ(x))

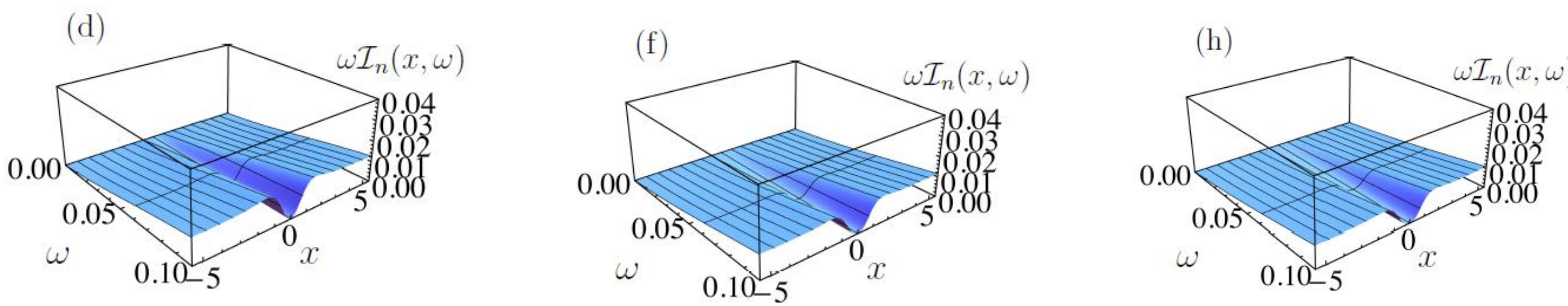


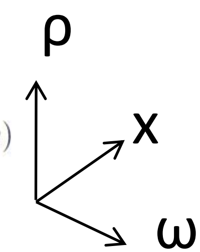
$$\rho(\mathbf{r},\omega) = \sum_l |\langle l|\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})|\mathrm{g}\rangle|^2 \delta(\omega - E_l + E_\mathrm{g})$$



V

V=Vc Fluctuation is enhanced near Vc!





