Baryonic popcorn: holographic nuclear interactions, nuclear matter and muti-instanton chains

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Introduction

- In recent years holography or gauge/gravity duality has provided a new tool to handle strong coupling problems.
- It has been spectacularly successful at explaining certain features of the quark-gluon plasma such as its low viscosity/entropy density ratio.
- An insightful picture, though not complete , has been developed for glueballs ,and mesons spectra.
- This naturally raises the question of whether one can apply holography to baryons and the "Strong interaction" namely to nuclear interactions and nuclear matter.

Questions to investigate in nuclear holography

- Is the large Nc and large λ world similar to reality
- Static properties of baryons
- Nuclear interactions
- The nuclear binding energy puzzle
- Nuclear matter at zero and finite temperature
- The structure of the QCD phase diagram

Nuclear binding energy puzzle

- The interactions between nucleons are strong so why is the nuclear binding non-relativistic, about 1.7% of M c^2 namely 16 Mev per nucleon.
- The usual explanation of this puzzle involves a near-cancellation between the attractive and the repulsive nuclear forces. [Walecka]
- Attractive due to σ exchange -400 Mev
- Repulsive due to ω exchange + 350 Mev
- Fermion motion + 35 Mev

Net binding per nucleon - 15 Mev

Outline

- Stringy holographic baryons
- The laboratory: the generalized Sakai Sugimoto model
- Baryons as flavor gauge instantons
- A brief review of static properties of Baryons
- Nuclear interaction: repulsion and attraction
- The DKS model and the binding energy puzzle

Outline

- Chains of baryons-generalities
- The 1d and 3d toy models of point charges.
- Exact ADHM 1d chain of instantons
- The two instanton approximation
- Phase transitions between lattice structures
- The phase diagram of QCD at large Nc
- Summary and open questions

Stringy holographic Baryons

Stringy Baryons in hologrphy

- How do we identify a baryon in holography ?

- Since a quark corresponds to a string, the baryon has to be a structure with N_c strings connected to it.

- Witten proposed a baryonic vertex in AdS_5xS^5 in the form of a wrapped D5 brane over the S^5.

- On the world volume of the wrapped D5 brane there is a CS term of the form

$$\text{Scs}= \int_{S^5\times R} a\wedge\frac{G_5}{2\pi}.$$

Baryonic vertex

- The flux of the five form is

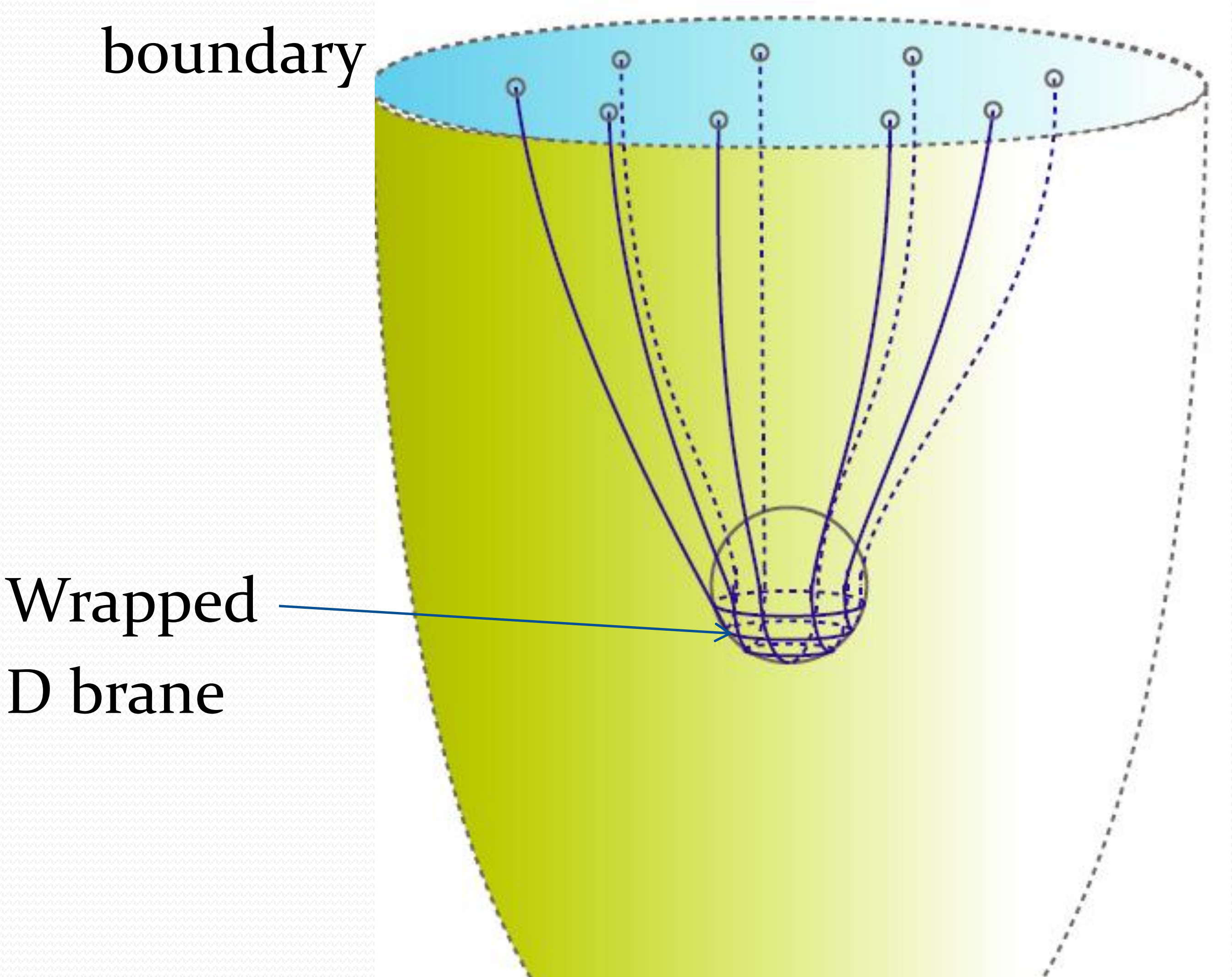
$$\boxed{\int_{\mathbf{S}^5} \frac{G_5}{2\pi} = N_c}$$

- This implies that there is a charge N_c for the abelian gauge field. Since in a compact space one cannot have non-balanced charges there must be

N_c strings attached to it.

