Gluon scattering amplitudes from gauge/string duality and integrability

Yuji Satoh
(University of Tsukuba)

Based on

Y. Hatsuda (DESY), K. Ito (TIT), K. Sakai (YITP) and Y.S.
JHEP 1004(2010)108; 1009(2010)064; 1104(2011)100

Y. Hatsuda (DESY), K. Ito (TIT) and Y.S.
JHEP 1202(2012)003; also to appear

1. Introduction

☆ Discovery of integrability in AdS/CFT

$\Rightarrow$ opened up new dimensions

1. Introduction

☆ Discovery of integrability in AdS/CFT

⇒ opened up new dimensions

- gauge/string duality beyond susy sectors
- quantitative analysis of
  gauge theory dynamics at strong coupling

1. Introduction

☆ Discovery of integrability in AdS/CFT

⇒ opened up new dimensions

- gauge/string duality beyond susy sectors
- quantitative analysis of
  gauge theory dynamics at strong coupling

- stimulating study of SYM
- insights into applications

⋮

In fact,

- spectrum of planar AdS/CFT for arbitrary coupling

  [Gromov-Kazakov-Vieira '09, Bombardelli-Fioravanti-Tateo '09, Arutyunov-Frolov '09, ...]

  - given by thermodynamic Bethe ansatz (TBA) eqs.
  - checked up to 5/6-loops of SYM

    [Bajnok-Hegedus-Janik-Lukowski '09, Arutyunov-Frolov-Suzuki '10 Balog-Hegedus '10, Eden-Heslop-Korchemsky-Smirnov-Sokatchev '12 Bajnok-Janik '12, Laurent-Serban-Volin '12]

In fact,

- spectrum of planar AdS/CFT for arbitrary coupling

  [Gromov-Kazakov-Vieira '09, Bombardelli-Fioravanti-Tateo '09, Arutyunov-Frolov '09, ...]

  - given by thermodynamic Bethe ansatz (TBA) eqs.
  - checked up to 5/6-loops of SYM

    [Bajnok-Hegedus-Janik-Lukowski '09, Arutyunov-Frolov-Suzuki '10 Balog-Hegedus '10, Eden-Heslop-Korchemsky-Smirnov-Sokatchev '12 Bajnok-Janik '12, Laurent-Serban-Volin '12]

Also,

- other aspects ?
  - gluon scattering amplitude/Wilson loops [this talk]
  - correlation fn.
  - quark - anti-quark potential/cusp anomalous dim.

⋮

gluon scatt. amplitudes at strong coupling

$\Uparrow$ [$\Leftarrow$ AdS/CFT]

minimal surfaces in AdS5 x S5

$\Uparrow$ [$\Leftarrow$ integrability]

thermodynamic Bethe ansatz (TBA) equations

In this talk,

- discuss maximally helicity violating (MHV) amplitudes/ Wilson loops of $\mathcal{N}=4$ SYM at strong coupling

  $\Uparrow$ underlying 2D integrable models and CFTs

- derive analytic expansions

  around certain kinematic points

  [regular polygonal Wilson loops]

Plan of talk

1. Introduction

2. Gluon scattering amplitudes at strong coupling

   [amplitude $\longrightarrow$ min. surface]

3. Minimal surfaces in AdS and integrability

   [min. surface $\longrightarrow$ TBA]

4. Analytic expansion of amplitudes at strong coupling

   [AdS_3 case, AdS_4 case]

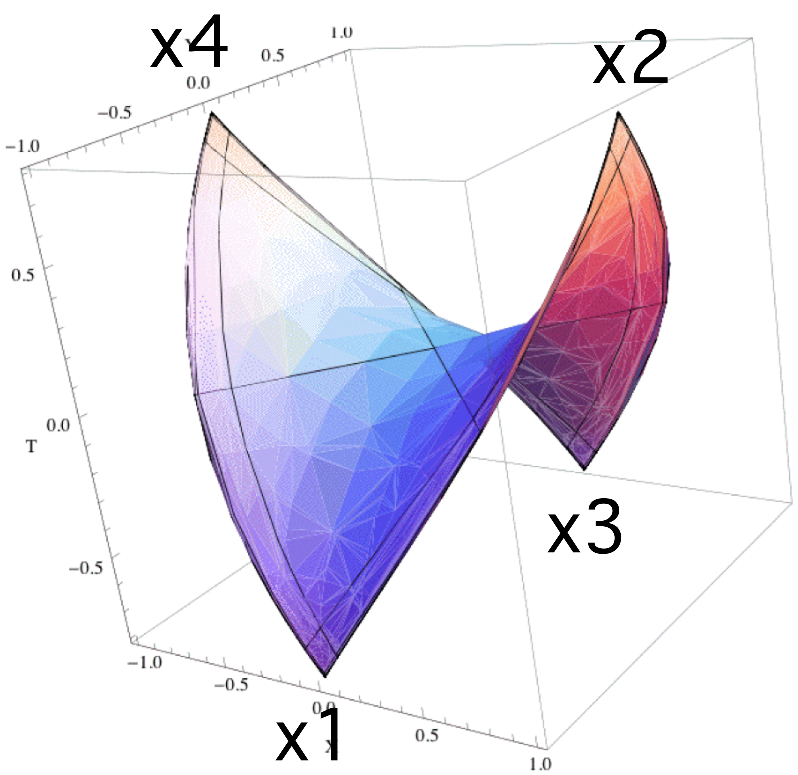
5. Summary

2. Gluon scattering amplitudes at strong coupling

[Alday-Maldacena '07]

amplitudes of $\mathcal{N}=4$ SYM at strong coupling	$\underset{\text{AdS/CFT}}{=}$	minimal surfaces in AdS

$$\mathcal{M} \sim e^{-\frac{\sqrt{\lambda}}{2\pi}(\text{Area})}$$



- $\mathcal{M}$: scalar part of MHV amplitude
- λ : 't Hooft coupling
- null boundary at AdS boundary

$$x^{\mu}_{i+1} - x^{\mu}_{i} = 2\pi k^{\mu}_{i} \quad \text{(momentum of particle)}$$

$\Rightarrow$ n-pt. amplitude $\approx$ n-cusp min. surface

4pt. amplitudes

- precisely matches BDS conjecture [Bern-Dixon-Smirnov '05]

  [all orders in perturbation]

4pt. amplitudes

- precisely matches BDS conjecture [Bern–Dixon–Smirnov '05]

[all orders in perturbation]

Insights into SYM

- amplitudes ≈ min. surfaces ≈ Wilson loops [Drummond–Korchemsky–Sokatchev '07, Brandhuber–Heslop–Travaglini '07]
- Remainder fn. : deviation from BDS formula ($n \geq 6$) [Bern et al. '08, Drummond–Henn–Korchemsky–Sokatchev '08]
- dual conformal symm.