V

V=0

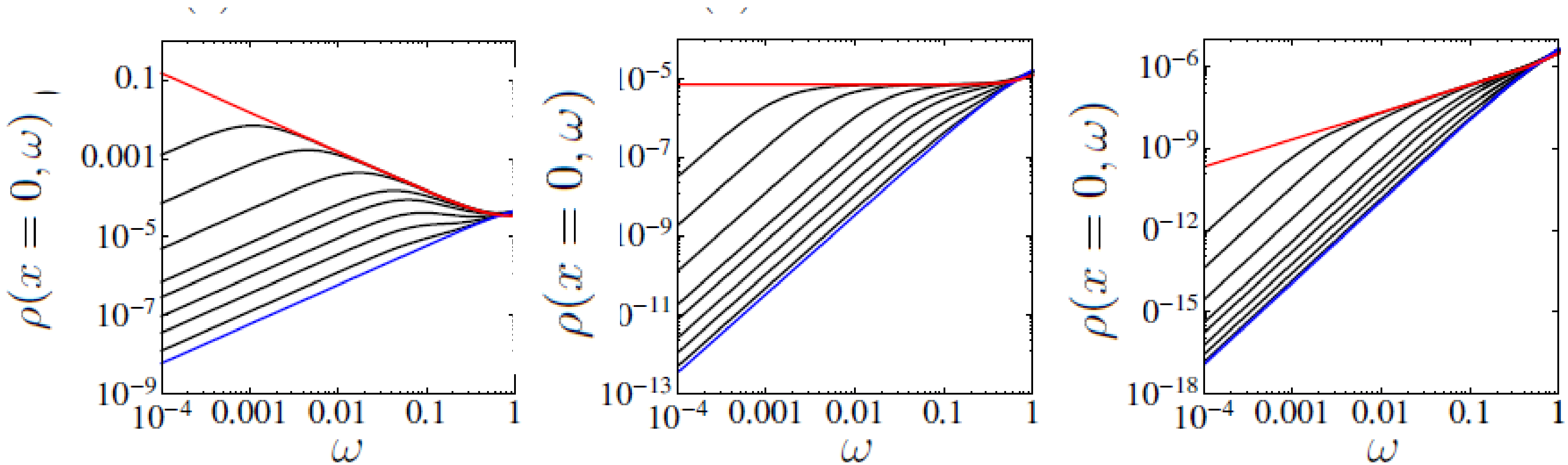
d=2

Watabe Thesis

ω dependence of spectral function at X=0 for soliton-emission instability

Blue curves for V=0, Black for 0<V<Vc, Red for V=Vc   Watabe   Thesis

d=1   d=2   d=3

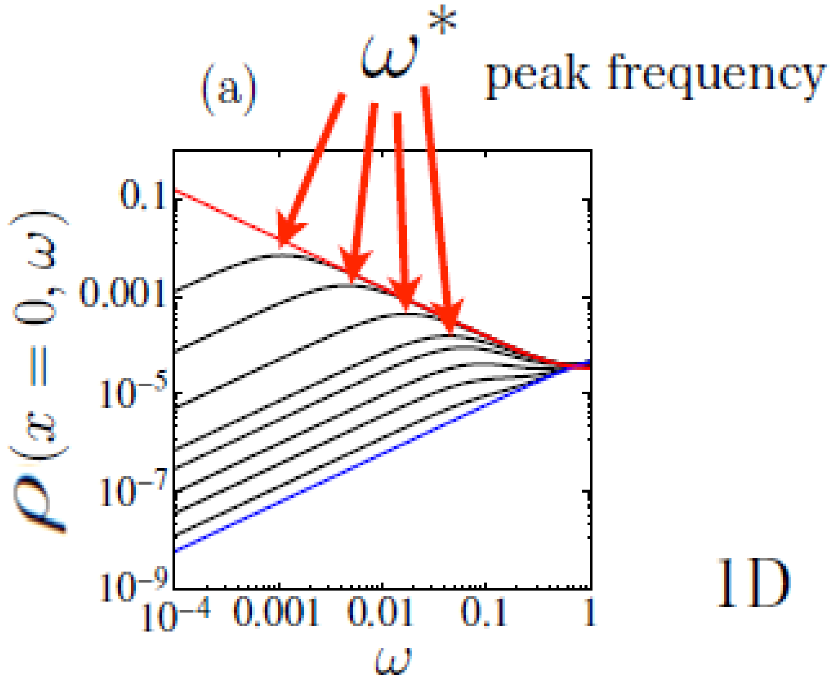


$$\rho(\boldsymbol{r},\omega) \propto \boxed{\omega^{d-2}} \quad V = V_c$$

$$\rho(\boldsymbol{r},\omega) \propto \boxed{\omega^{d}} \quad V < V_c$$

Note: the power of ω changes discontinuously at Vc
The exponent does not depend on x

