# New Derivation of QCD Sum Rules Based on Commutation Relations

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# Introduction I

○ Dipole sum rule in nuclear physics (Giant dipole resonance)

$$\sigma_{\rm tot} = \frac{4\pi^2 e^2}{\hbar c} \sum_\nu (E_\nu - E_0) |\langle \nu | D | 0 \rangle|^2$$

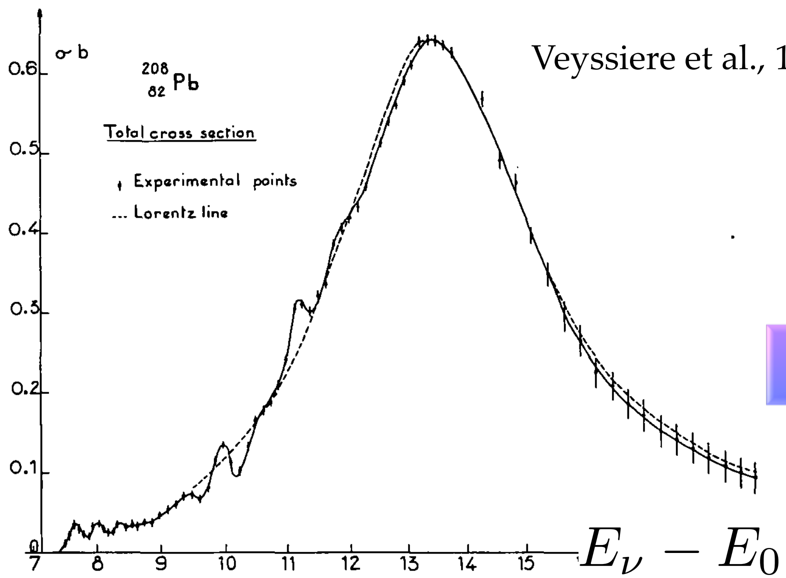
← Energy weighted sum

$$= \frac{4\pi^2 e^2}{\hbar c} \langle 0 | [D, [H, D]] | 0 \rangle$$

← Double commutator

$$= \frac{2\pi^2 e^2 \hbar}{mc} \frac{NZ}{A} (1 + K)$$

← Universal constant




Veyssiere et al., 1970

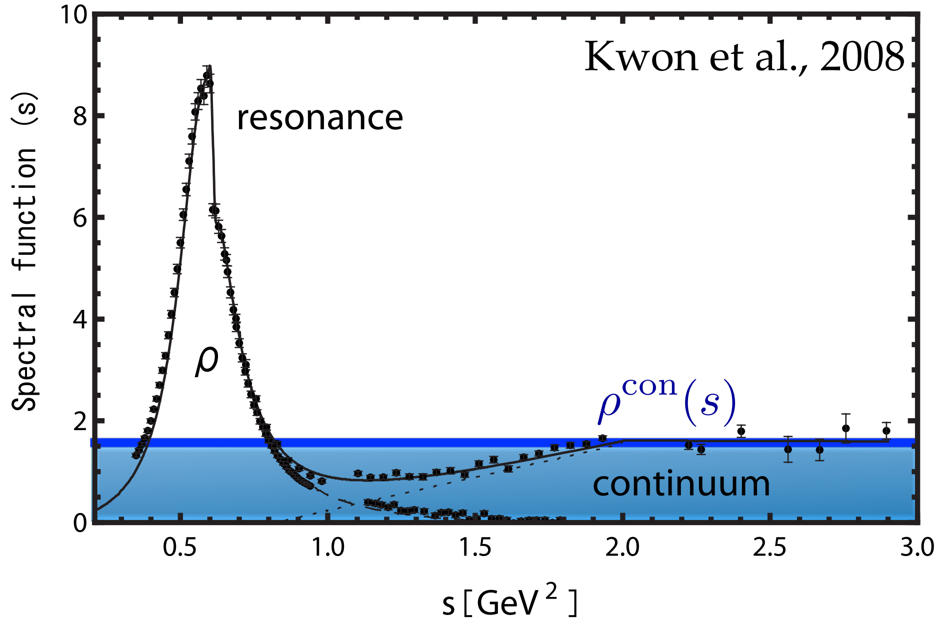
→ What about sum rules in QFT such as QCD ?

# Introduction II

○ QCD sum rules from OPE (演算子積展開) [Shifman, Vainshtein & Zhakharov (SVZ)]

$$\int_0^\infty \frac{ds}{2\pi}\, s\Big(\rho_V(s) - \rho_V^{\rm con}(s)\Big) = \langle 0| - \frac{m_q}{2}\bar{q}q - \frac{\alpha_s}{24\pi}G^a_{\mu\nu}G^{a\mu\nu}|0\rangle_{\rm NP}$$

Resonances ⇔ quark & gluon condensates $\langle 0|\bar{q}q|0\rangle_{\rm NP}$, $\langle 0|G^2|0\rangle_{\rm NP}$




Derive/ Generalize QCD sum rules from CRs without OPE

# Canonical quantization of QCD [Kugo & Ojima, '78]

$$\mathcal{L}_{\rm eff} = \bar{q}_f(i\not{D} - m_f)q_f - \frac{1}{4}G^{a\mu\nu}G^a_{\mu\nu} - \partial_\mu B^a A^{a\mu} + \frac{\alpha}{2}(B^a)^2 - i\partial^\mu \bar{c}^a D^{\rm ad}_\mu c^a$$

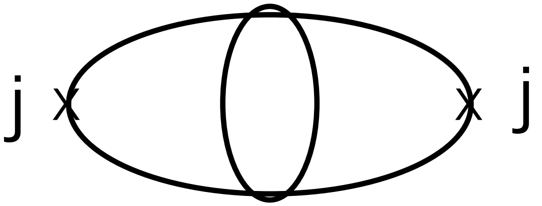
QCD Hamiltonian

$$\mathcal{H}_{\rm eff} = -gA^a_0\bar{q}_f\gamma^0 t^a q_f + \bar{q}_f(-i\gamma^k D_k + m_f)q_f + \frac{1}{2}\left((\vec{E}^a)^2 + (\vec{H}^a)^2\right)$$
$$+\vec{E}^a\cdot(\nabla A^a_0 - gf_{abc}\vec{A}^b A^c_0) + \partial_k B^a A^{ak} - \frac{\alpha}{2}(B^a)^2$$
$$+i\Pi^a_c\Pi^a_{\bar{c}} + gf_{abc}\Pi^a_c A^b_0 c^c - i\partial^k\bar{c}^a D^{\rm ad}_k c^a$$