$\Rightarrow$ remainder fn. = fn. of cross–ratios of cusp. coord. x_a^μ

[Drummond–Henn–Smirnov–Sokatchev '06, Drummond–Henn–Korchemsky–Sokatchev '07]

3. Minimal surfaces and integrability

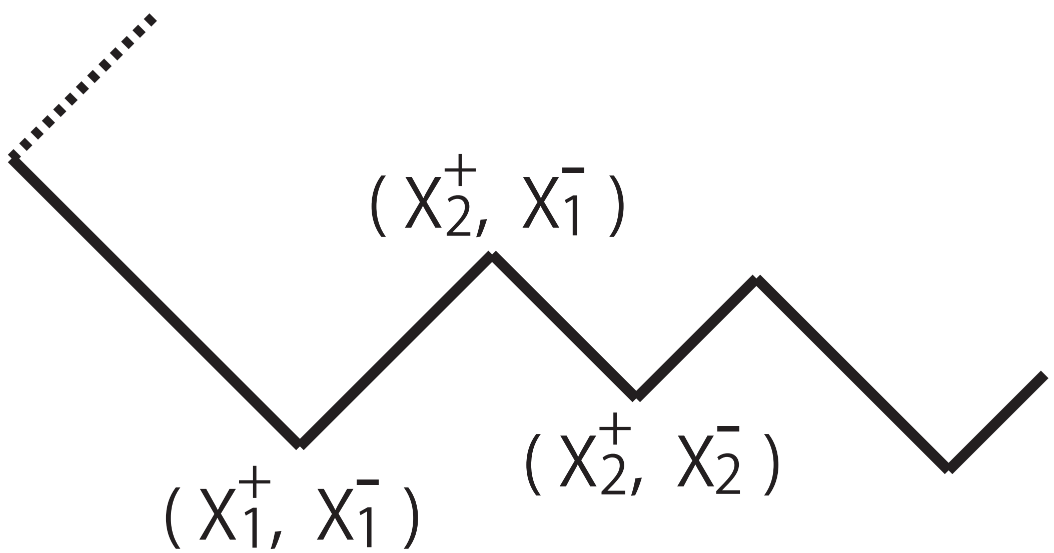
- difficult to construct min. surfaces w/ null bound. for n ≥ 5
  [cf. special 6-cusp sol., Sakai-Satoh '09]

- but possible to obtain A(area) w/o explicit solutions
  [Alday-Maldacena '09]
  - string e.o.m. ⇒ Hitchin system
  - "patching" 4-cusp sol.
  - analyzed min. surface in AdS3 (8pt.)

- amplitudes ⇐ thermodynamic Bethe ansatz (TBA) eq.
  [Alday-Gaiotto-Maldacena '09, Alday-Maldacena-Sever-Vieira '10
  Hatsuda-Ito-Sakai-YS '10]

Let us see this for 2n−cusp min. surface in AdS3 following [Alday−Maldacena−Sever−Vieira '10]

- 4D external momenta in $\mathbb{R}^{1,1}$
- # of cusps : even in AdS3 [mom. conservation]
- 2 light−cone coord. $x^{\pm}$ at ∂(AdS)



TBA eq. for AdS3 min. surface

[Alday–Maldacena–Sever–Vieira ’10, also, Hatsuda–Ito–Sakai–YS ’10]

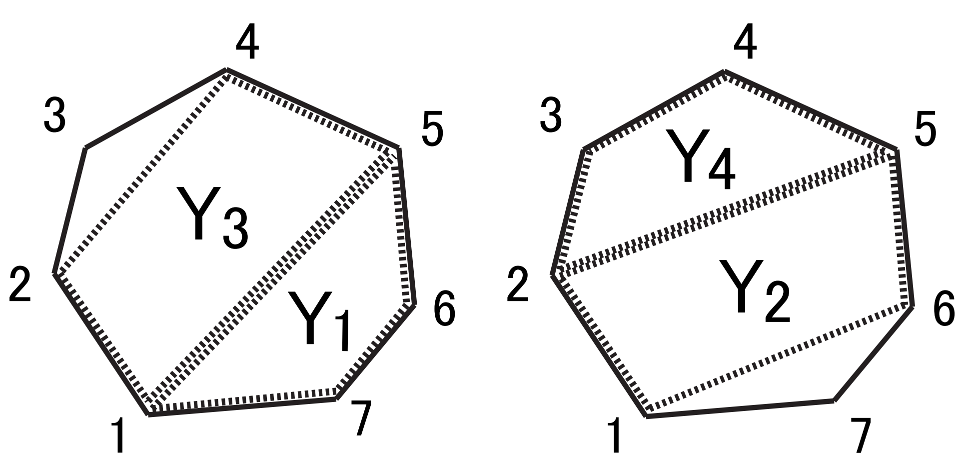
- to compute amplitudes (2n–pt.), first need to solve

$$\log Y_s(\theta) = -m_s \cosh\theta + \sum_r K_{sr} * \log(1+Y_r)$$

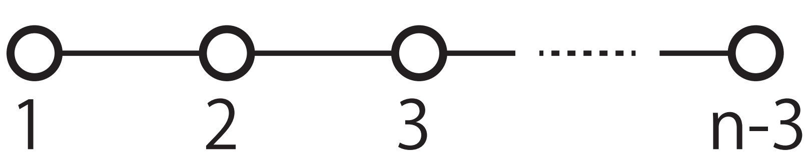
$$Y_0 = Y_{n-2} = 0 \qquad [s = 1, \ldots, n-3; \text{ real } m_s]$$

⇒ TBA eq. of hom. sine–Gordon model

[Hatsuda–Ito–Sakai–YS ’10, cf. Alday–Gaiotto–Maldacena ’09]



n = 7



- θ : spectral parameter
- Ys : (extended) cross–ratios of $x_a^{\pm}$

  e.g.) $Y_1(-\frac{\pi i}{2}) = \frac{x_{15}^+ x_{67}^+}{x_{56}^+ x_{17}^+}, \quad Y_1(0) = \frac{x_{15}^- x_{67}^-}{x_{56}^- x_{17}^-}$
- m_s : complex (mass) param.

  ≈ shape of surface ⇔ momenta
- $K_{sr} = I_{sr}/\cosh\theta$
- I_{sr} : incidence matrix for A_{n-3}

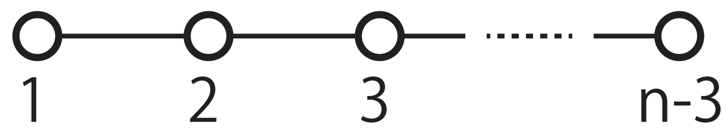
Y-system

- using asymptotics, analyticity of Y-fn.

  TBA eqs. are transformed into algebraic eqs.

$$Y_s^+ Y_s^- = (1+Y_{s+1})(1+Y_{s-1})$$

$$\left[\, Y_s^{\pm}(\theta) = Y_s\left(\theta \pm \frac{\pi i}{2}\right) \,\right]$$



T-system

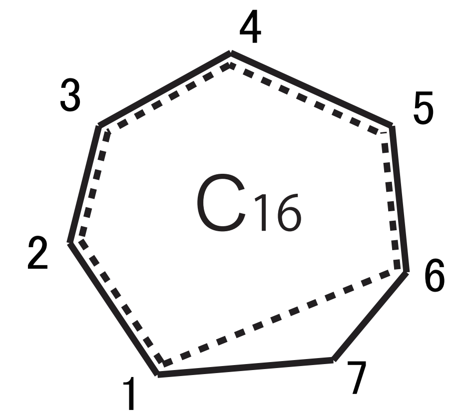
- Y-system is obtained from T-system for T-fn.

$$T_s^+ T_s^- = 1 + T_{s+1} T_{s-1}$$

$$\left[\, Y_s = T_{s+1} T_{s-1} \,\right]$$

Remainder function [overall coupling dependence : omitted]