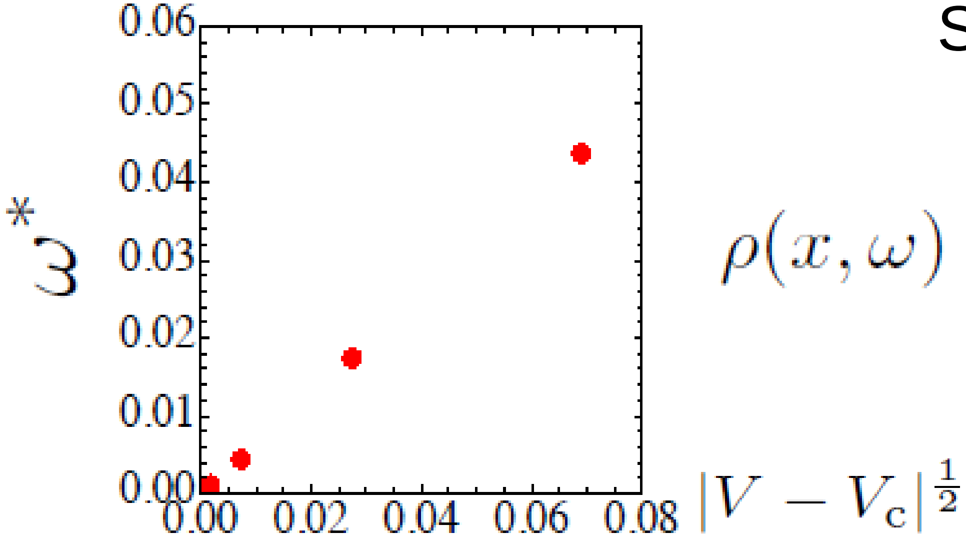
Crossover and scaling form



1D

$$\omega^* \propto |V - V_c|^{\frac{1}{2}}$$

Crossover frequency
(Dynamical scaling inherent to saddle-node bifurcation)

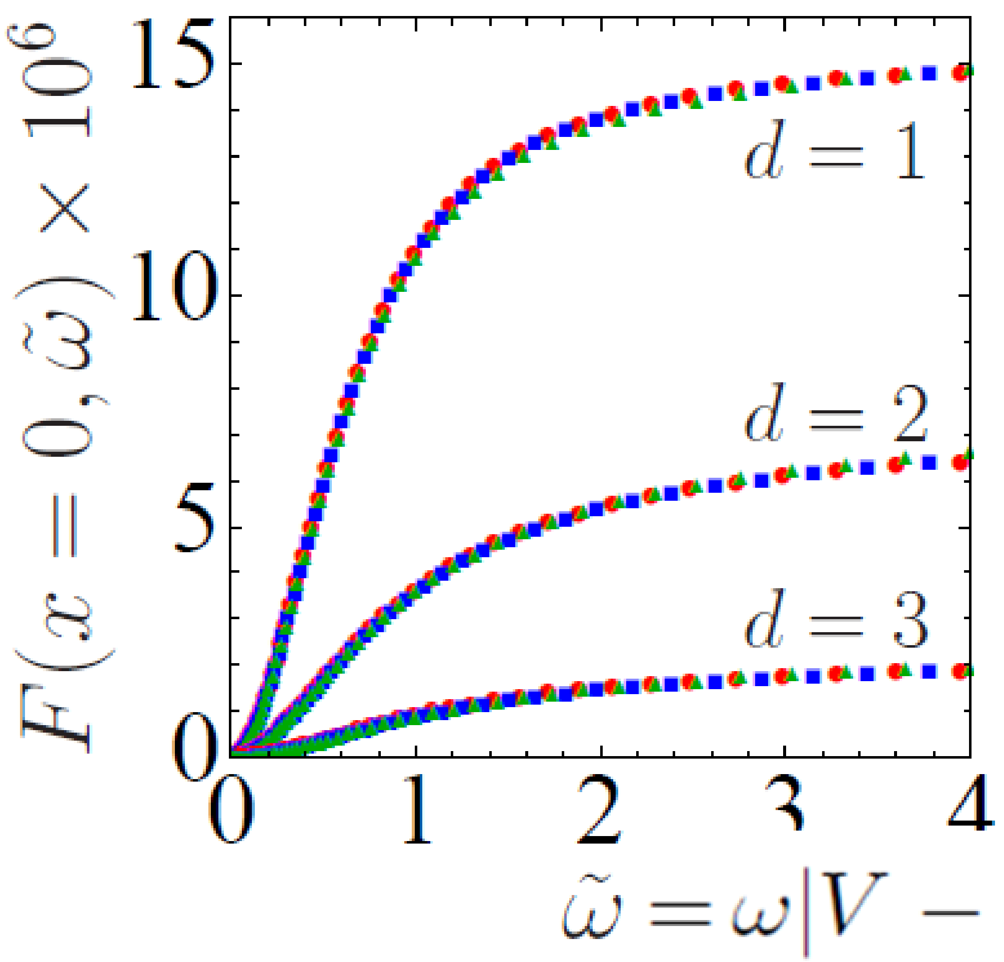


Scaling form (d=1,2,3)