- Heisenberg fields: $\underbrace{q_f,\ \bar{q}_f}_{\text{quarks}},\ \underbrace{A^a_\mu,\ B^a,\ E^a_k}_{\text{gluons}},\ \underbrace{c^a,\ \bar{c}^a,\ \Pi^a_c,\ \Pi^a_{\bar{c}}}_{\text{ghosts}}$
- CCRs: $\{q,\bar{q}\},\ [A^a,B^b],\ [A^a_i,E^{bj}],\ \{c^a,\Pi^b_c\},\ \{\bar{c}^a,\Pi^b_{\bar{c}}\}$
- BRST chargeless $Q_{\rm B}|{\rm phys}\rangle = 0$

# Sum rules for QCD current correlator

○ Spectral function




$$\rho(q^2) = -\frac{1}{3q^2}\sum_p (2\pi)^4\delta^{(4)}(q-p)\langle 0|j_\mu(0)|p\rangle\langle p|j^\mu(0)|0\rangle$$

○ Energy weighted sum rules at zero 3MOM $s \longrightarrow \omega^2$

$$\int_0^\infty \frac{\mathrm{d}s}{2\pi}\, s^n\rho(s) = -\frac{1}{3}\int \mathrm{d}^3x\, \langle 0|[[j_\mu(0,\vec{x}),\mathrm{H}]_{2n-1}, j^\mu(0)]|0\rangle$$

○ Renormalization of perturbative UV divergence

$$\int_0^\infty \frac{\mathrm{d}s}{2\pi}\, s^n(\rho(s)-\rho^{\mathrm{con}}(s)) = -\frac{1}{3}\int \mathrm{d}^3x\, \langle 0|[[j_\mu(0,\vec{x}),\mathrm{H}]_{2n-1}, j^\mu(0)]|0\rangle_{\mathrm{NP}}$$

# Commutators for QCD current correlator I

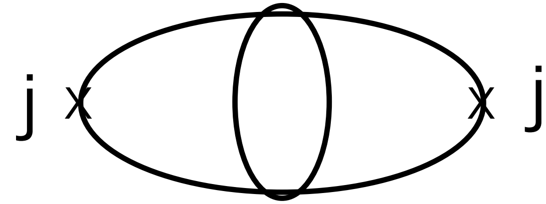
○ Basic commutator

$$[\bar{u}\gamma^\mu u, \mathrm{H}] = -i\bar{u}\gamma^\mu\gamma^0\gamma^k\overrightarrow{D}_k u - i\bar{u}\gamma^k\overleftarrow{D}_k\gamma^0\gamma^\mu u + m_u\bar{u}[\gamma^\mu, \gamma^0]u$$

○ Double commutator

$$[[\bar{u}\gamma^\mu u, \mathrm{H}], \mathrm{H}] = \bar{u}\gamma^\mu\gamma^k\overrightarrow{D}_k\gamma^{k'}\overrightarrow{D}_{k'}u - \bar{u}\gamma^k\overleftarrow{D}_k\gamma^0\gamma^\mu\gamma^0\gamma^k\overrightarrow{D}_k u$$
$$+2im_u\bar{u}\gamma^0\gamma^\mu\gamma^0\gamma^k\overrightarrow{D}_k u + \underline{ig\bar{u}\gamma^\mu\gamma^0\gamma^k E_k^a t^a u}$$
$$+m_u^2\bar{u}(\gamma^\mu - \gamma^{\mu\dagger})u + \mathrm{h.\,c.}$$

Canonical commutation relations alone, no OPE at all



# Commutators for QCD current correlator II

○ 1st moment

$$[[\bar{u}\gamma^{\mu}u, \mathrm{H}], \bar{u}\gamma_{\mu}u] = -4i\bar{u}\gamma^{k}\overleftrightarrow{D}_{k}u + 12m_{u}\bar{u}u$$

○ 2nd moment

$$\begin{aligned}[[\bar{u}\gamma^{\mu}u, \mathrm{H}]_2, [\bar{u}\gamma_{\mu}u, \mathrm{H}]] &= 20i\bar{u}\overleftarrow{D}^{k}\overleftarrow{D}_{k}\gamma^{k'}\overrightarrow{D}_{k'}u + 4i\bar{u}\overleftarrow{D}^{k}\gamma^{k'}\overrightarrow{D}_{k'}\overrightarrow{D}_{k}u \\ &- 16m_{u}\bar{u}\overleftarrow{D}^{k}\overleftarrow{D}_{k}u - 8igm_{u}\bar{u}\gamma^{0}\gamma^{k}E_{k}u \\ &+ 24im_{u}^{2}\bar{u}\gamma^{k}\overrightarrow{D}_{k}u - 24m_{u}^{3}\bar{u}u \\ &- 4g\bar{u}\gamma^{0}\gamma^{k'}\overleftarrow{D}_{k'}\gamma^{k}E_{k}u + 4g\bar{u}\gamma^{0}\overleftarrow{D}^{k}E_{k}u \\ &+ 4g\bar{u}\gamma^{0}\overrightarrow{D}_{k}^{\mathrm{ad}}E^{k}u \\ &- 4g^{2}(\bar{u}\gamma^{k}\gamma^{5}t^{a}u)(\bar{u}\gamma_{k}\gamma^{5}t^{a}u) + \Big(\mathrm{h.\ c.}\Big)\end{aligned}$$

# Weinberg sum rules

○ Difference between vector and axial-vector currents;

$$j_V^\mu = (\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d)/2 \;\;/\;\; j_A^\mu = (\bar{u}\gamma^\mu\gamma^5 u - \bar{d}\gamma^\mu\gamma^5 d)/2$$

○ Chiral transformation; $q = {}^t(u, d) \rightarrow \mathrm{e}^{i\theta^a\tau^a\gamma^5} q$