- Once Y-fn. are obtained [n odd]



$$R_{2n} := A \text{ [amp.]} - A_{BDS} \text{ [BDS formula]}$$

$$= \frac{7\pi}{12}(n-2) + A_{\text{periods}} + \Delta A_{BDS} + A_{\text{free}}$$

$$c_{16} = \frac{x_{23}\, x_{45}\, x_{16}}{x_{12}\, x_{34}\, x_{56}}$$

$$A_{\text{periods}} = -\frac{1}{4} m_r\, I^{-1}_{rs}\, \overline{m}_s$$

$$\Delta A_{BDS} = \frac{1}{4}\sum_{i,j=1}^{n} \log\frac{c^+_{i,j}}{c^+_{i,j+1}} \log\frac{c^-_{i-1,j}}{c^-_{i,j}} \ , \quad c^\pm_{i,j} = \frac{x^\pm_{i+2,i+1} x^\pm_{i+4,i+3} \cdots x^\pm_{j,i}}{x^\pm_{i+1,i} x^\pm_{i+3,i+2} \cdots x^\pm_{j,j-1}}$$

$$A_{\text{free}} = \sum_{s=1}^{n-3} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m_s \cosh\theta \log(1 + Y_s(\theta))$$

free energy of TBA system

$\Rightarrow$ non-trivial part : A_{free} , ΔA_{BDS} for given ms

- exact solutions $\begin{cases} m_s \to 0 & \text{: CFT limit} \\ m_s \to \infty \end{cases}$ regular polygon

- TBA system : solved numerically

- exact solutions $\begin{cases} m_s \to 0 & : \text{CFT limit} \\ m_s \to \infty \end{cases}$ regular polygon
- TBA system : solved numerically
  [but sometimes simple iteration fails]

- solutions to TBA system : not fully investigated
- momentum dependence : m_s ↔ momentum
- progress in analytic results at 2 loops

- exact solutions $\begin{cases} m_s \to 0 & : \text{CFT limit} \\ m_s \to \infty \end{cases}$ regular polygon
- TBA system : solved numerically
  [but sometimes simple iteration fails]

○ solutions to TBA system : not fully investigated

○ momentum dependence : ms ↔ momentum

○ progress in analytic results at 2 loops

Any analytic results at strong coupling except $m_s \to 0\ ,\ \infty$??

⇒ we discuss expansions around CFT lim.

$$l = ML \to 0 \quad (\, m_s = M_s L = \tilde{M}_s M L \,)$$

[M: mass scale, L: length scale]

4. Analytic expansion at strong coupling

[Hatsuda–Ito–Sakai–YS '11, Hatsuda–Ito–YS '11]

- basis of the expansion :

  - TBA for AdS3 2n–pt. amplitude $\Leftarrow$ HSG from $\dfrac{\widehat{\mathrm{su}}(n-2)_2}{[\widehat{\mathrm{u}}(1)]^{n-3}}$

    [w/ imaginary resonance param.]

    [Hatsuda–Ito–Sakai–YS '10]

  - relation between T–fn. and g–function (boundary entropy)

    [Bazhanov–Lukyanov–Zamolodchikov '94;
    Dorey–Runkel–Tateo–Watts '99; Dorey–Lishman–Rim–Tateo '05]

- set ms to be real to keep bd. integrability [Yang–Baxter eq.]

  $\Rightarrow$ recovered after expansion
  so as to maintain $\mathbb{Z}_{2n}$- symm. : $x_i \longrightarrow x_{i+1}$

Homogenous sine-Gordon (HSG) model

[Fernandez Pousa-Gallas-Hollowood-Miramontes '96]

- start w/ coset CFT $\widehat{\mathrm{su}}(N)_k/[\widehat{\mathrm{u}}(1)]^{N-1}$

  [for 2n-cusp AdS3 min. surface N = n-2, k=2]

- HSG model : integrable deformation of coset CFT

$$S_{HSG} = S_{\mathrm{gWZNW}} + \beta \int d^2x \ \Phi$$

Φ : comb. of weight 0 adjoint op. [multi-param. deformation]

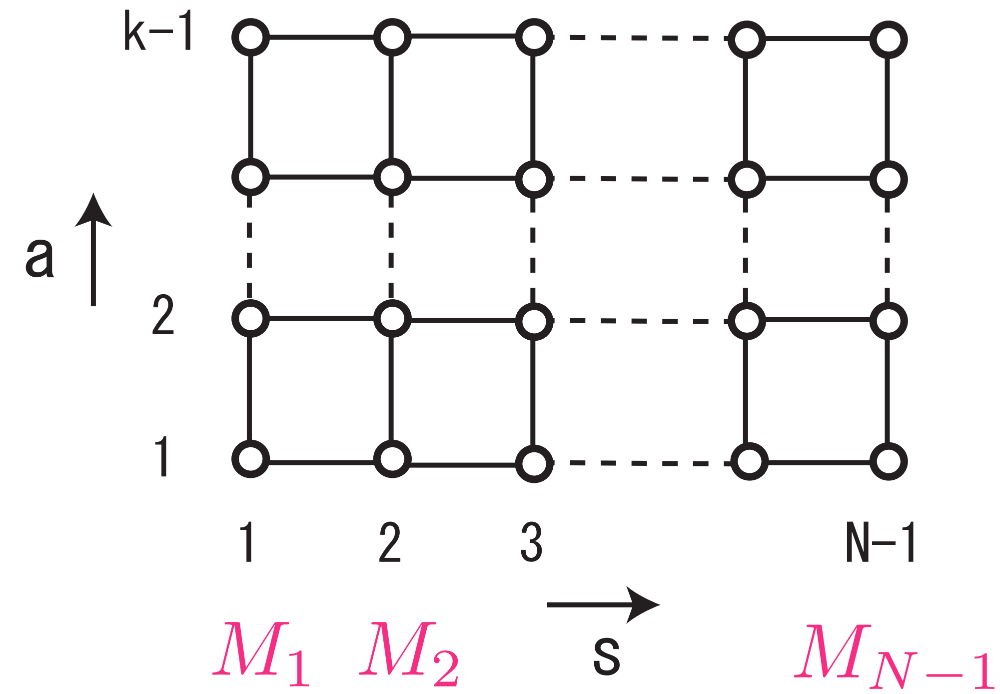
$$\Delta = \bar{\Delta} = \frac{N}{N+k} , \qquad \beta = -\kappa\, M^{2(1-\Delta)}$$

- simplest case from $\widehat{\mathrm{su}}(2)_k/\widehat{\mathrm{u}}(1)$

  $\Rightarrow$ complex sine-Gordon/minimal A_{k-1} affine Toda

- spectrum [copies of min. affine Toda]

$$M_s^a = \frac{\sin \frac{\pi a}{k}}{\sin \frac{\pi}{k}} M_s \ , \qquad M_s = M \tilde{M}_s$$



min. surface ← 2D integrable model ← CFT

Expansion of A_{free} : $[\, R_{2n} \sim A_{\text{free}} + \Delta A_{BDS} \,]$

- free energy around CFT limit $\Leftarrow$ CFT perturbation

- for the case of 2n–cusp min. surface in AdS3

$$A_{\text{free}} = \frac{\pi}{6} c_n + f_n^{\text{bulk}} + \sum_{k=2}^{\infty} f_n^{(k)} \, l^{\frac{4k}{n}}$$

$$c_n = \frac{(n-2)(n-5)}{n} \quad \text{[central charge]}$$

$$f_n^{\text{bulk}} = \frac{1}{4} m_r \, I_{rs}^{-1} \, m_s$$

$f_n^{(k)} \Leftarrow$ k-pt. fn.of Φ

$\Delta A_{BDS} \quad (\leftarrow \; c^{\pm}_{ij} \;)$

- can show cross-ratios $c^{\pm}_{ij}$: nothing but T-fn.