$$\rho(x,\omega) = \omega^{d-2} F(x, \omega |V - V_c|^{-\frac{1}{2}})$$

Numerical confirmation of Scaling relation

$$\rho(x,\omega) = \omega^{d-2} F(x, \omega |V - V_c|^{-\frac{1}{2}})$$



Red: $(V_c-V)/V_c=6\times10^{-5}$

Blue: $(V_c-V)/V_c=1\times10^{-3}$

Green: $(V_c-V)/V_c=1\times10^{-2}$

Enhancement of fluctuation is due to saddle-node bifurcation

Stability criterion of superfluidity

$$\omega \to 0, \quad \rho(\mathbf{r},\omega) \to \begin{cases} O(\omega^\beta) \ \text{ with } \beta < d & \text{when } V = V_\mathrm{c} \\ O(\omega^d) & \text{when } V < V_\mathrm{c} \end{cases}$$

Fourier Transform

$$C(\mathbf{r},t) = \langle \hat{n}(\mathbf{r},t)\hat{n}(\mathbf{r},0) + \hat{n}(\mathbf{r},0)\hat{n}(\mathbf{r},t)\rangle/2 - \langle \hat{n}(\mathbf{r},0)\rangle^2$$

$$= \int_0^\infty \mathrm{d}\omega \rho(\mathbf{r},\omega)\cos\omega t$$

$$t \to \infty, \quad C(\mathbf{r},t) \to \begin{cases} O(1/t^{\beta+1}) \quad \text{with} & \text{for } {}^\exists\mathbf{r}, \quad v = v_\mathrm{c} \\ O(1/t^{d+1}) & \text{for } {}^\forall\mathbf{r}, \quad v < v_\mathrm{c} \end{cases},$$

Zero mode of Bogoliubov equation

For V≦Vc

$$\varepsilon \to 0 \qquad \begin{pmatrix} u(\boldsymbol{r}) \\ v(\boldsymbol{r}) \end{pmatrix} = \begin{pmatrix} \Psi(\boldsymbol{r}) \\ \Psi^*(\boldsymbol{r}) \end{pmatrix}$$

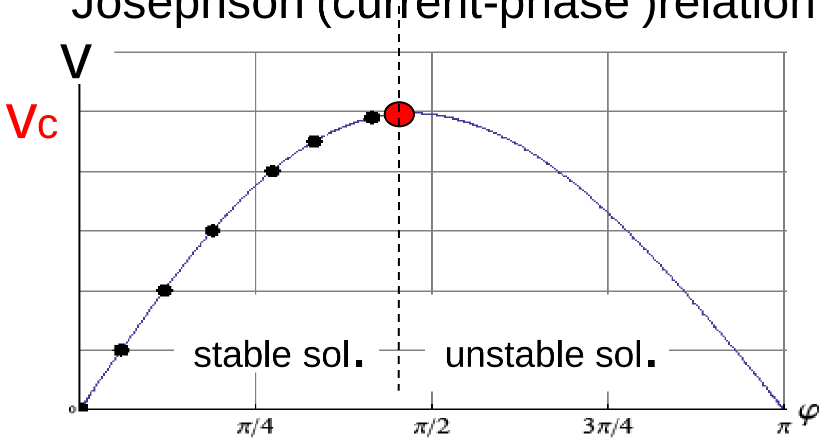
A zero mode(Goldstone mode)
Fetter 1972

For V=Vc

$$\varepsilon \to 0 \quad \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \frac{\partial}{\partial \varphi} \begin{pmatrix} \Psi(x;\varphi) \\ -\Psi^*(x;\varphi) \end{pmatrix}$$

Another zero mode
Pham-Brachet 2002 Takahashi-Kato2009

Josephson (current-phase)relation



At v=vc, stable and unstable solutions are degenerate.