○ Non-OPE derivation of Weinberg sum rules

- $1^{\mathrm{st}}$ moment

$$\int_0^\infty \frac{\mathrm{d}s}{2\pi}\, s\rho_{V(A)}(s) = -\frac{1}{3}\int \mathrm{d}^3x\, \langle 0|[[j^\mu_{V(A)}(0,x),\mathrm{H}], j_{V(A)\mu}(0,0)]|0\rangle$$

- $2^{\mathrm{nd}}$ moment

$$\int_0^\infty \frac{\mathrm{d}s}{2\pi}\, s^2\rho_{V(A)}(s) = -\frac{1}{3}\int \mathrm{d}^3x\, \langle 0|[[j^\mu_{V(A)}(0,x),\mathrm{H}]_2, [j_{V(A)\mu}(0,0),\mathrm{H}]]|0\rangle$$

# Weinberg sum rules

○ Difference between vector and axial-vector currents;

$$j_V^\mu = (\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d)/2 \;\; / \;\; j_A^\mu = (\bar{u}\gamma^\mu\gamma^5 u - \bar{d}\gamma^\mu\gamma^5 d)/2$$

○ Chiral transformation; $q = {}^t(u, d) \to \mathrm{e}^{i\theta^a\tau^a\gamma^5} q$

○ Non-OPE derivation of Weinberg sum rules

- 1st moment

$$\int_0^\infty \frac{\mathrm{d}s}{2\pi}\, s(\tilde{\rho}_A(s) - \rho_V(s)) = \langle 0|\frac{4m_u}{3}\bar{u}u + \frac{4m_d}{3}\bar{d}d|0\rangle_{\mathrm{NP}}$$

- 2nd moment

$$\int_0^\infty \frac{\mathrm{d}s}{2\pi}\, s^2(\tilde{\rho}_A(s) - \rho_V(s)) = \langle 0|8\pi\alpha_s(\bar{q}_L\gamma^\mu t^a\tau_z q_L)(\bar{q}_R\gamma_\mu t^a\tau_z q_R)|0\rangle_{\mathrm{NP}}$$

⇔ consistent with SVZ sum rules from OPE

# Sum rules for vector correlation I

○ 1st moment bare sum rule

$$j_V^\mu = (\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)/2$$

$$\int_0^\infty \frac{\mathrm{d}s}{2\pi}\, s\rho_V(s) = -\frac{1}{3}\int \mathrm{d}^3x\, \langle 0|[[j_V^\mu(0,x),\mathrm{H}], j_{V\mu}(0,0)]|0\rangle$$

$$= \langle 0|\frac{i}{3}\bar{u}\overleftrightarrow{D}_k\gamma^k u - m_u\bar{u}u|0\rangle + (u \Leftrightarrow d)$$

○ Renormalized sum rule

• Dirac Eq. + Lorentz invariance

$$\langle 0|\bar{q}\gamma^\mu\overrightarrow{D}^\nu q|0\rangle = \frac{g^{\mu\nu}}{4}\langle 0|\bar{q}\overrightarrow{\not{D}}q|0\rangle$$

$$\Rightarrow \quad \int_0^\infty \frac{\mathrm{d}s}{2\pi}\, s\Big(\rho_V(s) - \rho_V^{\mathrm{con}}(s)\Big) = \langle 0| -\frac{m_u}{2}\bar{u}u - \frac{m_d}{2}\bar{d}d|0\rangle_{\mathrm{NP}}$$

• No gluon condensate in (axial) vector sum rule!?

$$\langle 0|G^{a\mu\nu}G^a_{\mu\nu}|0\rangle_{\mathrm{NP}}$$

# Commutator by BJL limit [Bjorken, Johnson, Low, 66]

○ Causality

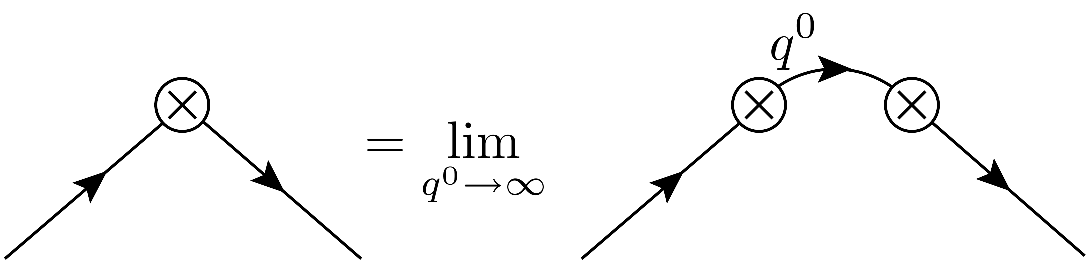
$$[A(x), B(0)]_{\mathrm{ET}} = \sum C_i \hat{O}_i(x, \nabla_x)\delta^{(3)}(x)$$

○ Commutator btw asymptotic states

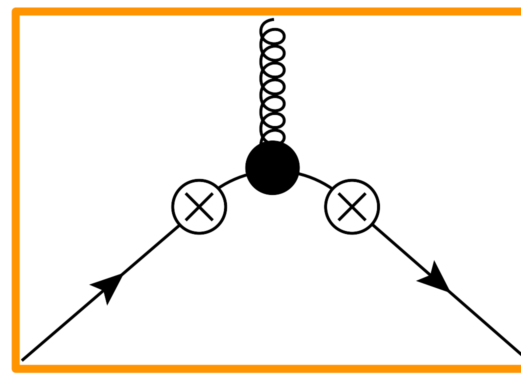
$$\int \mathrm{d}^3x \langle\alpha|[A(x), B(0)]_{\mathrm{ET}}|\beta\rangle = \lim_{q^0\to\infty}(-iq^0)\int \mathrm{d}^4x\, \mathrm{e}^{iq^0x^0}\langle\alpha|\mathrm{T}^*[A(x), B(0)]|\beta\rangle$$

○ Point splitting regularization

➔ Anomalous matrix elements (Higher orders in $\alpha_s$)