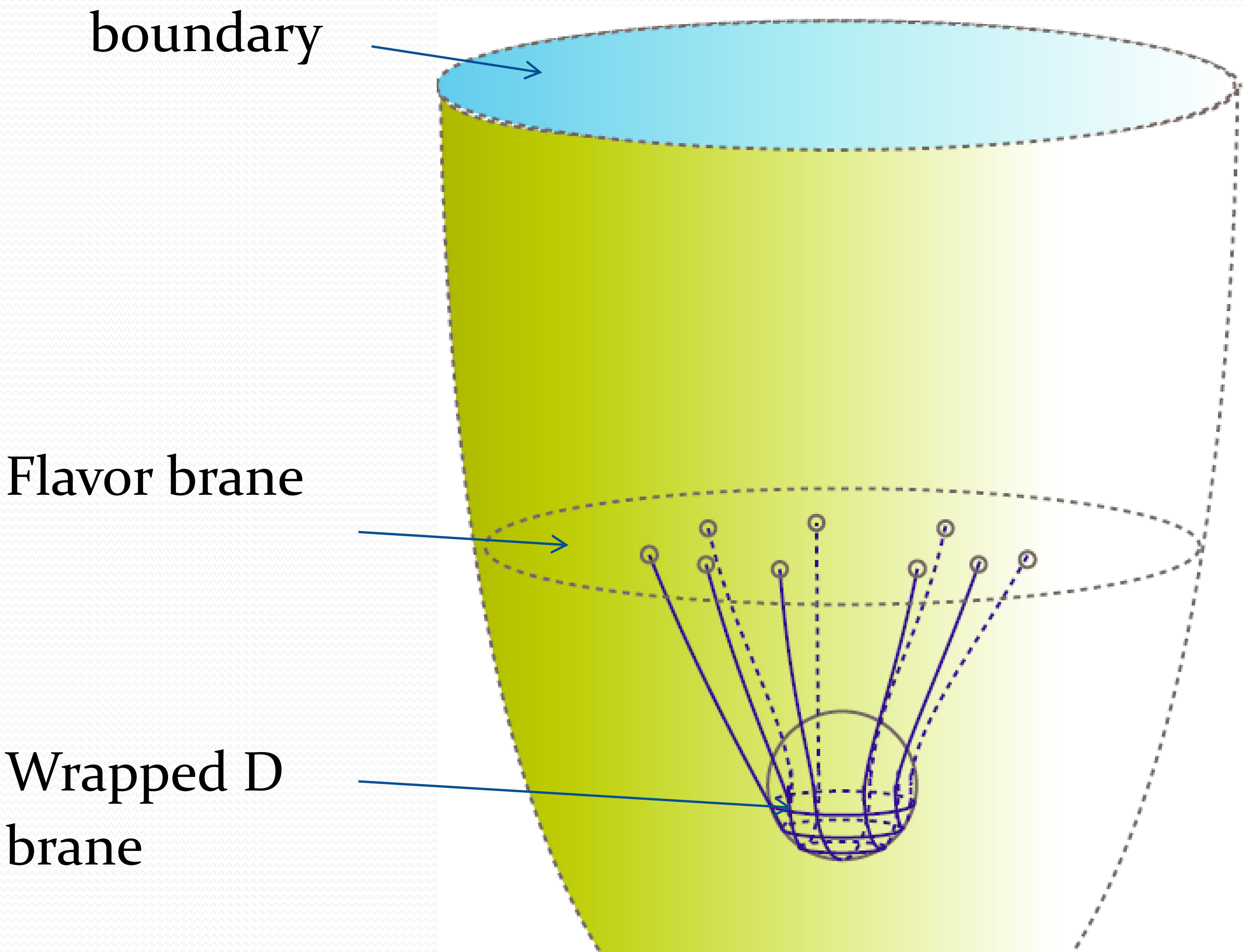
External baryon

- External baryon – Nc strings connecting the baryonic vertex and the boundary



Dynamical baryon

- Dynamical baryon – Nc strings connecting the baryonic vertex and flavor branes



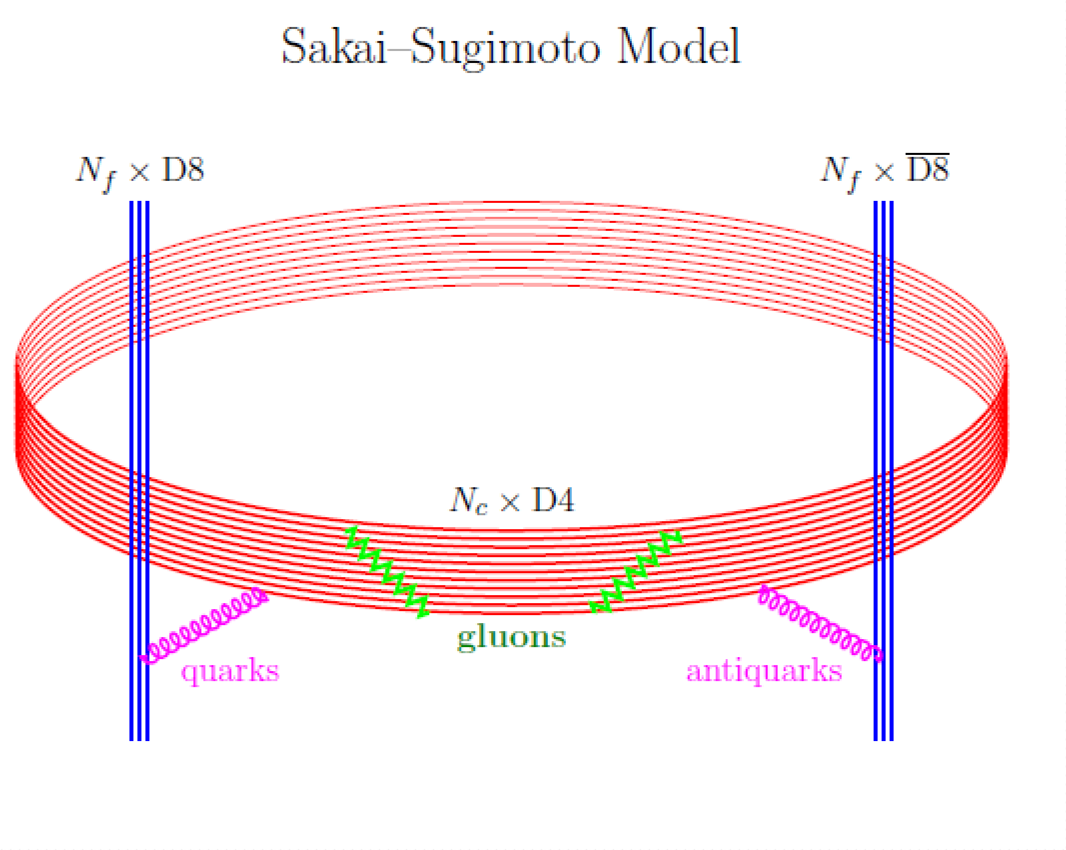
Baryons as instantons in the generalized Sakai Sugimoro model

Baryons in a confining gravity background

- Holographic baryons have to include a baryonic vertex embedded in a gravity background ``dual" to the YM theory with flavor branes that admit chiral symmetry breaking

- A suitable candidate is the Sakai Sugimoto model which is based on the incorporation of D8 anti D8 branes in Witten's model

The brane setup of the Sakai Sugimoto model



Structure of geometries with confining dual

[Witten, Sakai & Sugimoto, . . .]

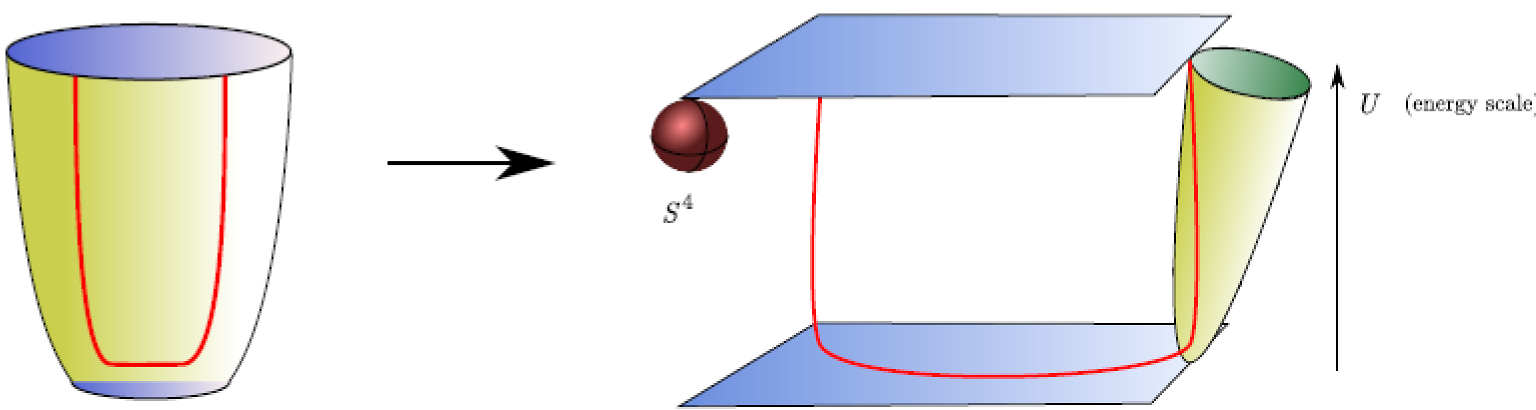
$$ds^2 = \left(\frac{U}{R}\right)^{3/2} \left[\eta_{\mu\nu} dX^\mu dX^\nu + f(U) d\theta^2\right] + \left(\frac{R}{U}\right)^{3/2} \left[\frac{dU^2}{f(U)} + U^2 d\Omega_4\right]$$

world-volume
our 3+1 world

$$f(U) = 1 - \left(\frac{U_\Lambda}{U}\right)^3$$

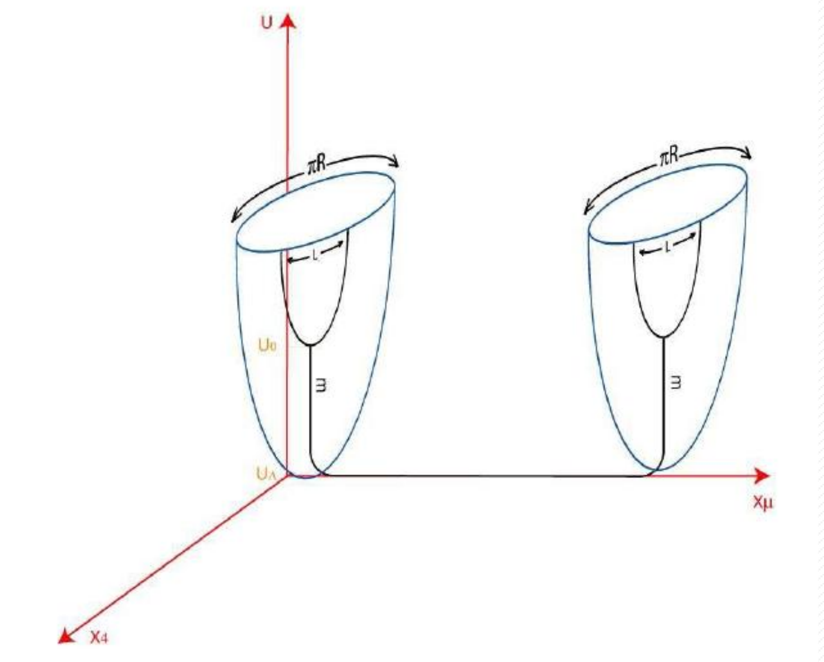
θ is a compact
Kaluza-Klein circle

U: radial direction
bounded from
below $U \geq U_\Lambda$

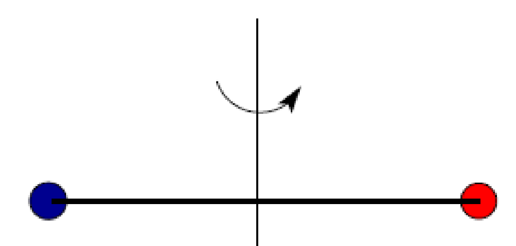


Mesons in the gSS

- The holographic meson= string in curved space that connect the tip of the U shat at two points in x



- Is mapped into a rotated string with massive endpoints



The location of the baryonic vertex

- We need to determine the location of the baryonic vertex in the radial direction.

- In the leading order approximation it should depend on the wrapped brane tension and the tensions of the Nc strings.

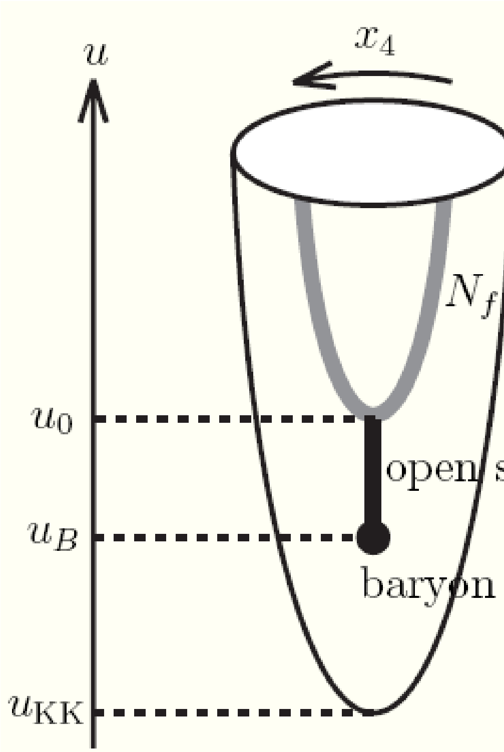
- We can do such a calculation in a background that corresponds to confining (like gSS) and to deconfining gauge theories. Obviously we expect different results for the two cases.