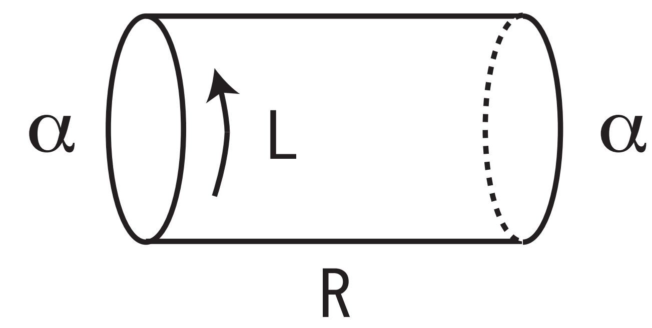
$$c^{+}_{ij} = T^{[i+j]}_{|i-j|-1}(0), \qquad c^{-}_{ij} = T^{[i+j+1]}_{|i-j|-1}(0)$$

where $T^{[k]}_{s}(\theta) := T_s\left(\theta + \frac{\pi i}{2} k\right)$

$$\Rightarrow \qquad \Delta A_{BDS} = \frac{1}{4} \sum_{i,j=1}^{n} \log \frac{T^{[i+j]}_{|i-j|-1}}{T^{[i+j+1]}_{|i-j-1|-1}} \log \frac{T^{[i+j]}_{|i-j-1|-1}}{T^{[i+j+1]}_{|i-j|-1}}$$

everything fits into language of 2D integrable model

T–function ⇐ g–function (boundary entropy)



- g–fn. ≈ $\log \mathcal{G}^{(0)}_{|\alpha\rangle}$

$$Z_{\langle\alpha|\alpha\rangle} = \langle\alpha|e^{-RH}|\alpha\rangle = \sum_{k=0}^{\infty}\left(\mathcal{G}^{(k)}_{|\alpha\rangle}(l)\right)^2 e^{-RE_k}$$

[counts ground state degeneracy]

- integral eq. for g–fn. is known :

  similar to TBA eq. , including boundary contributions

  [Dorey–Lishman–Rim–Tateo ’05; Poszgay ’10; Woynarovich ’10]

- comparing this w/ TBA eq. following

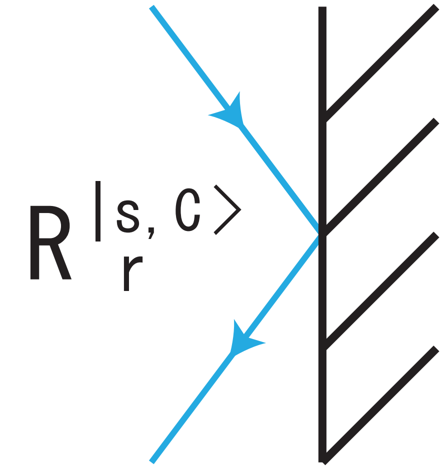
  [Dorey–Runkel–Tateo–Watts ’99; Dorey–Lishman–Rim–Tateo ’05]

$$\Rightarrow \quad \boxed{\mathcal{G}^{(0)}_{|s,C\rangle}\Big/\mathcal{G}^{(0)}_{|1\rangle} = T_s\left(\frac{i\pi}{2}C\right)}$$

- boundaries ≈ reflection factors

  $|1\rangle$: trivial boundary

  $|s, C\rangle$: $\Leftarrow$ $R_r^{|s,C\rangle}(\theta) = R_r^{|1\rangle}(\theta) / Z_r^{|s,C\rangle}(\theta)$



[deforming factor]   [Sasaki ’93]

- need to find $Z_r^{|s,C\rangle}$
  - satisfying boundary bootstrap, unitarity, crossing symm.
  - corresponding precisely to T_s

$$\Rightarrow \qquad Z_r^{|s,C\rangle}(\theta) := \Big((1+C)_\theta (1-C)_\theta\Big)^{\delta_{sr}}$$

$$(x)_\theta := \frac{\sinh \frac{1}{2}(\theta + i\frac{\pi}{2}x)}{\sinh \frac{1}{2}(\theta - i\frac{\pi}{2}x)}$$

12年10月17日水曜日

Expansion of T-function

- periodicity $\Leftarrow$ T-system

$$\Rightarrow \quad T_s(\theta) = \sum_{p,q=0}^{\infty} t_s^{(p,q)} l^{(1-\Delta)(p+q)} \cosh\left(\frac{2p\theta}{n}\right)$$

- boundary CFT perturbation for g-fn. [Dorey-Runkel-Tateo-Watts '99; Dorey-Lishman-Rim-Tateo '05]

$$\Rightarrow \frac{t_s^{(2,0)}}{t_s^{(0,0)}} = -\frac{\kappa_n G(\tilde{M}_j) B(1-2\Delta,\Delta)}{2(2\pi)^{1-2\Delta}} \left(\frac{\sin(\frac{3(s+1)\pi}{n})}{\sin(\frac{(s+1)\pi}{n})} \sqrt{\frac{\sin(\frac{\pi}{n})}{\sin(\frac{3\pi}{n})}} - \sqrt{\frac{\sin(\frac{3\pi}{n})}{\sin(\frac{\pi}{n})}} \right)$$

$$t_s^{(0,0)} = \frac{\sin\left(\frac{(s+1)\pi}{n}\right)}{\sin(\frac{\pi}{n})} \quad , \quad \langle \Phi(z)\Phi(0) \rangle = \frac{G^2(\tilde{M}_s)}{|z|^{4\Delta}}$$

[given by modular S-matrix]

- $t_s^{(2,0)},\ t_s^{(0,4)} \quad \Leftarrow \quad t_1^{(2,0)} \quad \propto \quad \kappa_n G(\tilde{M}_s)$

T-system

Expansion of 2n-pt. remiander function

Combining all,

$$R_{2n} = R_{2n}^{(0)} + l^{\frac{8}{n}} R_{2n}^{(4)} + \mathcal{O}(l^{\frac{12}{n}})$$