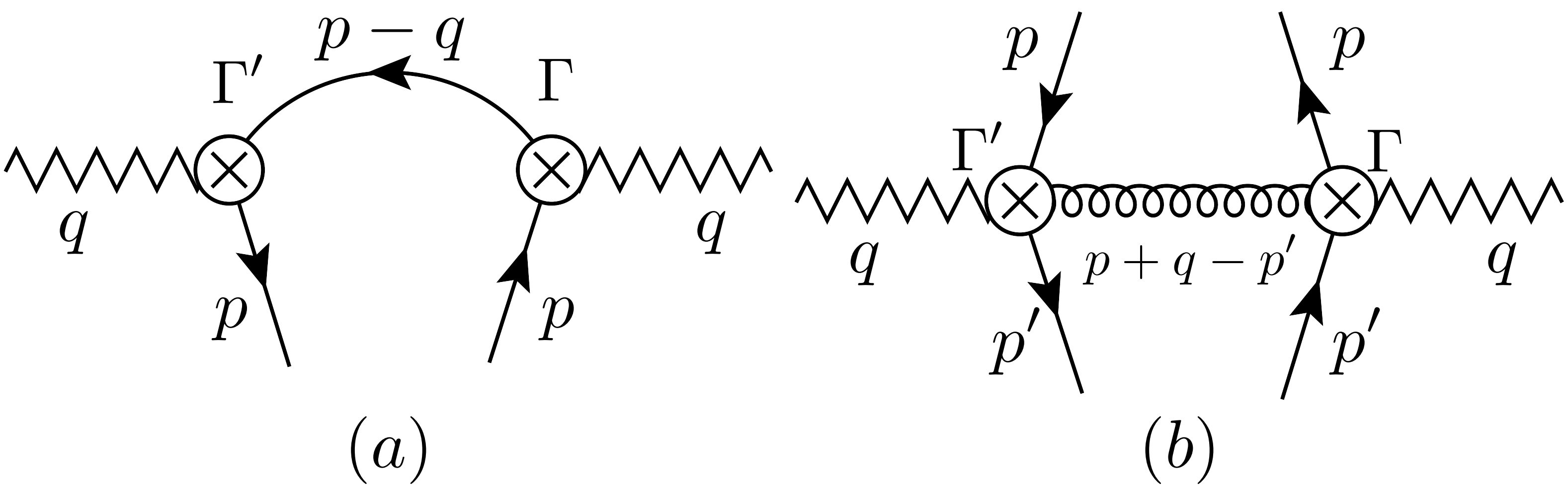



Deviation from CCRs (Commutator anomaly)



# Commutator anomaly I

○ Tree diagrams = Results of CCRs $\langle \text{quark}|[\bar{q}\Gamma q, \bar{q}\Gamma' q]_{\text{ET}}|\text{quark}\rangle$




• Diagram (a)

$$\int \mathrm{d}^3x \, \langle \text{quark}|[\bar{q}\Gamma q(x), \bar{q}\Gamma' q(0)]_{\text{ET}}|\text{quark}\rangle$$

$$= (-iq^0)\Big(\langle \text{quark}|\bar{q}_i \Gamma_{ij} \Gamma'_{kl} q_l|\text{quark}\rangle S_F^{jk}(p-q)$$

$$-\langle \text{quark}|\bar{q}_k \Gamma'_{kl} \Gamma_{ij} q_j|\text{quark}\rangle S_F^{li}(p+q)\Big)$$

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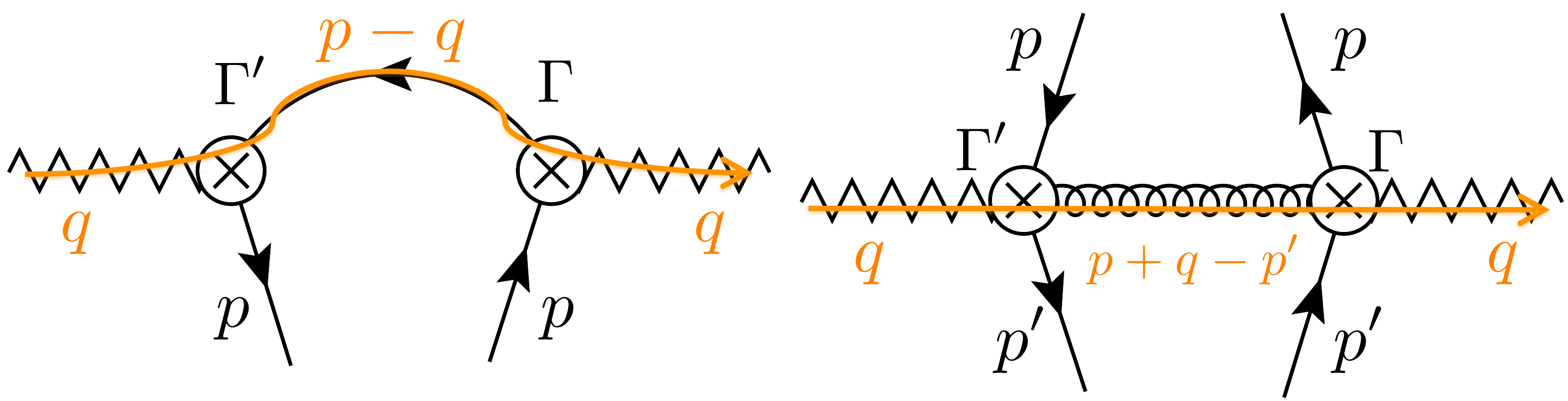