- The location of the baryonic vertex in the radial direction is determined by ``static equillibrium”.

$$S = -T_4 \int dt d\Omega_4 e^{-\phi} \sqrt{-\det g_{\mathrm{D4}}} - N_c T_f \int dt du \sqrt{-\det g_{\mathrm{string}}}$$

- The energy is a decreasing function of x=uB/uKK and hence it will be located at the tip of the flavor brane

$$\mathcal{E}_{\mathrm{conf}}(x; x_0) = \frac{1}{3}x + \int_x^{x_0} \frac{dy}{\sqrt{1-y^{-3}}}$$

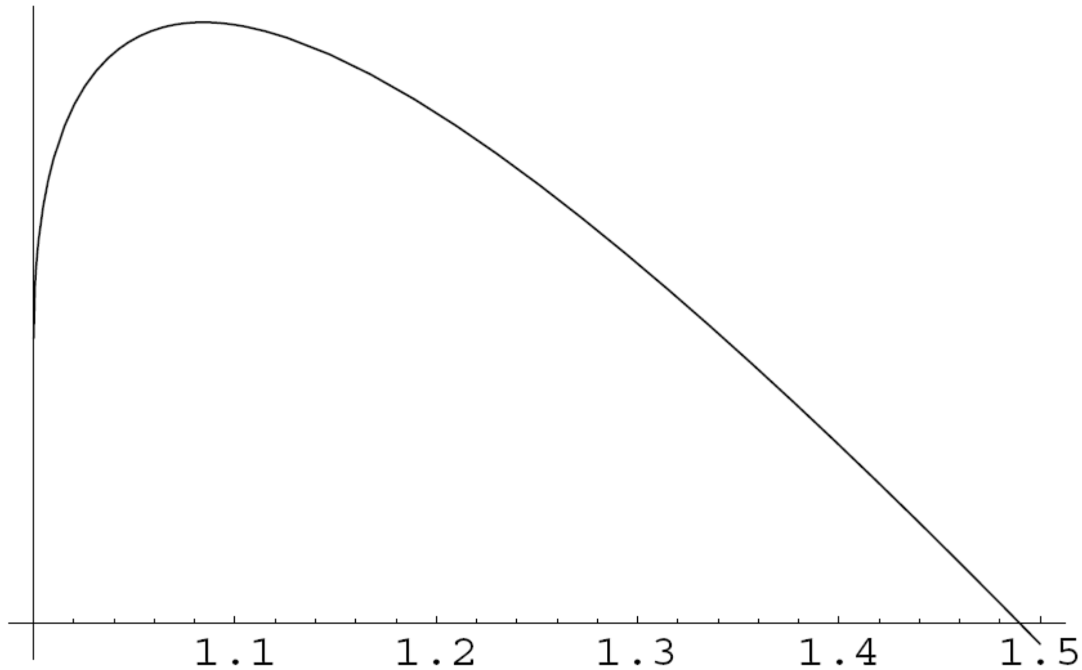


- It is interesting to check what happens in the deconfining phase.
- For this case the result for the energy is

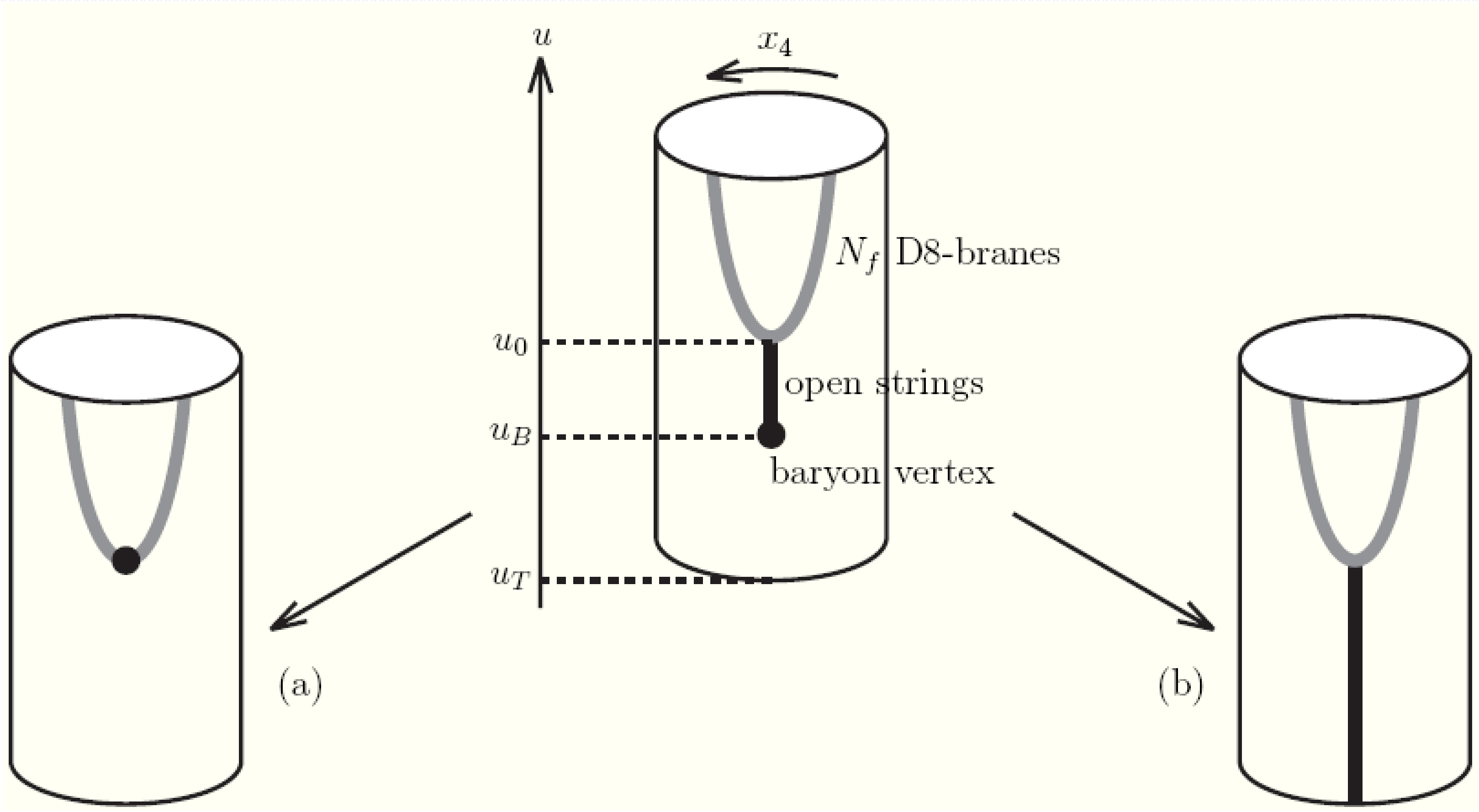
$$\mathcal{E}_{\text{deconf}}(x;x_0) = \frac{1}{3}x\sqrt{1-\frac{1}{x^3}} + (x_0 - x)$$

- For $x > x_{cr}$ low temperature stable baryon
- For $x < x_{cr}$ high temperature dissolved baryon

The baryonic vertex falls into the black hole



The location of the baryonic vertex at finite temperature



Baryons as Instantons in the SS model (review)

- In the SS model the b.v is immersed in the flavor branes.
- The baryon takes the form of an instanton in the 5d U(N_f) gauge theory.

D4 wrapped on S^4 $\simeq$ **instanton on D8** $\simeq$ **Skyrmion**

[Witten, Gross-Ooguri 1998] [Atiyah-Manton 1989] [Skyrme 1961]

Realization of Atiyah-Manton: $U(x^\mu) \equiv P\exp\left\{-\int_{-\infty}^{\infty} dz A_z(x^\mu, z)\right\}$

Skyrmion **Instanton**

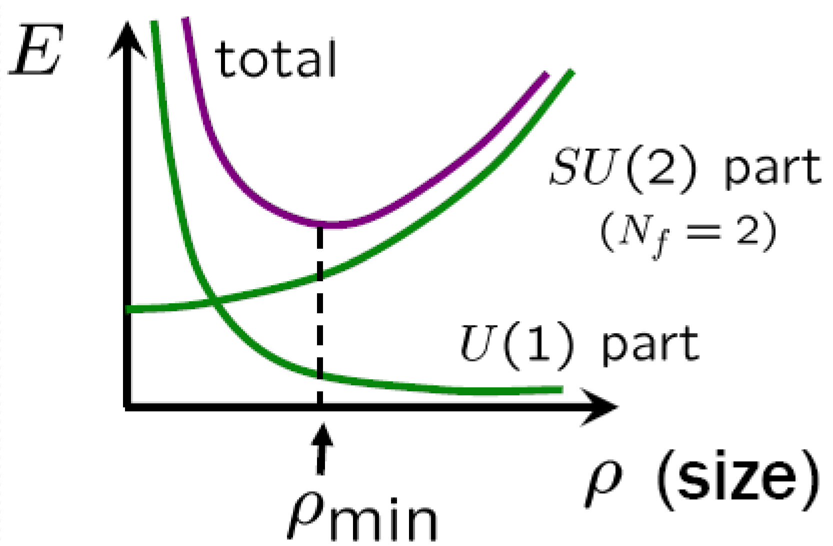
- The instanton is a BPST-like instanton in the (x_i,z) 4d curved space. In the leading order in λ it is exact.

baryon # **Instanton #**

$$\boxed{N_B = \frac{1}{8\pi^2}\int \mathrm{tr} F \wedge F}$$

Baryon (Instanton) size

- For $N_f = 2$ the SU(2) yields a rising potential
- The coupling to the U(1) via the CS term has a run away potential .
- The combined effect



"stable" size but unfortunately of the order of $\lambda^{-1/2}$ so stringy effects cannot be neglected in the large λ limit.