$\Downarrow$ regular polygon

$$R_{2n}^{(0)} = \frac{\pi}{4n}(n-2)(3n-2) - \frac{n}{2}\sum_{s=1}^{(n-3)/2} \log^2\left(\frac{\sin(\frac{(s+1)\pi}{n})}{\sin(\frac{s\pi}{n})}\right)$$

$$R_{2n}^{(4)} = \frac{\pi}{6} C_n^{(2)} \kappa_n^2 G^2(\tilde{M}_j) - \frac{n}{4}\left[\sum_{s=1}^{(n-3)/2} A_{n,s} - 2\left(\frac{t_{(n-3)/2}^{(2,0)}}{t_{(n-3)/2}^{(0,0)}}\right)^2 \sin^2\left(\frac{\pi}{n}\right)\right]$$

$$\begin{aligned} A_{n,s} := & \left[\left(\frac{t_{s-1}^{(2,0)}}{t_{s-1}^{(0,0)}}\right)^2 + \left(\frac{t_s^{(2,0)}}{t_s^{(0,0)}}\right)^2\right]\cos\left(\frac{2\pi}{n}\right) - \frac{2t_{s-1}^{(2,0)} t_s^{(2,0)}}{t_{s-1}^{(0,0)} t_s^{(0,0)}} \\ & + \left[\left(\frac{t_{s-1}^{(2,0)}}{t_{s-1}^{(0,0)}}\right)^2 - \left(\frac{t_s^{(2,0)}}{t_s^{(0,0)}}\right)^2 - 4\left(\frac{t_{s-1}^{(0,4)}}{t_{s-1}^{(0,0)}} - \frac{t_s^{(0,4)}}{t_s^{(0,0)}}\right)\right]\log\left(\frac{t_s^{(0,0)}}{t_{s-1}^{(0,0)}}\right) \end{aligned}$$

$$C_n^{(2)} = 3(2\pi)^{\frac{2(n-4)}{n}} \gamma^2\left(\frac{n-2}{n}\right)\gamma\left(\frac{4-n}{n}\right) \ , \ \gamma(x) = \Gamma(x)/\Gamma(1-x)$$

- fn. of $t_1^{(2,0)} \propto \kappa_n G(\tilde{M}_s)$

To further express amplitudes by momenta

- need
  - $\kappa_n G\,, \kappa_n \Phi \;\Leftrightarrow\; m_s$ [mass-coupling/relevant op. relation]
  - invert relation btw Y-fn. (cross-ratio) and m_s

    $$\Rightarrow \; m_s = m_s(k_a) \qquad [k_a \;:\; \text{momenta}]$$

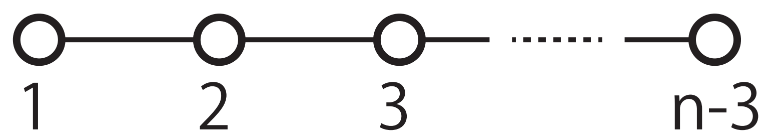
  - recover phases of m_s

To further express amplitudes by momenta

- need
  - $\kappa_n G \, , \kappa_n \Phi \;\Leftrightarrow\; m_s$ [mass-coupling/relevant op. relation]
  - invert relation btw Y-fn. (cross-ratio) and m_s

    $$\Rightarrow \; m_s = m_s(k_a) \qquad [k_a \; : \text{ momenta}]$$

  - recover phases of m_s



- this can be done for single mass cases

  [Zamolodchikov '95; Fateev '94]

  e.g.,
  - $$M_s = \delta_{s,1} M \;\Rightarrow\; \frac{\widehat{su}(2)_1 \oplus \widehat{su}(2)_{n-3}}{\widehat{su}(2)_{n-2}} = \mathcal{M}_{n-1,n}$$
  - $$M_s = (\delta_{s,1} + \delta_{s,n-3}) M$$ [minimal models]

    $$\Rightarrow \frac{\widehat{su}(2)_1 \oplus \widehat{su}(2)_{n/2-2}}{\widehat{su}(2)_{n/2-1}} = \mathcal{M}_{n-2,n}$$

⇒ amplitudes along certain trajectories in momentum space

8-pt. remainder function [simplest for AdS3]   A1 ○

- integral representation of 8-pt. remainder fn.

[Alday-Maldacena '09]

- HSG model ⇒ Ising model
  ⇒ all order expansion in l^2

$$R_8 = \sum_{k=0}^{\infty} R_8^{(2k)}(\varphi) l^{2k} \qquad [\, l = ML, \ m = l\, e^{i\varphi} \,]$$

$$R_8^{(0)}(\varphi) = \frac{5\pi}{4} - \frac{\log^2 2}{2}$$

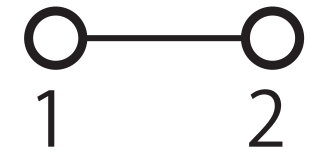
$$R_8^{(2)}(\varphi) = \frac{1}{8\pi} - \frac{\log 2}{16}$$

$$R_8^{(4)}(\varphi) = \frac{2\log 2 - 1}{1024} + \left(\frac{2\log 2 + 3}{3072} - \frac{7\zeta(3)}{192\pi^3} \right) \cos 4\varphi$$

$$\vdots$$

[Hatsuda-Ito-Sakai-YS '10]

10 pt. remainder function



- For 10–pt., these completely fix $\kappa_n G \Leftrightarrow m_s$

$$R_{10} = R_{10}^{(0)} + R_{10}^{(4)} \cdot l^{8/5} + \mathcal{O}(l^{12/5}) \quad [\, l = ML,\ m_s = \tilde{M}_s l \,]$$

complex

$$R_{10}^{(0)} = \frac{39}{20}\pi - \frac{5}{2}\log^2\left(2\cos\frac{\pi}{5}\right)$$

$\Leftarrow$ regular polygon

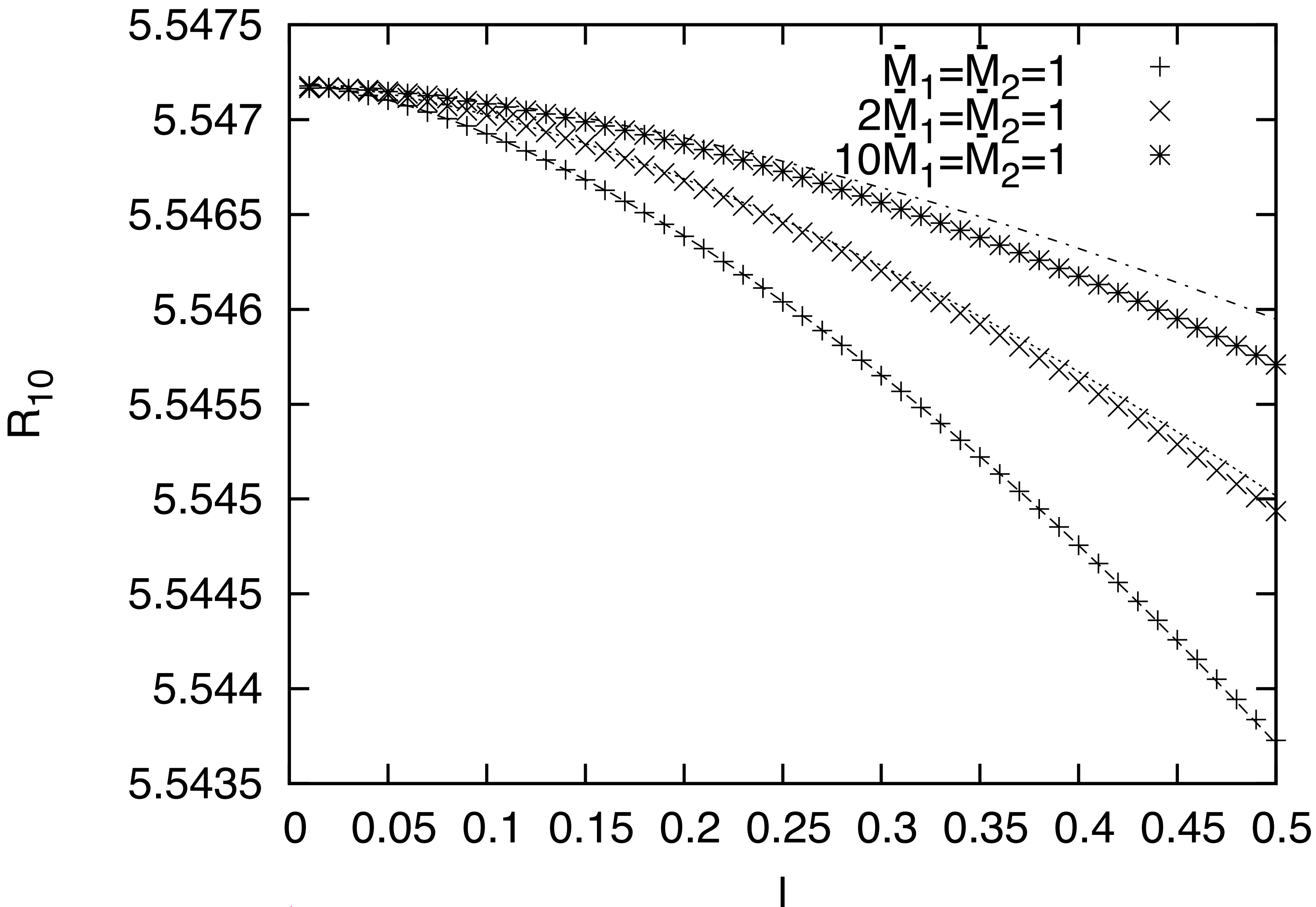
$$R_{10}^{(4)} = \left(-\frac{1}{5}\tan\frac{\pi}{5} + C_1\right) \cdot \left|t_1^{(2,0)}\right|^2$$