- BJL limit; $q \to (\infty, 0)$

**Free propagators in BJL limit = CCRs**

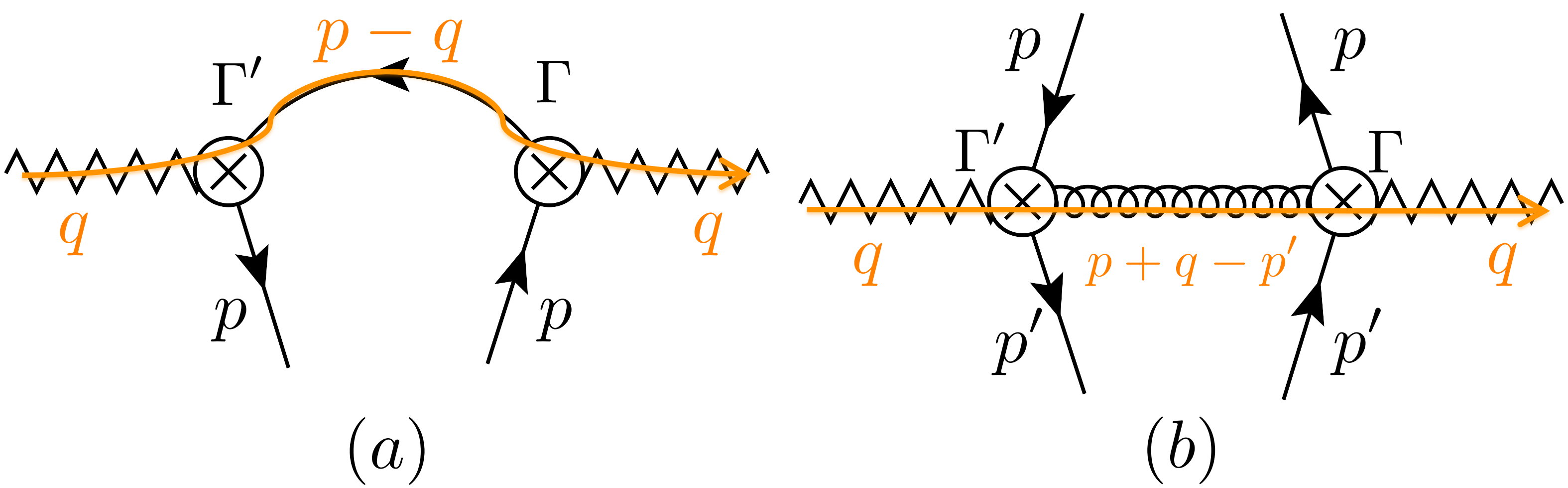
$$\int \mathrm{d}^3x \, \langle \text{quark}|[\bar{q}\Gamma q(x), \bar{q}\Gamma' q(0)]_{\text{ET}}|\text{quark}\rangle$$

$$= (-iq^0)\Big(\langle \text{quark}|\bar{q}_i \Gamma_{ij} \Gamma'_{kl} q_l|\text{quark}\rangle \underline{S_F^{jk}(p-q)}$$

$$-\langle \text{quark}|\bar{q}_k \Gamma'_{kl} \Gamma_{ij} q_j|\text{quark}\rangle \underline{S_F^{li}(p+q)}\Big)$$

# Commutator anomaly I

○ Tree diagrams = Results of CCRs $\langle \mathrm{quark} | [\bar{q}\Gamma q, \bar{q}\Gamma' q]_{\mathrm{ET}} | \mathrm{quark} \rangle$



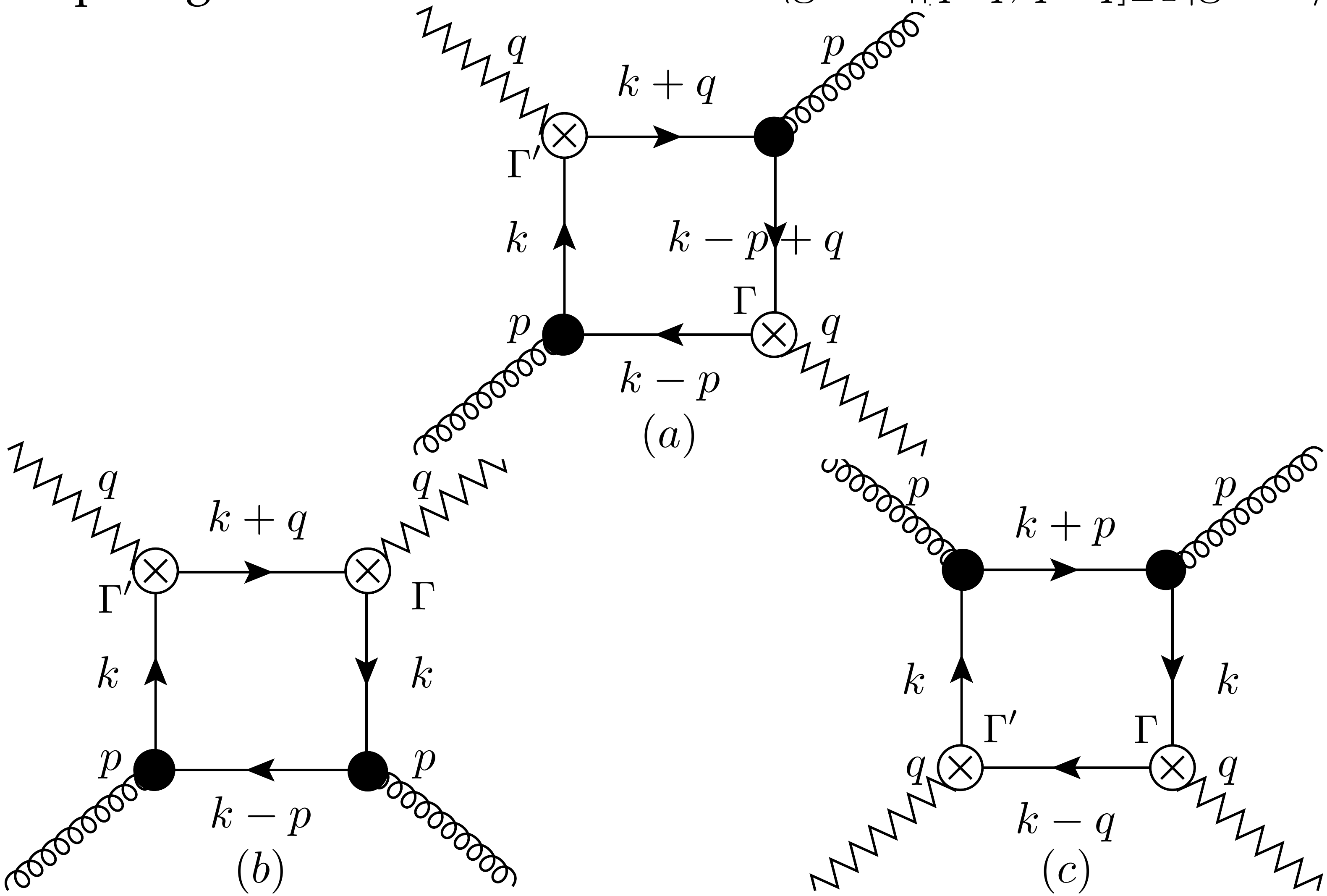

- BJL limit; $q \to (\infty, 0)$

$$\int \mathrm{d}^3x \, \langle \mathrm{quark} | [\bar{q}\Gamma q(x), \bar{q}\Gamma' q(0)]_{\mathrm{ET}} | \mathrm{quark} \rangle$$
$$= \langle \mathrm{quark} | \bar{q}_i \Gamma_{ij} \gamma^0_{jk} \Gamma'_{kl} q_l | \mathrm{quark} \rangle - \langle \mathrm{quark} | \bar{q}_k \Gamma'_{kl} \gamma^0_{li} \Gamma_{ij} q_j | \mathrm{quark} \rangle$$