Cf analogous to elementary diff. eq.

$$\left(\frac{dy}{dx} - \varphi_1\right)\left(\frac{dy}{dx} - \varphi_2\right) = 0$$

$$y = \exp(\varphi_1 x), \quad \exp(\varphi_2 x) \qquad \text{for } \varphi_1 \neq \varphi_2$$

$$y = \exp(\varphi_1 x), \quad \frac{\partial \exp(\varphi_1 x)}{\partial \varphi_1} \qquad \text{for } \varphi_1 = \varphi_2$$

Spectral function of local density

$$\rho(\mathbf{r},\omega)=\sum_{l}|\langle l|\hat{\psi}^{\dagger}(\mathbf{r})\hat{\psi}(\mathbf{r})|\mathrm{g}\rangle|^{2}\delta(\omega-E_{l}+E_{\mathrm{g}})$$

$$\rho(\mathbf{r},\omega)=\sum_{i}|\Psi(\mathbf{r})u_i^{*}(\mathbf{r})-\Psi^{*}(\mathbf{r})v_i^{*}(\mathbf{r})|^{2}\delta(\omega-\varepsilon_{i})$$

Fetter's solution (Goldstone mode)

$$\begin{pmatrix} u(\boldsymbol{r}) \\ v(\boldsymbol{r}) \end{pmatrix} = \begin{pmatrix} \Psi(\boldsymbol{r}) \\ \Psi^{*}(\boldsymbol{r}) \end{pmatrix} \longrightarrow \Psi(\mathbf{r})u_i^{*}(\mathbf{r})-\Psi^{*}(\mathbf{r})v_i^{*}(\mathbf{r}) = 0$$

does not couple to density fluctuation

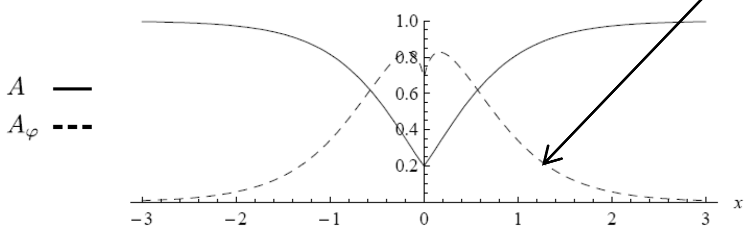
The other zero mode

$$\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \frac{\partial}{\partial\varphi}\begin{pmatrix} \Psi(x;\varphi) \\ -\Psi^{*}(x;\varphi) \end{pmatrix} \longrightarrow \Psi(\mathbf{r})u_i^{*}(\mathbf{r})-\Psi^{*}(\mathbf{r})v_i^{*}(\mathbf{r}) \neq 0$$