Baryons as instantons in the SS model

- The probe brane world volume 9d → 5d upon Integration over the S^4. The 5d DBI+ CS is approximated

$$S = S_{\rm YM} + S_{\rm CS} \ ,$$

$$S_{\rm YM} = -\kappa \int d^4x dz \ \mathrm{tr} \left[\frac{1}{2} h(z) \mathcal{F}_{\mu\nu}^2 + k(z) \mathcal{F}_{\mu z}^2 \right]$$

$$S_{\rm CS} = \frac{N_c}{24\pi^2} \int_{M^4 \times \mathbb{R}} \omega_5^{U(N_f)}(\mathcal{A}) \ .$$

where

$$h(z) = (1+z^2)^{-1/3} \ , \quad k(z) = 1 + z^2$$

Baryons in the SS model

- One decomposes the flavor gauge fields to SU(2) and U(1)
- In a $1/\lambda$ expansion the leading term is the YM action
- Ignoring the curvature the solution of the SU(2) gauge field with baryon #= instanton #=1 is the BPST instanton

$$A_M(x) = -i f(\xi)\, g\partial_M g^{-1} \ ,$$

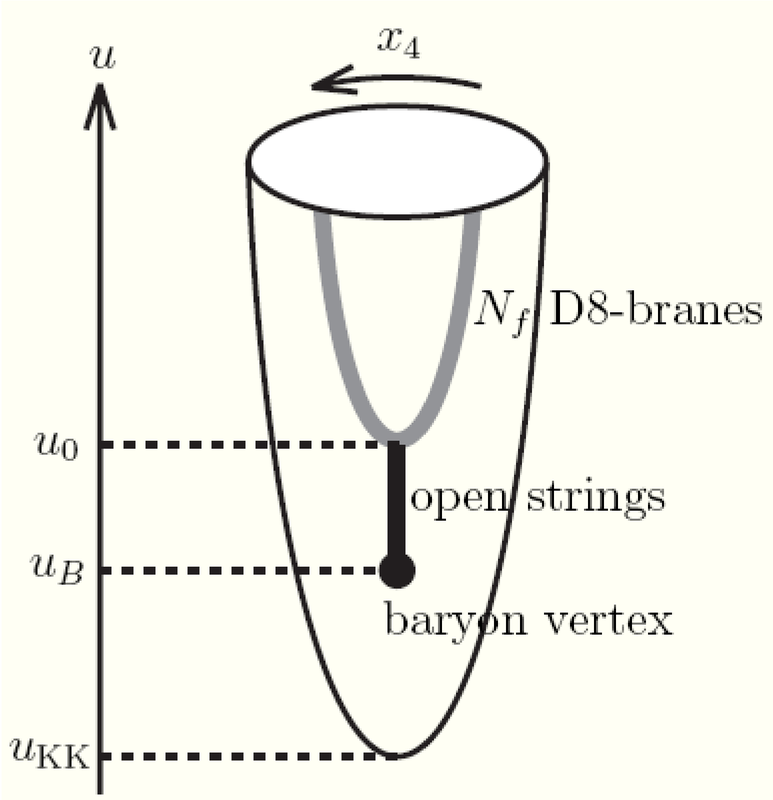
$$f(\xi) = \frac{\xi^2}{\xi^2+\rho^2} \ , \quad \xi = \sqrt{(\vec{x}-\vec{X})^2 + (z-Z)^2} \ ,$$

$$g(x) = \frac{(z-Z) - i(\vec{x}-\vec{X})\cdot\vec{\tau}}{\xi} \ ,$$

Baryons in the generalized SS model

- With the generalized non-antipodal with non trivial m_{sep} namely for u_o different from $u_{\Lambda} = U_{kk}$ with general

$$\zeta = u_o / u_{KK}$$

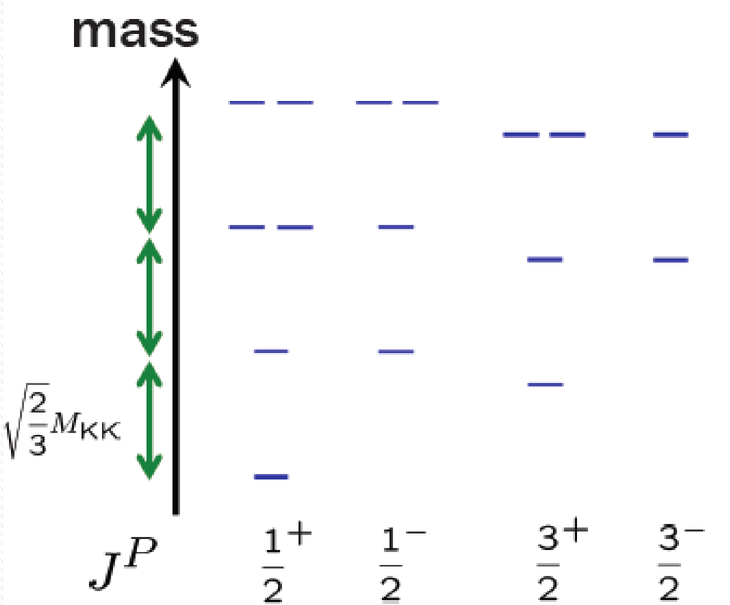


- We found that the size scales in the same way with λ. We computed also the baryonic properties

Baryonic spectrum

Theory

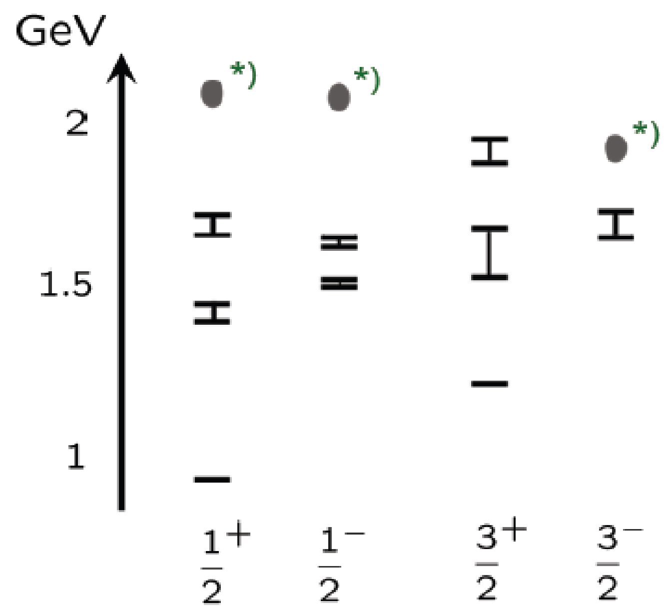
$$M \simeq M_0 + \left(\sqrt{\frac{(l+1)^2}{6} + \frac{2}{15}N_c^2} + \sqrt{\frac{2}{3}}(n_\rho + n_z) \right) M_{\rm KK}$$



Experiment

($I = J$ states from PDG)

*) Evidence for existence is poor



Note: We only consider the mass difference, since $\mathcal{O}(N_c^0)$ term in M_0 is not known,

$$M_0 = (\text{classical soliton mass}) + \mathcal{O}(N_c^0)$$
$$\sim \mathcal{O}(N_c)$$

The spectrum of nucleons and deltas

- The spectrum using best fit approach

N baryons	(n_ρ, n_z)	$M_{\mathrm{KK}} M_{1,n_\rho,n_z}$	Δ baryons	(n_ρ, n_z)	M_{KK}
$n(940)$	$(0,0)$	1027	$\Delta(1232)$	$(0,0)$	1282
$N(1440)$	$(1,0)$	1374	$\Delta(1600)$	$(1,0)$	1629
$N(1535)$	$(0,1)$	1374	$\Delta(1700)$	$(0,1)$	1629
$N(1650)$	$(1,1)$	1721	$\Delta(1920)$	$(2,0),(0,2)$	1976
$N(1710)$	$(2,0),(0,2)$	1721	$\Delta(1940)$	$(1,1)$	1976
$N(2090)$	$(2,1),(0,3)$	2068			
$N(2100)$	$(1,2),(3,0)$	2068			

Table 3: The baryon masses by the use of the minimal χ^2 fitting.

Hadronic properties of the generalized model

	our model	experiment	discrepancy[%]
m_ρ	746 MeV	776 MeV	-3.86
m_{a_1}	1160 MeV	1230 MeV	-5.31
$\frac{m_{\Delta(1232)}}{m_{n(940)}}$	1.51	1.31	15.2
$\sqrt{\langle r^2 \rangle_{I=0}}$	0.813 fm	0.806 fm	0.920
$\sqrt{\langle r^2 \rangle_A}$	0.594 fm	0.674 fm	-11.9
$g_{I=0}$	1.99	1.76	13.1
$g_{I=1}$	8.41	9.41	-10.7

Holographic nuclear interaction

holographic nuclear interaction

- In real life, the nucleon has a fairly large radius , $R_{nucleon} \sim 4/M_{\rho meson}$.

- But in the holographic nuclear physics with $\lambda \gg 1$, we have the opposite situation

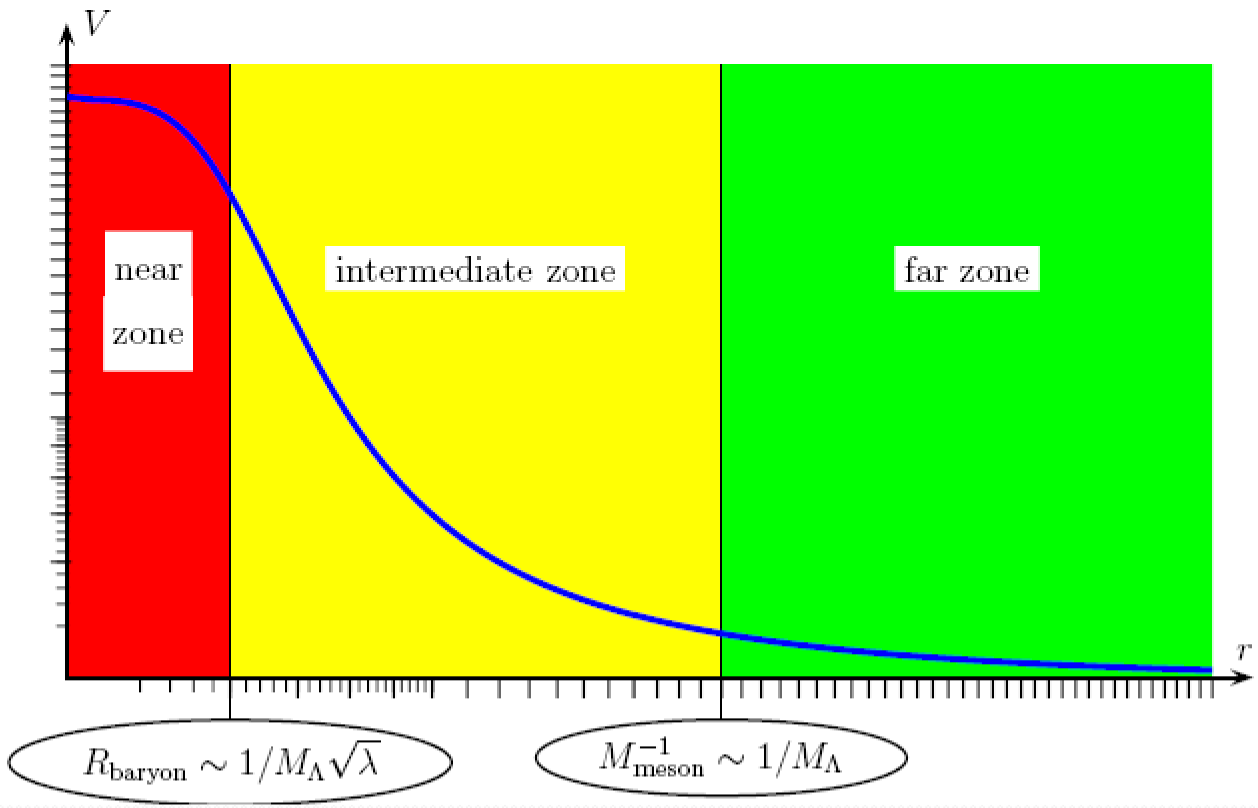
  $$R_{baryon} \sim 1/ (\sqrt{\lambda}\ M),$$

- *Thanks to this hierarchy, the nuclear forces between* two baryons at distance r from each other fall into

3 distinct zones

Zones of the nuclear interaction

- The 3 zones in the nucleon-nucleon interaction



Intermediate Zone of the nuclear interaction

- In the intermediate zone $R_{baryon} \ll r \ll (1/M)$

- The baryons do not overlap much and the fifth dimension is approximately flat.
- At first blush, the nuclear force in this zone is simply the 5D Coulomb repulsive force between two point sources,

$$\boxed{V(r) = \frac{N_c^2}{4\kappa} \times \frac{1}{4\pi^2 r^2} = \frac{27\pi N_c}{2\lambda M_\Lambda} \times \frac{1}{r^2}}$$

Nuclear attraction

- We expect to find a holographic attraction due to the interaction of the instanton with the fluctuations of the embedding which is the dual of the scalar fields

- The attraction term should have the form

$$\boxed{L_{attr} \sim \phi \, Tr[F^2]}$$

- In the antipodal case (the SS model) there is a symmetry under δx_4 -> $-\delta x_4$ and since asymptotically x_4 is the transverse direction

$$\phi \sim \delta x_4$$

such an interaction term does not exist.