$$t_1^{(2,0)} = C_2\left(\tilde{M}_1^{4/5} + \tilde{M}_2^{4/5} - C_3 \tilde{M}_1^{2/5}\tilde{M}_2^{2/5}\right)$$

$$C_1 = 20\cos^4\left(\frac{2\pi}{5}\right)\left(1 - 5^{-1/2}\log\left(2\cos\frac{\pi}{5}\right)\right)$$

$$C_2 = \frac{1}{4 \cdot 6^{1/5}}\Gamma(-1/5)\left[10\cos\frac{\pi}{5}\gamma(3/5)\gamma(4/5)\right]^{1/2}$$

$$C_3 = 2 - \left(\frac{3}{\pi^2}\right)^{1/5}\gamma(1/4)^{4/5}$$



($\varphi_1 = \pi/20,\ \varphi_2 = \pi/5$)

⇑ CFT lim.

l- dependence of 10–pt. remainder function $[\, m_s = \tilde{M}_s e^{i\varphi_s} l \,]$

- dashed lines : $R_{10}^{(0)} + R_{10}^{(4)} l^{8/5}$
- good agreement w/ numerics

5. AdS4 minimal surfaces and W minimal models

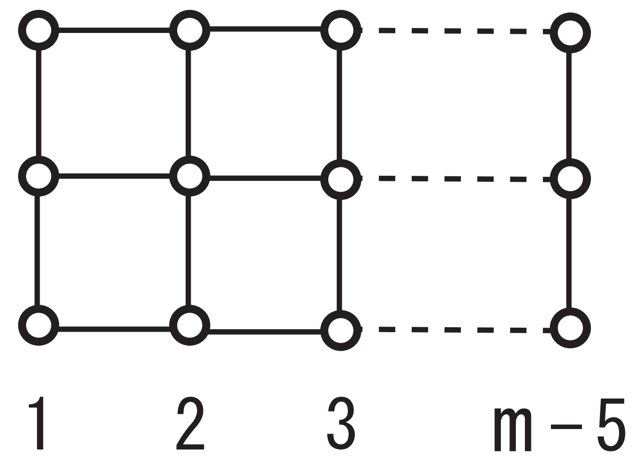
[Hatsuda-Ito-YS, to appear]

- Y-system for AdS4 m-cusp sol. [Alday-Maldacena-Sever-Vieira '10]

$$\frac{Y^-_{2,s}Y^+_{2,s}}{Y_{1,s}Y_{3,s}} = \frac{(1+Y_{2,s+1})(1+Y_{2,s-1})}{(1+Y_{1,s})(1+Y_{3,s})}$$

$$\frac{Y^-_{3,s}Y^+_{1,s}}{Y_{2,s}} = \frac{(1+Y_{3,s+1})(1+Y_{1,s-1})}{1+Y_{2,s}}$$

$$\frac{Y^-_{1,s}Y^+_{3,s}}{Y_{2,s}} = \frac{(1+Y_{1,s+1})(1+Y_{3,s-1})}{1+Y_{2,s}}$$



$$Y_{1,s} = Y_{3,s} \qquad \left[Y^{\pm}_{a,s}(\theta) = Y_{a,s}(\theta \pm \frac{\pi}{4} i), \quad s = 1, ..., m-5 \right]$$

- TBA for AdS4 m-pt. amplitude $\Leftarrow$ HSG from $\dfrac{\widehat{\mathrm{su}}(m-4)_4}{[\widehat{\mathrm{u}}(1)]^{m-5}}$

12年10月17日水曜日

- simple iteration for numerics

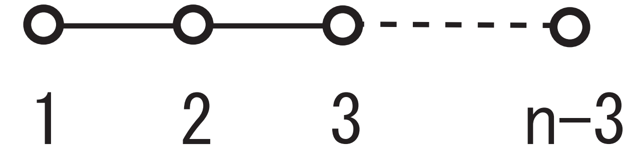
  near CFT lim. does not seem to work, generally

  [cf. Castro Alvaredo-Fring '04]

- analytic expansion near CFT lim. : possible

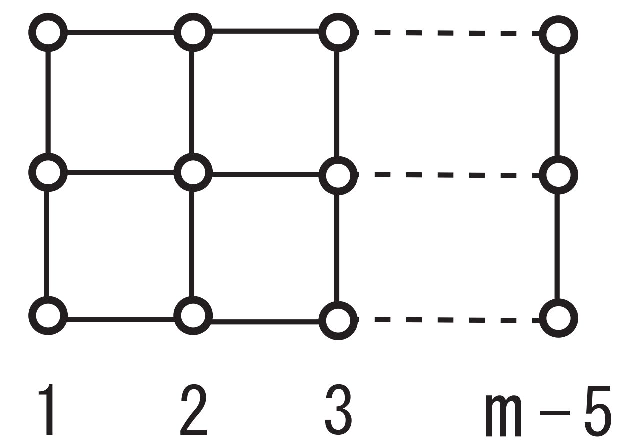
single mass cases

- AdS3 case: su(2) (non−)unitary cosets



$\Downarrow$

AdS4 case: su(4) (non−)unitary cosets



- in particular,
  - $M_s = \delta_{s,1} M \;\Rightarrow\; \dfrac{\widehat{\mathrm{su}}(4)_1 \oplus \widehat{\mathrm{su}}(4)_{m-5}}{\widehat{\mathrm{su}}(4)_{m-4}} = WA_3^{(m-1,m)}$
  - $M_s = (\delta_{s,1} + \delta_{s,m-5}) M$ [W−minimal models]

$$\Rightarrow \frac{\widehat{\mathrm{su}}(4)_1 \oplus \widehat{\mathrm{su}}(4)_{m/2-5}}{\widehat{\mathrm{su}}(4)_{m/2-4}} = WA_3^{(m-2,m)}$$