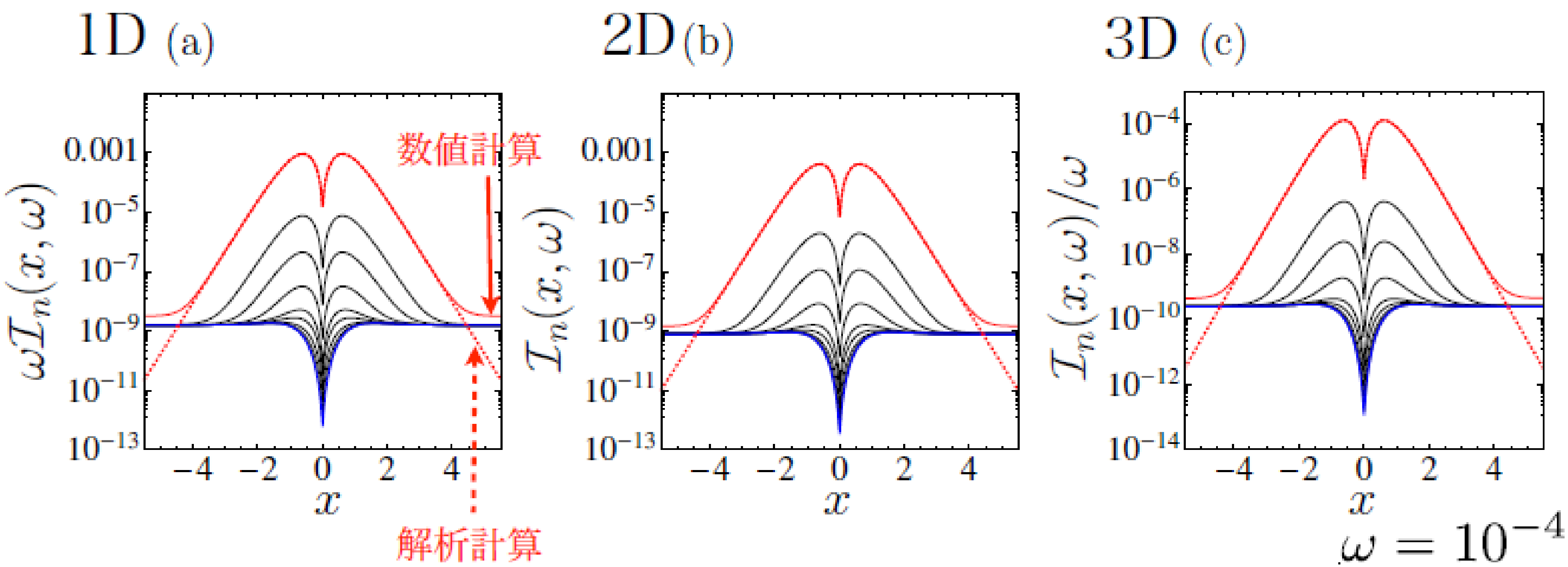
couples to density fluctuation

At V=Vc, density fluctuation occurs at low energy

Localized density fluc.



Spatial dependence near Vc



Each curve represents the results for each V(red curve for Vc)

$$\rho(x,\omega) \to \begin{cases} \omega^d f_d(x,V) & V < V_\mathrm{c}, \\ \boxed{\nu_d \omega^{d-2} \left| \dfrac{\partial |\Psi(x,\varphi)|^2}{\partial \varphi} \right|^2 \qquad V = V_\mathrm{c},} \end{cases}$$

Discussion: Why is density fluctuation enhanced near critical velocity?

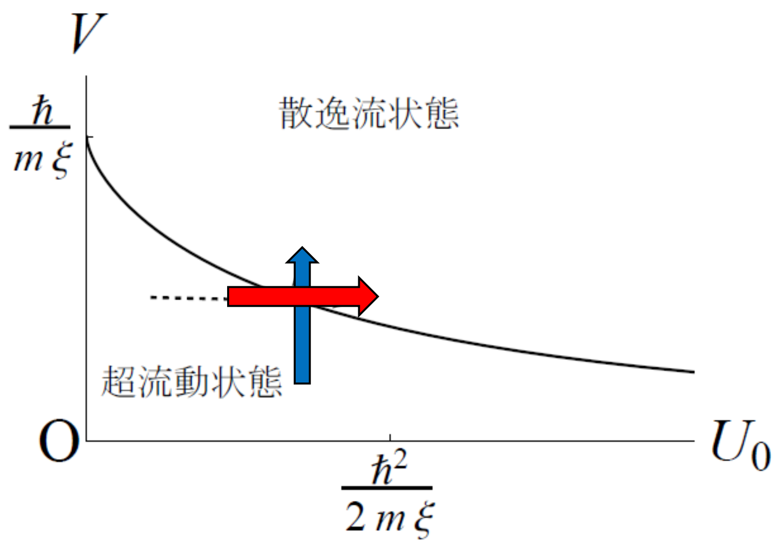
Near critical velocity, the state changes drastically with change V→V+δV

≒

Near critical velocity, the state changes drastically with change $U_0 \rightarrow U_0+\delta U_0$

↓

Near critical velocity, the density fluctuation would be enhanced



summary

- **dynamical density fluctuation** at low ω and **autocorrelation of the local density** at long time difference are enhanced near critical velocity in a soliton-emission instability.

- Enhancement of fluctuation at critical velocity is due to appearance of the zero mode that couples to density fluctuation.

- Our result suggests that superfluidity requires BEC with finite compressibility and suppressed dynamical density fluctuations.