Attraction versus repulsion

- In the generalized model the story is different.
- Indeed the 5d effective action for A_M and ϕ is

$$S_{5d} = \int d^4x dw \left[N_c \lambda M_\Lambda [\frac{u}{u_0} tr[F_{MN}^2 + \frac{u^9}{u_0^9} \frac{1}{1-\zeta^{-3}} (\partial_M \phi)^2] - N_c[\phi(TrF_{MN}^2) - \mathcal{L}_{CS}] \right]$$

- For instantons F=*F so there is a competition between

repulsion	attraction
$A\, TrF^2$	$\phi Tr\, F^2$

- The attraction potential also behaves as

$$V_{scalar} \sim 1/r^2$$

Attraction versus repulsion

- The ratio between the attraction and repulsion in the intermediate zone is

$$C_{a/r} \equiv \frac{-V^{\text{attractive}}}{V^{\text{repulsive}}} = \frac{1}{9} \times \left(1 - \zeta^{-3}\right),$$

$$\zeta = \frac{u_0}{u_\Lambda}.$$

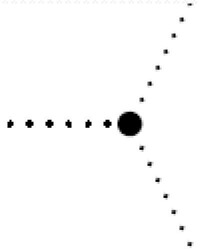
Nuclear potential in the far zone

- We have seen the repulsive hard core and attraction in the intermediate zone.
- To have stable nuclei the attractive potential has to dominate in the far zone.
- In holography this should follow from the fact that the lightest isoscalar scalar is lighter than the corresponding lightest vector meson.
- In SS model this is not the case.
- Maybe the dominance of the attraction associates with two pion exchange(sigma)?.

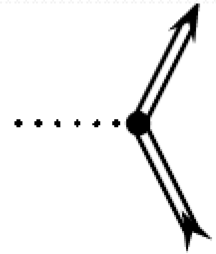
Multi meson exhange at large λ

What are the effects of large λ

- Baryon mass increases, $M_{baryon} \sim \lambda N_c M_{meson}$, while baryon radius shrinks, $R_{baryon} \sim 1/\lambda^{1/2} \times 1/M_{meson}$.

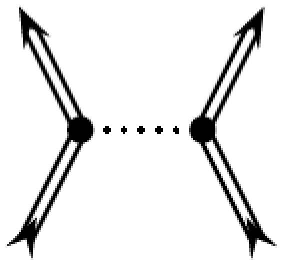


$$g_{MMM} \sim \frac{1}{\sqrt{\lambda N_c}},$$



$$g_{MBB} \sim \frac{N_c}{\sqrt{\lambda N_c}}.$$

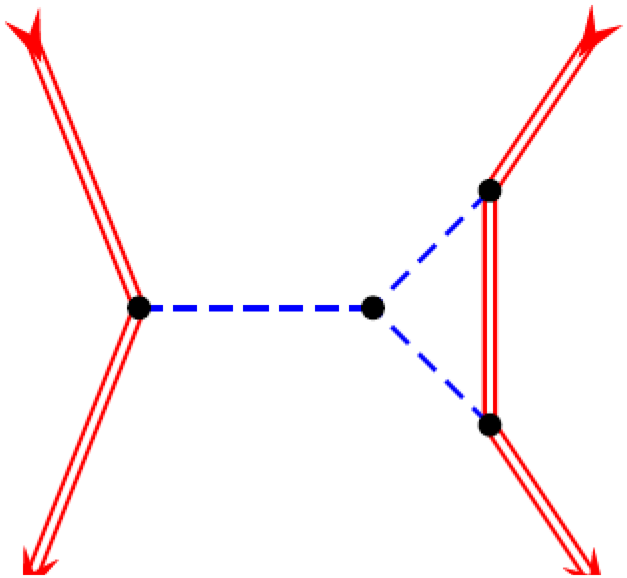
Meson's couplings decrease as $\lambda^{-1/2}$:



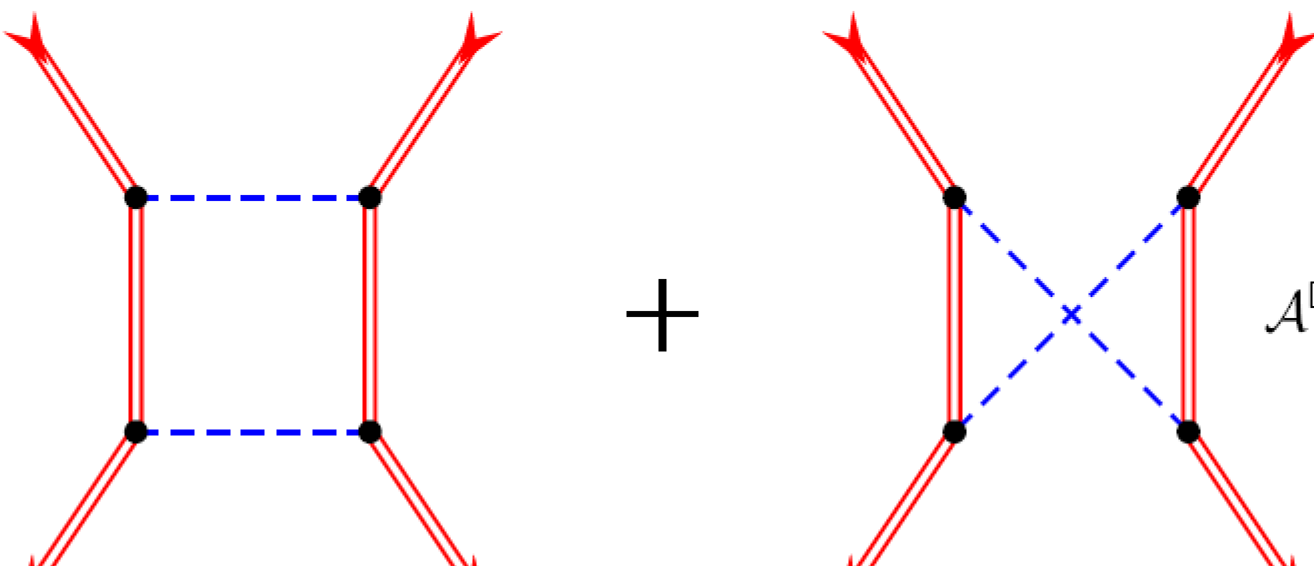
$$V \propto \frac{N_c}{\lambda} \qquad (\text{for fixed } r)$$

The role of the large λ limit

- At one loop there are two types of diagrams



$$\mathcal{A}^{\Delta} \sim g^3_{MB\bar{B}} \times g_{MMM} \sim \frac{N_c}{\lambda^2}$$



$$\mathcal{A}^{\Box} \sim g^4_{MB\bar{B}} \sim \frac{N_c^2}{\lambda^2}$$

The role of the large λ limit

- However, *for non-relativistic* baryons, the box and the crossed-box diagrams almost cancel each other, with the un-canceled part having

$$\mathcal{A}^{\Box}_{\text{uncanceled}} \sim \frac{N_c}{\lambda^2} \sim \mathcal{A}^{\Delta} \sim \frac{1}{\lambda}\mathcal{A}^{\text{tree}}$$

- In other words, the contribution of the double-meson exchange carries the same power of Nc but is suppressed by a factor 1/λ

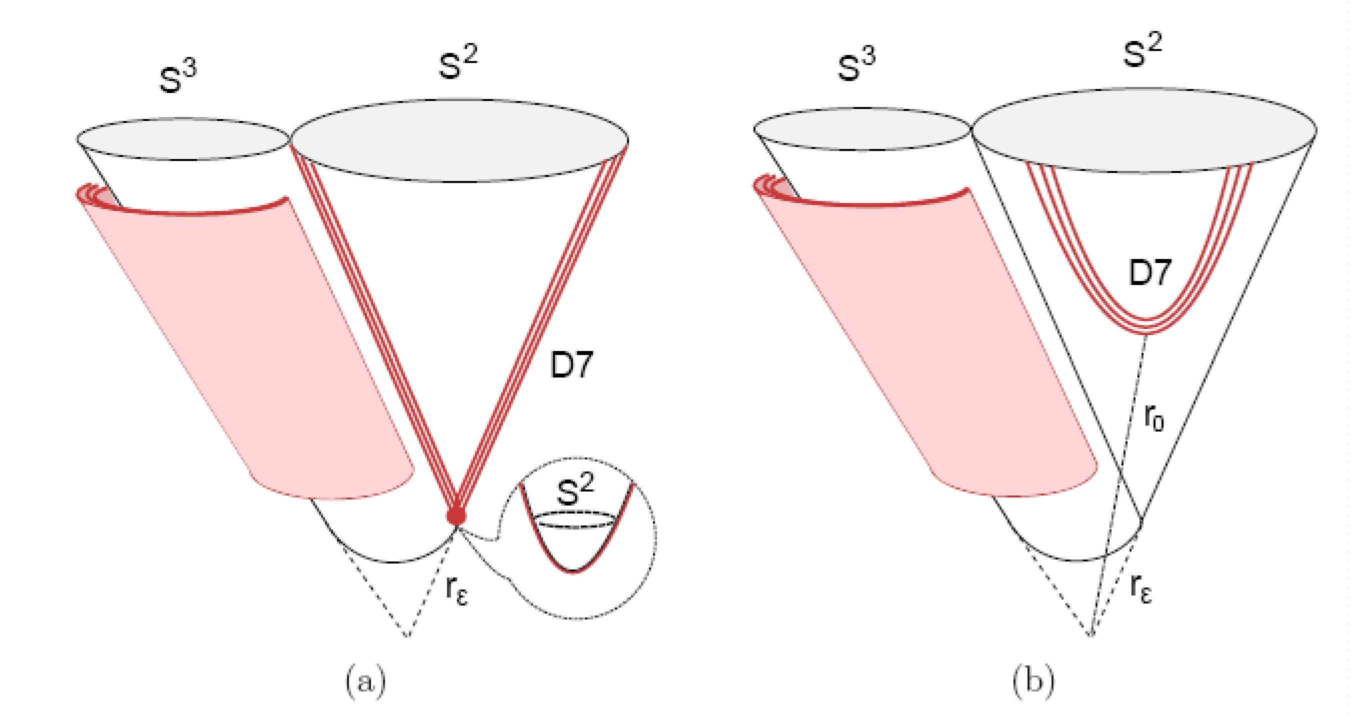
Nuclear interaction in the DKS model

A holographic analog of Walecka's model?