6-pt. remainder function [simplest for AdS4/5]

[cf. Alday-Gaiotto-Maldacena '09]

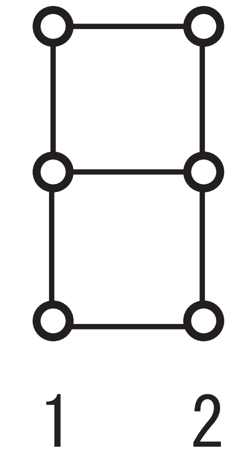
- HSG model

  ⇒ Z4 parafermion CFT

  deformed by one relevant op. [no mixing]

- expansion near CFT limit agrees w/ partially numerical results in [Hatsuda-Ito-Sakai-YS '11]

7-pt. remainder function



- similarly to 10-pt. for AdS3,
  mass-coupling relation is completely determined

$$R_7 = R_7^{(0)} + R_7^{(4)} \cdot l^{16/7} + \mathcal{O}(l^{24/7}) \qquad [\, l = ML, \; m_s = \tilde{M}_s l \,]$$

$$R_7^{(0)} = \frac{3}{7}\pi - \frac{7}{4}\left[\log^2\left(2\cos\frac{\pi}{7}+1\right) + \mathrm{Li}_2\left(\frac{2\cos\frac{\pi}{7}}{2\cos\frac{\pi}{7}+1}\right)\right]$$

$$R_7^{(4)} = (C_1 + C_2^2)\cdot \left|\kappa G\right|^2$$

$$\kappa G = \tilde{M}_1^{8/7} + \tilde{M}_2^{8/7} + C_3 \tilde{M}_1^{4/7}\tilde{M}_2^{4/7}$$

$$C_1 = \frac{\pi}{2(2\pi)^{2/7}}\gamma^2(3/7)\gamma(1/7)$$

$$C_2 = \frac{B(1/7,3/7)}{2(2\pi)^{1/7}}\frac{\sin\frac{3\pi}{7}}{\sin\frac{3\pi}{7}}\left(\sqrt{\frac{\sin\frac{\pi}{14}}{\sin\frac{5\pi}{14}}} - \sqrt{\frac{\sin\frac{5\pi}{14}}{\sin\frac{\pi}{14}}}\right)$$

$$C_3 = \left(\frac{2^{11}\pi^8}{3^2}\right)^{1/14}\left[\gamma\left(\frac{4}{7}\right)\gamma\left(\frac{6}{7}\right)\gamma\left(\frac{1}{14}\right)\right]^{1/2}\left[\frac{\Gamma(\frac{7}{8})}{\Gamma(\frac{3}{4})\Gamma(\frac{5}{8})}\right]^{8/7} - 2$$

AdS5 case

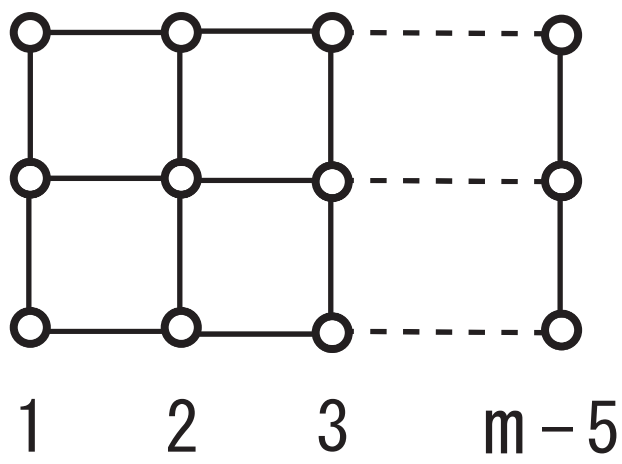
[Alday-Maldacena-Sever-Vieira ' 10]

- Y-system for AdS5 m-cusp sol.

$$\frac{Y_{2,s}^{-}Y_{2,s}^{+}}{Y_{1,s}Y_{3,s}} = \frac{(1+Y_{2,s+1})(1+Y_{2,s-1})}{(1+Y_{1,s})(1+Y_{3,s})}$$

$$\frac{Y_{3,s}^{-}Y_{1,s}^{+}}{Y_{2,s}} = \frac{(1+Y_{3,s+1})(1+Y_{1,s-1})}{1+Y_{2,s}}$$

$$\frac{Y_{1,s}^{-}Y_{3,s}^{+}}{Y_{2,s}} = \frac{(1+Y_{1,s+1})(1+Y_{3,s-1})}{1+Y_{2,s}}$$



$$Y_{1,s} \neq Y_{3,s}$$

- $Y_{1,s}$ couples to $Y_{3,s}$ on l.h.s. ["non-standard" Y-system]

- underlying integrable model: not identified yet

6. Comparison with 2-loop results

AdS3 case

- analytic results at 2 loops: given for AdS3 amplitudes

[Heslop-Khoze' 10 ; Gaiotto-Maldacena-Sever-Vieira ' 11]

- for the same cross-ratios, have similar expansion

$$R_{2n}^{\text{2-loop}} = R_{2n}^{\text{2-loop}\,(0)} + l^{\frac{8}{n}} R_{2n}^{\text{2-loop}\,(4)} + \mathcal{O}(l^{\frac{12}{n}})$$

rescaled remainder fn.

[Brandhuber-Heslop-Khoze-Travaglini ' 09]

- for comparison, useful to define

$$\overline{R}_{2n} := \frac{R_{2n} - R_{2n,\text{UV}}}{R_{2n,\text{UV}} - (n-2)R_6}$$

  - normalized s.t. $\overline{R}_{2n} \to 0 \quad (l \to 0)$, $\to -1 \quad (l \to \infty)$

- dependence on m_s (momentum) :

  encoded in $\kappa_n G(\tilde{M}_s)$ at leading order

$\Rightarrow$ ratio of $\overline{R}_{2n}$: number

2n	8	10	12	14	16	18	$\cdots$
$\bar{R}_{2n}^{\text{strong}}/\bar{R}_{2n}^{\text{2-loop}}$	1.026	0.984	0.961	0.946	0.937	0.930	$\cdots$

$$\Rightarrow \; 0.905 - 0.118/n \quad (n \gg 1)$$

close to 1 [not 1000, 0.0001 ...]

any mechanism ?