⇔ consistent with CCRs

# Commutator anomaly II

○ Loop diagrams ≠ Results of CCRs $\langle \mathrm{gluon}|[\bar{q}\Gamma q, \bar{q}\Gamma' q]_{\mathrm{ET}}|\mathrm{gluon}\rangle$



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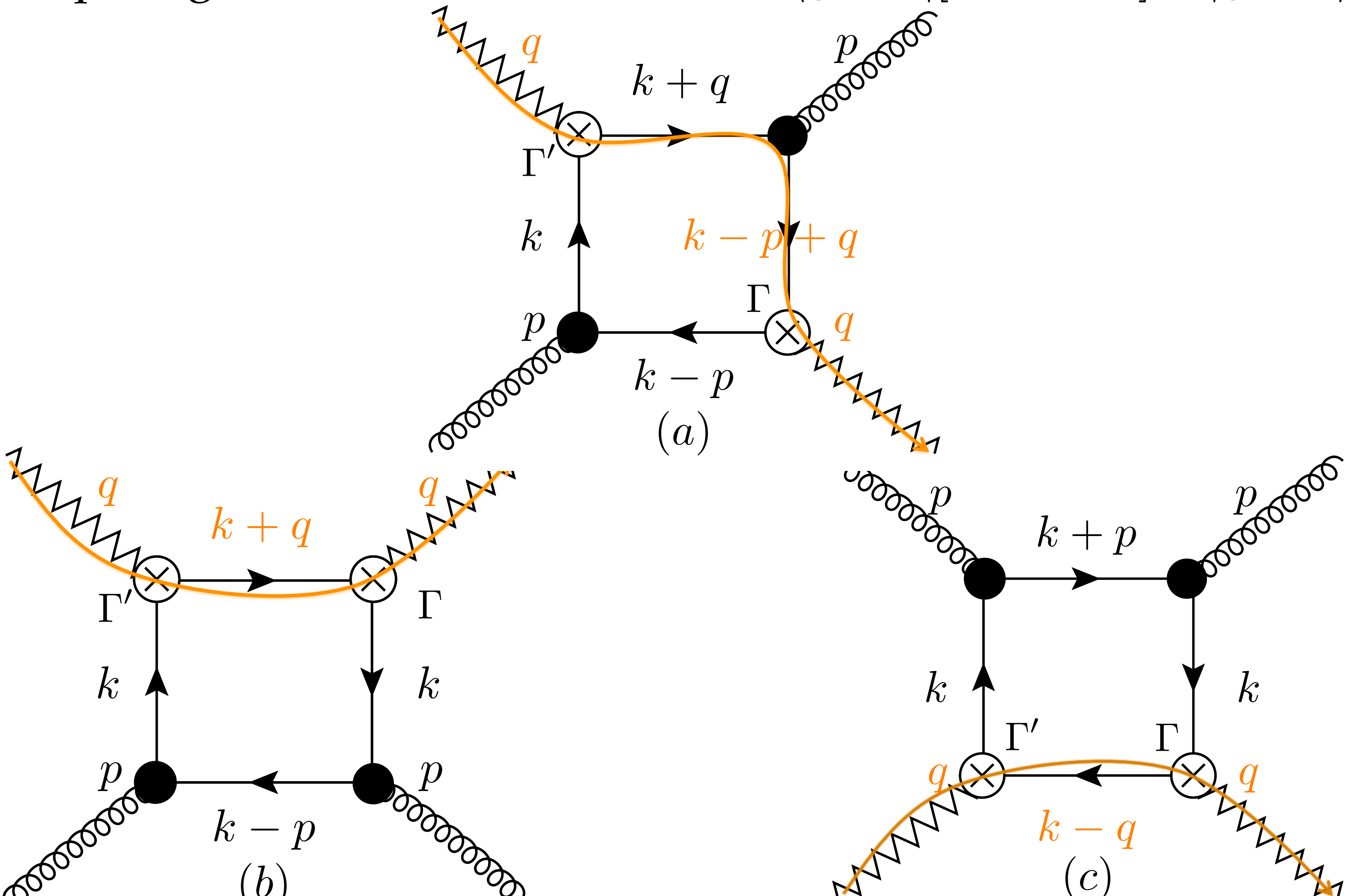