- Can one find another holographic laboratory apart from the SS model where the lightest scalar particle is lighter than the lightest vector particle which interacts with the baryon.
- Can we find a model of an almost cancelation?
- Generically, similar to the gSS model in other holographic models the vector is lighter.
- The Goldstone mechanism may provide a lighter scalar.

The DKS model

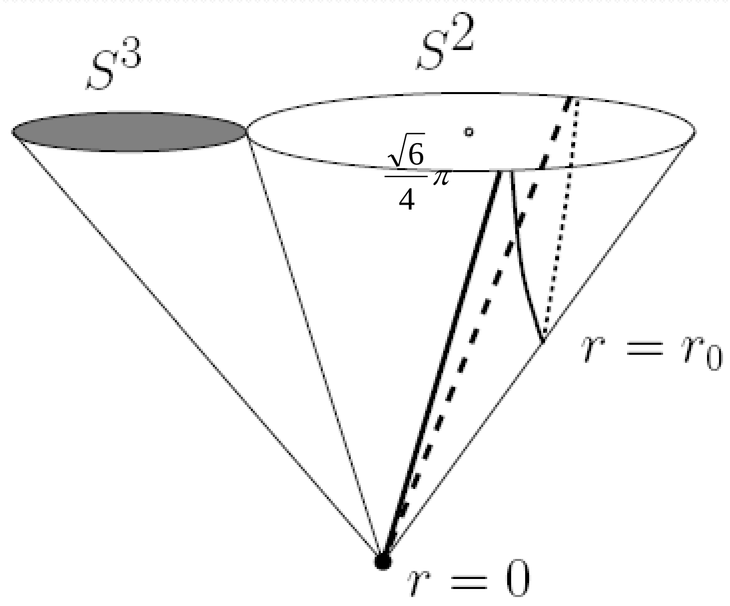
- In the DKS model we placeNf D7 and anti-D7 branesin the Klebanov Strassler model.



- In the undeformed conifold the D7 anti D7 branes spontaneously break the conformal symmetry

Spontaneous breaking of scale invariance

- Adding brane anti-branes to the Klebanov Witten model is different than in the SS model. The asymptotic difference is fixed $\frac{\sqrt{6}}{4}\pi$ independent of ro



- The mode of changing ro is a ``dilaton" a Goldstone boson associated with breaking scale invariance.

Baryons in the DKS model

- The baryons are D3-branes wrapping the S3 of the conifold with M strings connecting the D3 and the flavor branes
- When ro is significantly close to re the geometry can be effectively approximated by the flat one and creates only a mild force. The string tension wins, and the D3-brane

  is pulled towards the D7–D7 branes and dissolves there becoming an instanton
- The model has the following hierarchy of ligh particles:
- The mass of glueballs remains the same as in the KS and therefore is r_0-independent. The typical scale of the glueball mass is

$$m_{\rm gb} \sim \frac{r_\epsilon}{\alpha' \lambda}, \qquad \lambda = g_s M$$

Meson masses in the DKS model

- In the regime ro $\gg$ r$_\varepsilon$ the theory is (almost) conformal and therefore the mass of mesons can depend only on the scale of symmetry breaking ro

$$\boxed{m_{\rm meson} \sim \frac{r_0}{\alpha'\lambda} = \frac{r_0}{r_\epsilon} m_{\rm gb}}$$

- The pseudo-Goldstone boson σ is parametrically lighter

$$\boxed{m_\sigma \sim \frac{r_\epsilon^2}{r_0^2} m_{\rm gb}\,.}$$

The net baryonic potential

- The net potential in the far zone in this case can be written in the form

$$\boxed{V = \frac{1}{4\pi g_s}\left(g_\omega^2 \frac{\mathrm{e}^{-m_\omega|x|}}{|x|} - g_\sigma^2 \frac{\mathrm{e}^{-m_\sigma|x|}}{|x|}\right)}.$$

- For ro ~ rϱ the approximate cancelation of the attractive and the repulsive force can occur naturally
- It is valid only for |x| large enough.

- If mσ < mω, the potential is attractive at large distances no matter what the couplings are.

Binding energy and near cacelation

- On the other hand if gσ is small enough, at distances shorter than 1/mω the vector interaction "wins" and the potential becomes repulsive.
- The binding energy $$E_{\text{binding}} \sim \kappa \frac{r_\epsilon}{g_s^2 M \alpha'}$$

is suppressed by a small dimensionless number κ, which is related to the smallness of the coupling gσ and the fact that mσ and mω are of the same order.

- κ is phenomenologically promising as it represents the near-cancelation of the attractive and repulsive forces responsible for the small binding energy in hadron physics.

Nuclear matter in large Nc is necessarily in a solid phase

The crystal structure of holographic nuclear matter

- Is nuclear matter at large Nc the same as for finite Nc?
- Let's take an analogy from condensed matter – some atoms that attract at large and intermediate distances but have a hard core- repulsion at short ones.
- The parameter that determines the state at T=o p=o is

$$\frac{K}{U} \approx 11\Lambda_B^2, \qquad \Lambda_B = \frac{\hbar}{r_c\sqrt{2M\epsilon}}$$

de Bour parameter

$$K \sim \frac{\pi^2\hbar^2}{2M_{\text{atom}}(\text{well diameter})^2}$$

is the kinetic term rc is the radius of the atomic hard core and ε is the maximal depth of the potential.

The solid structure of holographic nuclear matter

When Λ_B exceeds 0.2-0.3 the crystal melts.

For example,

- Helium has $\Lambda_B = 0.306$, $K/U \approx 1$ quantum liquid
- Neon has $\Lambda_B = 0.063$, $K/U \approx 0.05$; a crystalline solid
- For large Nc the leading nuclear potential behaves as

$$\boxed{\begin{aligned} V(\vec{r}, I_1, I_2, J_2, J_2; N_c) \;=\; & N_c \times A_C(r) + N_c \times A_S(r)(\mathbf{I}_1\mathbf{I}_2)(\mathbf{J}_1\mathbf{J}_2) \\ & + N_c \times A_T(r)(\mathbf{I}_1\mathbf{I}_2)\left[3(\mathbf{nJ}_1)(\mathbf{nJ}_2) - (\mathbf{J}_1\mathbf{J}_2)\right] \\ & + O(1/N_c). \end{aligned}}$$

- Since the well diameter is Nc independent and the mass M scales as~Nc

$$\boxed{\frac{K}{U} \propto \frac{N_c^{-1}}{N_c^{+1}} = \frac{1}{N_c^2}.}$$

The solid structure of holographic nuclear matter

- The maximal depth of the nuclear potential is ~ 100 Mev so we take it to be $\epsilon \sim N_c \times 30$, the mass scales as

$M_N \sim N_c \times 300 \text{ MeV}$. $r_c \sim 0.7$

Consequently

$$\boxed{\Lambda_B = \frac{\hbar}{r_c\sqrt{2M\epsilon}} \sim \frac{2}{N_c} \Longrightarrow \frac{K}{U} \sim \frac{45}{N_c^2}}$$

Hence the critical value is $N_c=8$

Liquid nuclear matter Nc<8

Solid Nuclear matter Nc>8

Lattices (chains) of holographic nuclear matter

. Lattice nuclear matter

- To zeroth order in $1/\lambda$ the SU(Nf) gauge fields are self-dual $\Rightarrow$ ADHM solutions with 4Nf degenerate moduli per baryon (the 4D locations of the instantons, their radii, and the SU(Nf orientations)
- At first order, the degeneracy is lifted by Coulomb interactions (via abelian electric and scalar fields) and the curvature which enters g(z)
- As we have just seen at large Nc nuclear matter is a solid.

Lattice nuclear matter

- We study 3 types of toy models of lattices
- (i) Baryons as point charges of 1d and 3d.
- (ii) 1d exact instanton chains
- (iii) Two instanton inteaction approximation

- We investigate the phase diagram of holographic solid nuclear matter.
- In particular: whether at high enough density instantons spill to the holgraphic spatial dimension?
- Is the GS configuration abelian on non-abelian?