AdS4 case

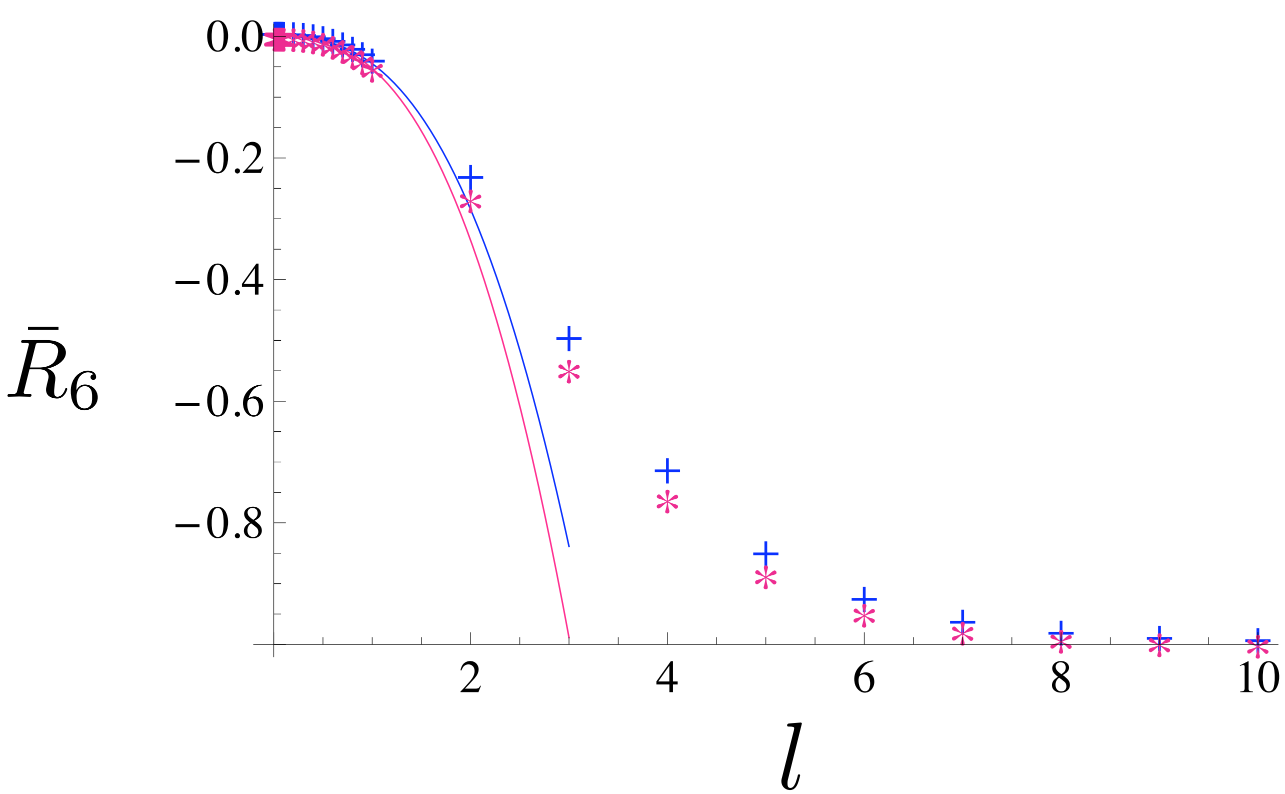
- analytic results of 6-pt remainder fn. at 2 loops :
  read from those for AdS5 [Del Duca-Duhr-Smirnov '10, Zhang '10, Goncharov-Spradlin-Vergu-Volovich '10]
- can compare expansions around CFT lim.,
  rescaled remainder fn.

AdS4 case

- analytic results of 6–pt remainder fn. at 2 loops :
  read from those for AdS5 [Del Duca–Duhr–Smirnov '10, Zhang '10, Goncharov–Spradlin–Vergu–Volovich '10]

- can compare expansions around CFT lim., rescaled remainder fn.

$$\Rightarrow \quad \bar{R}_6^{\text{strong}} / \bar{R}_6^{\text{2-loop}} \approx 1.179 + \mathcal{O}(l^4)$$



numerics for real $m = l$

+ : 2 loops

* : strong coupling

leading analytic expansion

— : 2 loops

— : strong coupling

7. Summary

- Gluon scatt. amplitudes of $\mathcal{N}=4$ SYM at strong coupling

  $\Uparrow$ minimal surfaces in AdS [AdS/CFT]

  $\Uparrow$ TBA eq. [integrability]

6. Summary

- Gluon scatt. amplitudes of $\mathcal{N}=4$ SYM at strong coupling

  $\Uparrow$ minimal surfaces in AdS [AdS/CFT]

  $\Uparrow$ TBA equations [integrability]

- Underlying 2D integrable (HSG) model

  $\Rightarrow$ analytic expansions of amplitudes/Wilson loops around CFT lim. [regular polygon]

  - A_{free} $\Leftarrow$ bulk CFT perturbation
  - ΔA_{BDS} $\Leftarrow$ T-function
  - T-/Y-fn. $\Leftarrow$ g-fn. $\Leftarrow$ boundary CFT perturbation

- Why minimal surface ⇔ TBA ? [cf. ODE/IM]

  T-function ⇔ g-function (boundary entropy) ?

- Why strong coupling ⇔ 2 loops : close

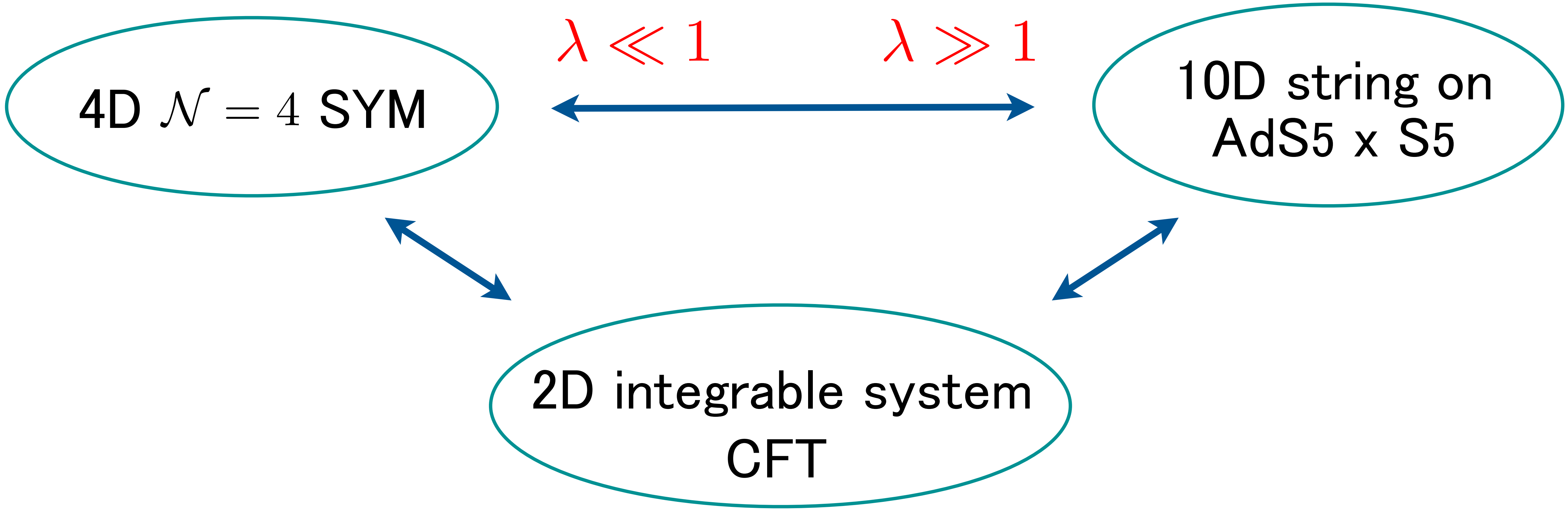
  [full structure of amplitudes]

- General AdS5 case ?

- Strong coupling corrections ?

  [cf. Alday-Gaiotto-Maldacena-Sever-Vieira '10]

⋮



[λ : 't Hooft coupling]