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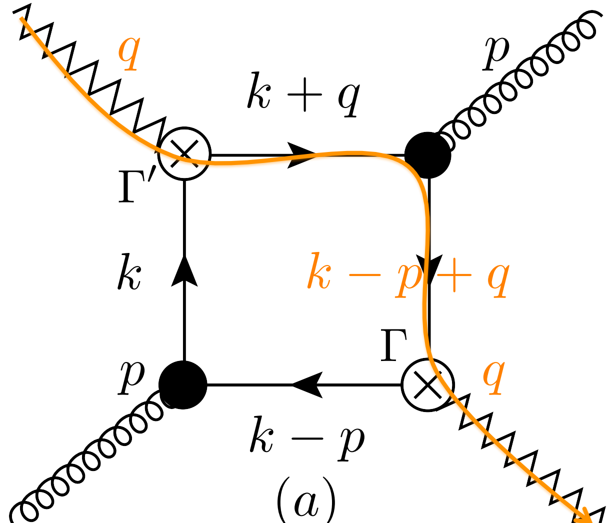
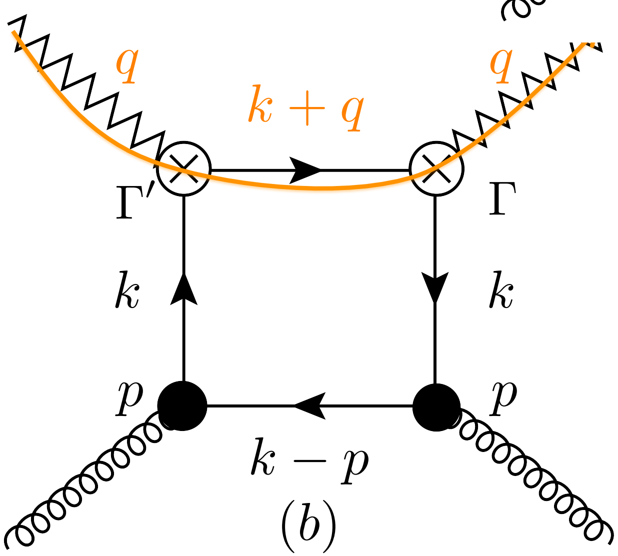
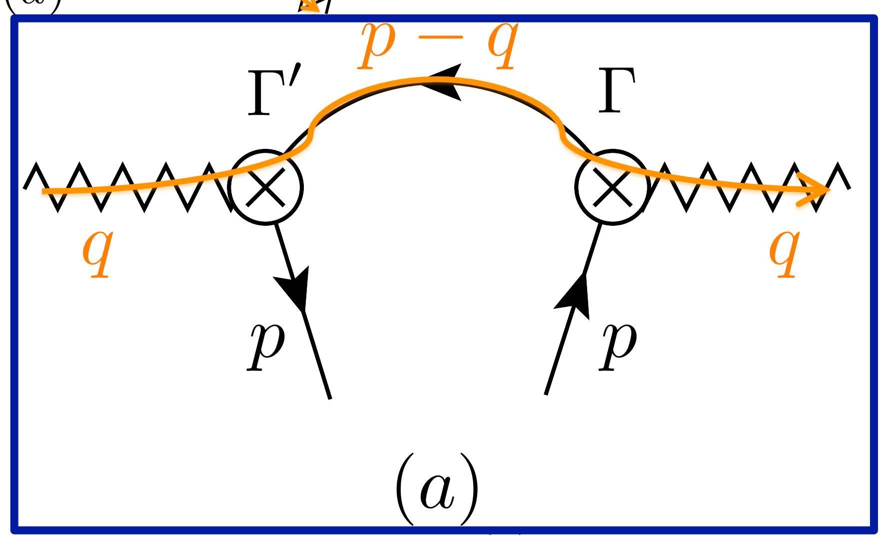
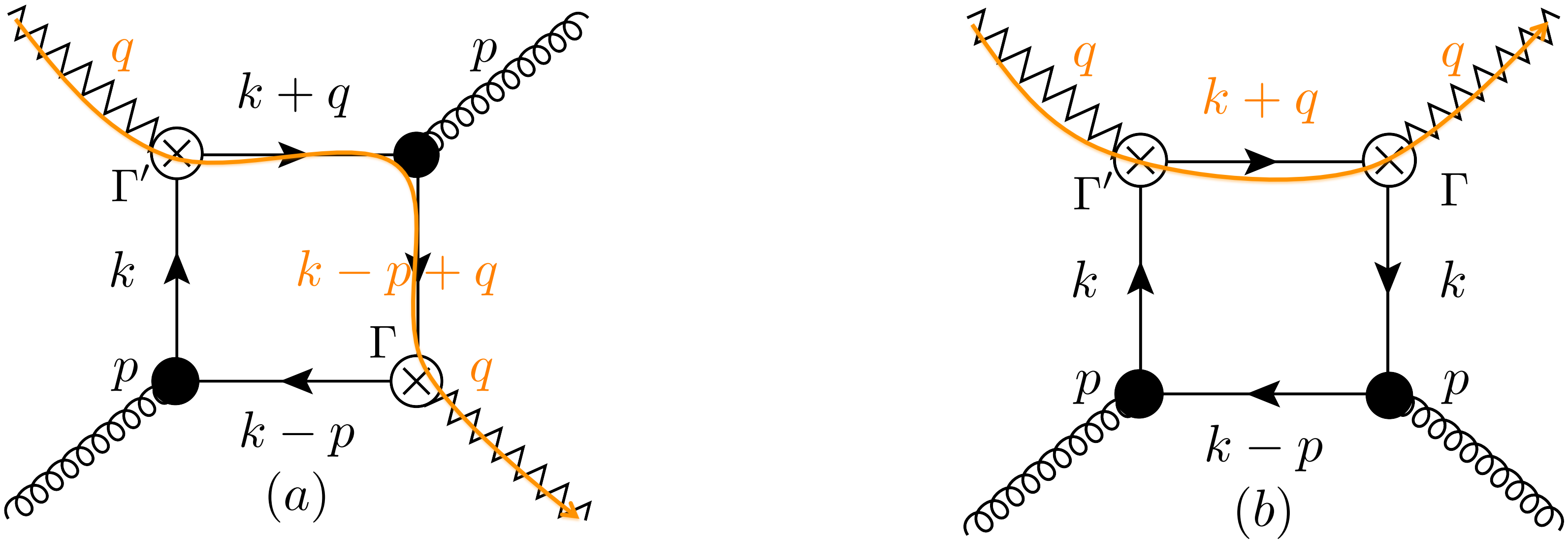


# Commutator anomaly II

○ Loop diagrams ≠ Results of CCRs  $\langle \mathrm{gluon} | [\bar{q}\Gamma q, \bar{q}\Gamma' q]_{\mathrm{ET}} | \mathrm{gluon} \rangle$




reproduce $\langle \mathrm{gluon} | \bar{q}\Gamma\gamma^0\Gamma' q - \bar{q}\Gamma'\gamma^0\Gamma q | \mathrm{gluon} \rangle$