Forcing the system to be one dimensional

- Recall the 5d flavor gauge action of the gSS model

$$S = \int d^4x\, dz \left[\frac{1}{g^2(z)} \left(\tfrac{1}{2} \operatorname{tr}(F^{\mu\nu} F_{\mu\nu}) + \operatorname{tr}(D^\mu \Phi D_\mu \Phi) \right) \right.$$
$$\left. + N_c \times \omega_5(F, A) + N_c c(z) \times \operatorname{tr}(\Phi \{F^{\mu\nu}, F_{\mu\nu}\}) \right]$$

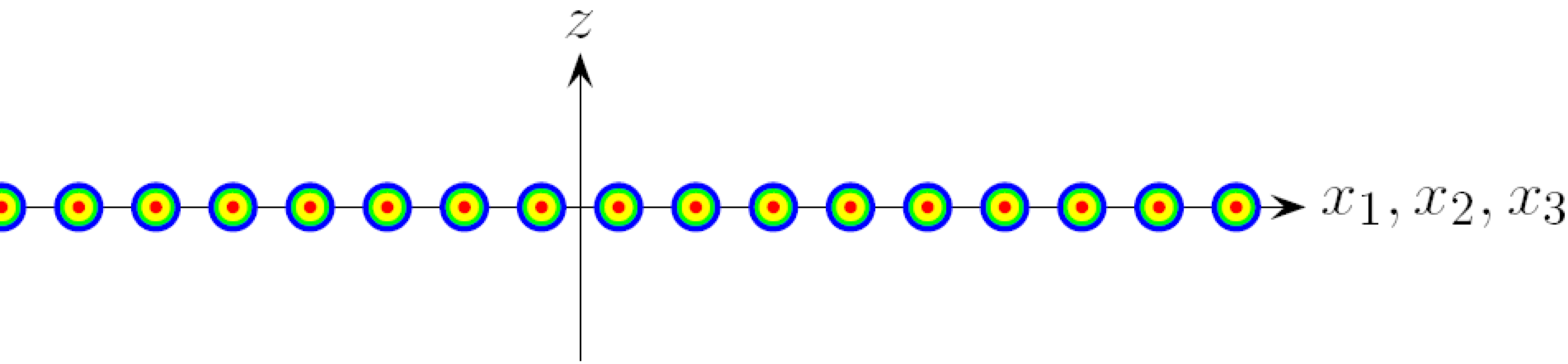
$$\text{where } \frac{8\pi^2}{g^2(z)} = \lambda N_c M_m \Big(1 + (M_m z)^2 + O((M_m z)^4) \Big),$$
$$c(z) = O(1), \text{ details not important except } c < 1.$$

- We force the system to be a 1d chain by adding a harmonic potential to the charge in the other directions

$$\frac{8\pi^2}{g^2_{\mathrm{YM}}} = N_c \lambda M \left(1 + M^2 z^2 + M'^2 (x_1^2 + x_2^2)^2 \right)$$

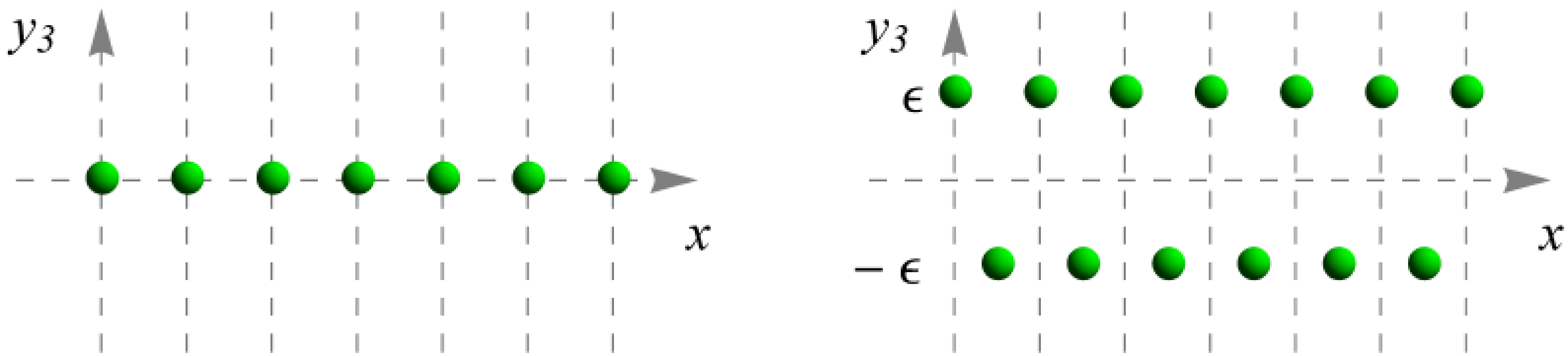
The general structure of holographic nuclear matter

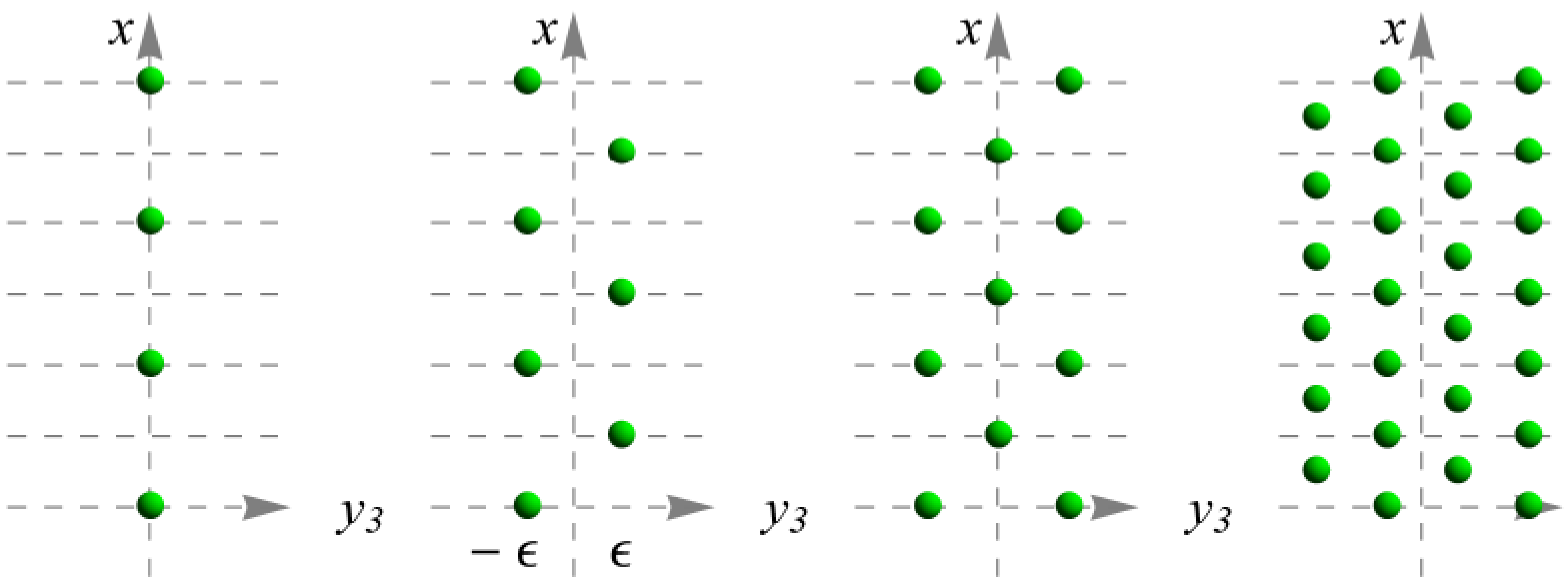
- At low densities g(z) dominates. Each instanton falls to the bottom of the U , i.e to the z=0, hyperplane. The instantons form a 3d lattice



- This phase of the holographic nuclear matter is dual to the baryonic crystal of large Nc QCD

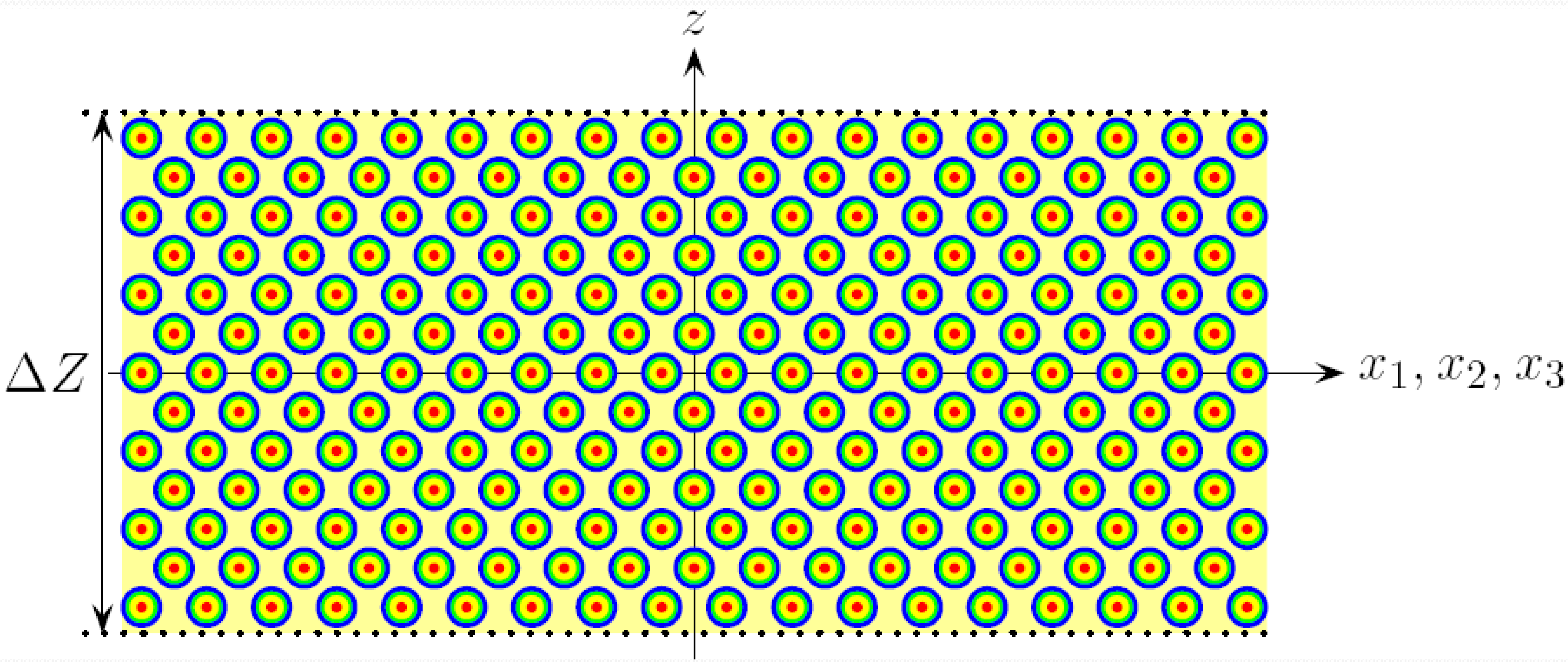
Zig Zag chain, transition to 2,3,4 layers





The general structure of holographic nuclear matter

- At higher density the 1/r^2 repulsion pushes the instantons into the holographic dimension forming a 4d lattice



- In the z direction the lattice has a width ΔZ>> lattice spacing ⟶ many baryons on each 3d point .

The general structure of holographic nuclear matter

- From the 3D point of view, the 4D lattice means overlapping baryons

$\Downarrow$

- Quarks are no longer confined to individual baryons

$\Downarrow$

- The 4D instanton lattice of the holographic QCD is dual to the quarkyonic phase of the nuclear matter. (Quark fermi liquid — weakly coupled for large Nc — with baryon-like excitations near the fermi surface.)