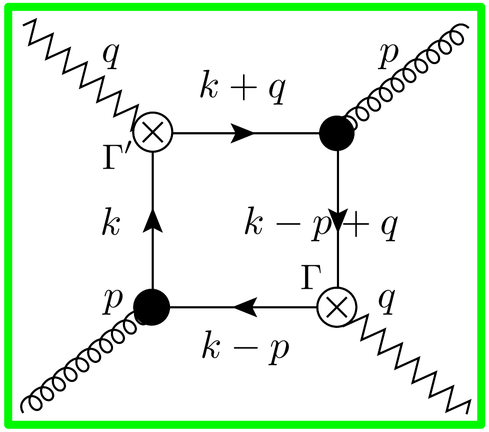
Deviation written by pure gluonic operator $\alpha_s G^a_{\mu\nu} G^{a\mu\nu}$

# Sum rules for vector correlation II

$$j_V^\mu = (\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d)/2$$

○ Bare sum rule

$$\int_0^\infty \frac{\mathrm{d}s}{2\pi} s\rho_V(s) = -\frac{1}{3}\int \mathrm{d}^3x \, \langle 0|[[j_V^\mu(0,x),\mathrm{H}], j_{V\mu}(0,0)]|0\rangle$$




○ BJL modified commutator

$$[[\bar{u}\gamma^\mu u, \mathrm{H}], \bar{u}\gamma_\mu u] = -4i\bar{u}\gamma^k \overleftrightarrow{D}_k u + 12 m_u \bar{u}u$$

$$[[\bar{u}\gamma^\mu u, \mathrm{H}], \bar{u}\gamma_\mu u] = -4i\bar{u}\gamma^k \overleftrightarrow{D}_k u + 12 m_u \bar{u}u + \frac{\alpha_s}{2\pi} G^a_{kk'} G^a_{kk'}$$

○ Renormalized sum rule

$$\int_0^\infty \frac{\mathrm{d}s}{2\pi} s\left(\rho_V(s) - \rho_V^{\mathrm{con}}(s)\right) = \langle 0| - \frac{m_q}{2}\bar{q}q - \frac{\alpha_s}{24\pi} G^a_{\mu\nu} G^{a\mu\nu}|0\rangle_{\mathrm{NP}}$$

⇔ consistent with SVZ sum rules from OPE

# Summary of sum rules by QCD commutator

○ 1st and 2nd Weinberg sum rules

$$\int_0^\infty \frac{\mathrm{d}s}{2\pi}\, s(\tilde{\rho}_A(s) - \rho_V(s)) = \langle 0| \frac{4m_u}{3}\bar{u}u + \frac{4m_d}{3}\bar{d}d|0\rangle_{\mathrm{NP}}$$

$$\int_0^\infty \frac{\mathrm{d}s}{2\pi}\, s^2(\tilde{\rho}_A(s) - \rho_V(s)) = \langle 0|8\pi\alpha_s(\bar{q}_L\gamma^\mu t^a\tau_z q_L)(\bar{q}_R\gamma_\mu t^a\tau_z q_R)|0\rangle_{\mathrm{NP}}$$

○ Sum rule for vector and axial-vector currents

$$\int_0^\infty \frac{\mathrm{d}s}{2\pi}\, s\Big(\rho_V(s) - \rho_V^{\mathrm{con}}(s)\Big) = \langle 0| \underline{-\frac{m_q}{2}\bar{q}q} \underline{- \frac{\alpha_s}{24\pi}G^a_{\mu\nu}G^{a\mu\nu}}|0\rangle_{\mathrm{NP}}$$

$$\int_0^\infty \frac{\mathrm{d}s}{2\pi}\, s(\tilde{\rho}_A(s) - \tilde{\rho}_A^{\mathrm{con}}(s)) = \langle 0|\frac{5}{6}m_q\bar{q}q - \frac{\alpha_s}{24\pi}G^a_{\mu\nu}G^{a\mu\nu}|0\rangle_{\mathrm{NP}}$$

CCR commutator ➔ Chiral condensate

Commutator anomaly ➔ Gluon condensate

# Comparison btw SVZ approach and our approach

| SVZ approach | Our approach |
| --- | --- |
| OPE as an operator identity | Commutators btw currents and the effective Hamiltonian |
| Perturbative calculation of Wilson coefficients | Perturbative calculation of commutator anomalies |
| Subtraction of perturbative contribution from the unit operator | Subtraction of perturbative vacuum graph |

# New sum rules based on CRs (Details ➔ Poster talk)

○ Energy weighted sum rules at zero 3MOM $s \to \omega^2$

- Odd

$$\int_0^\infty \frac{ds}{2\pi} s^n(\rho(s) - \rho^{\mathrm{con}}(s)) = -\frac{1}{3}\int d^3x \,\langle 0|[[j_\mu(0,\vec{x}), \mathrm{H}]_{2n-1}, j^\mu(0)]|0\rangle_{\mathrm{NP}}$$

- Even

$$\int_0^\infty \frac{ds}{2\pi} s^{n-\frac{1}{2}}(\rho(s) - \rho^{\mathrm{con}}(s)) = \frac{2}{3V}\langle 0|([Q_\mu, \mathrm{H}]_{n-1})^2|0\rangle_{\mathrm{NP}}$$

where $Q^\mu = \int d^3x \, j^\mu(0,\vec{x})$

○ 1st moment

$$\int_0^\infty \frac{ds}{2\pi} s^{\frac{1}{2}}(\rho(s) - \rho^{\mathrm{con}}(s)) = \frac{8}{3}\chi_{\mathrm{NP}}$$

Resonances ⇔ Charge fluctuations $\chi_{\mathrm{NP}} = \langle 0|Q_0^2|0\rangle_{\mathrm{NP}}/V$

# Summary

○ New derivation of QCD sum rules by commutator approach

⇔ simple & straightforward generalization of dipole sum rule

based on {
- Kugo-Ojima operator formalism
- commutator anomaly
- suitable subtractions of UV divergences

○ Weinberg and (axial) vector sum rules are derived

⇔ consistent with SVZ sum rules based on OPE

○ New sum rules are derived

- Hadronic resonances ⇔ Charge fluctuations

<u>Future</u>

- Hadrons in dense matters
  - Nucleon sum rules in finite chemical potential
- Application to other strongly interacting systems
  e.g., ultra-cold atom gases and graphene

Thank you for your kind attention!!