Point charge approximation

(ia) Phase transitions in chain of point charges

- We want to consider a 1 D chain of point charges.
- For that we turn on a potential in the transverse directions x1, x2 and z.

$$\boxed{\frac{8\pi^2}{g_{\rm YM}^2} = N_c \lambda M \left(1 + M^2 z^2 + M'^2 (x_1^2 + x_2^2)^2\right)}$$

- We put a preference to dislocations in z via $M' \gg M$
- At low density the chain is straight
- When the density is increased we find

Phase transitions for chains of point particles

- Let us now study the transitions quantitatively
- The instanton density is replaced by

$$I(\mathbf{x}) = \sum_{n=-\infty}^{\infty} \delta^{(4)}(\mathbf{x} - n\mathbf{d})$$

- For the straight chain the non abelian energy per instanton

$$E_{\mathrm{NA}} = N_c \lambda M \int_0^d \mathrm{d}x_3 \int \mathrm{d}^3x \; I(x) \left(1 + M'^2(x_1^2 + x_2^2) + M^2 z^2\right) = N_c \lambda M \left(1 + M'^2 (\vec{x}_\perp)^2 + M^2 z^2\right)$$

- The minimum is at x1=x2=z=0
- The Coulomb energy is

$$E_{\mathrm{C}} = \frac{N_c}{4\lambda M} \sum_{n\neq 0} \frac{1}{(nd)^2} = \frac{N_c}{\lambda M} \frac{\pi^2}{12 d^2}$$

The energy for a zig zag configuration

- For a zig-zag with displacement ε the total Ec is

$$E_C = \frac{N_c}{4\lambda M}\left(\sum_{\text{even } n\neq 0}\frac{1}{(nd)^2} + \sum_{\text{odd } n}\frac{1}{(nd)^2+(2\epsilon)^2}\right) = \frac{N_c}{\lambda M}\left(\frac{\pi^2}{48d^2} + \frac{\pi}{16\epsilon d}\tanh\frac{\pi\epsilon}{d}\right)$$

- Expanding in e^2 we get

$$E = E_0 + N_c\lambda M^3\epsilon^2 + \frac{N_c}{\lambda M}\left(-\frac{\pi^4\epsilon^2}{48d^4} + \frac{\pi^6\epsilon^4}{120d^6} + O(\epsilon^6)\right)$$

- Thus there is a critical separation distance

$$\boxed{d = d_c \equiv \frac{\pi}{2\cdot 3^{1/4}M\sqrt{\lambda}}}\,.$$

- For spacing slightly smaller that dc the system admits a zig-zag structure with

$$\langle\epsilon\rangle \simeq \pm\frac{\sqrt{5}}{\pi}\sqrt{d_c(d_c-d)}$$

Transitions to multi-layers

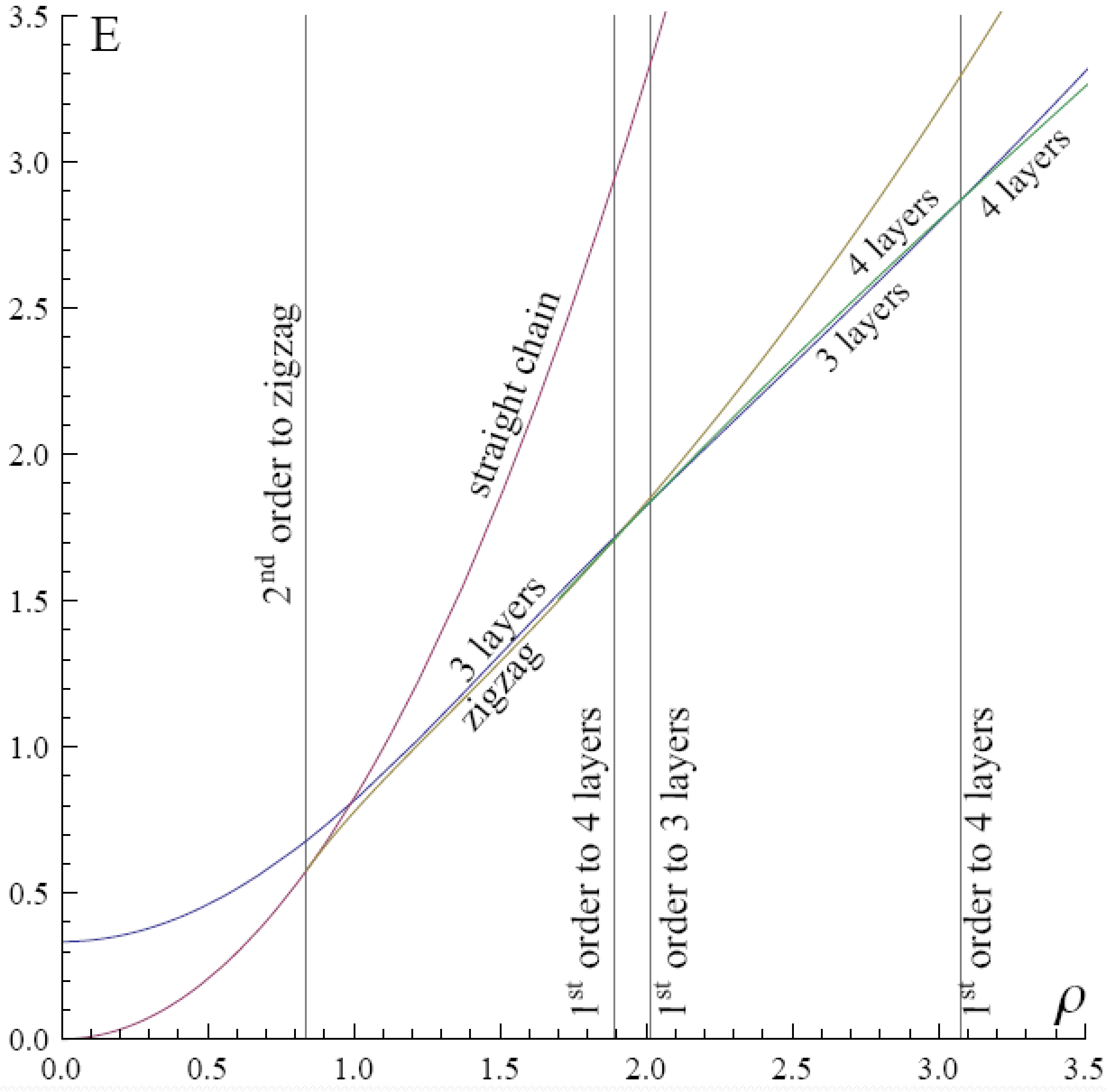
- At higher densities the following sequence of transitions take place

  1 ➡ 2 ➡ 4 ➡ 3 ➡ 4

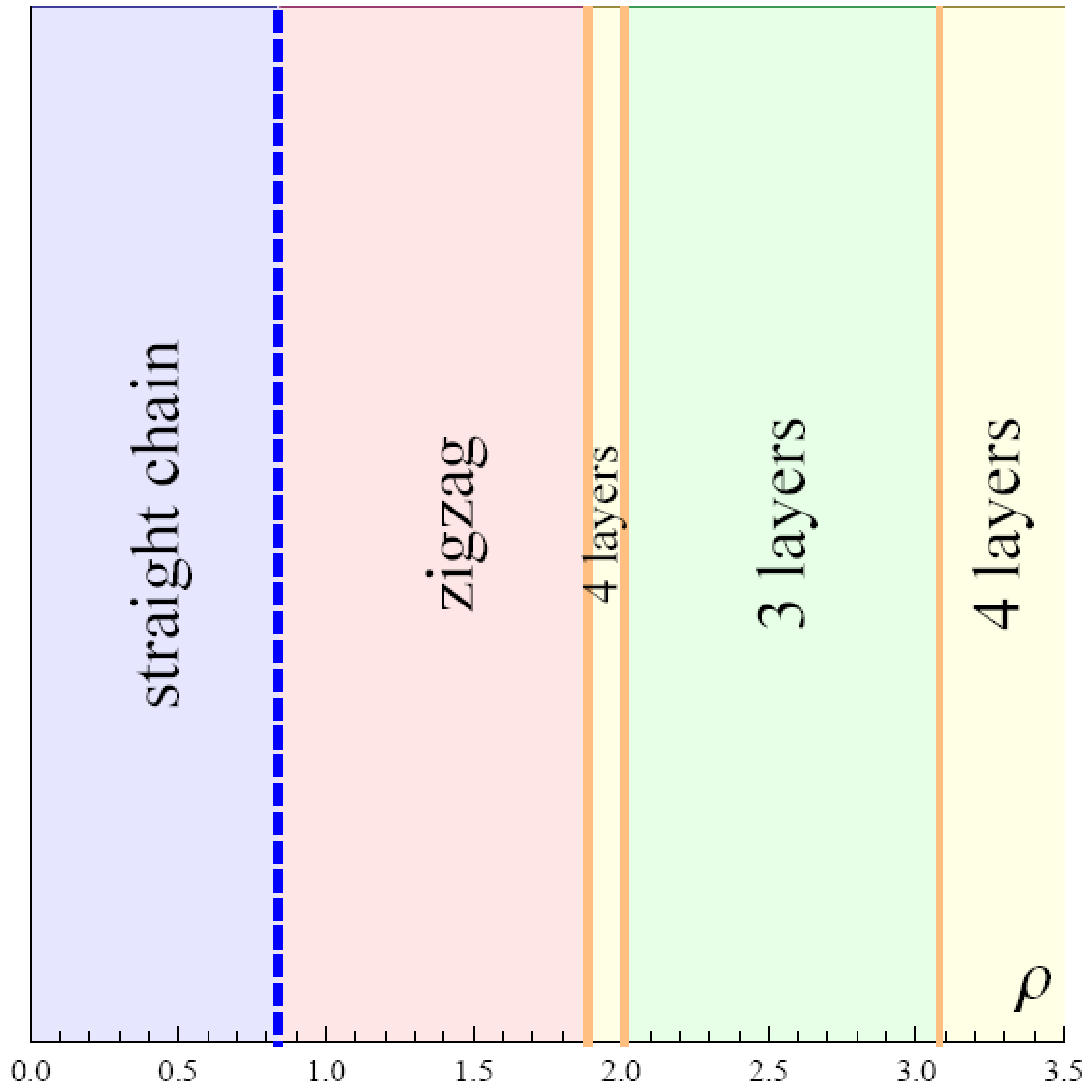
- The structure of the phase transition is given in the figure of the energy as a function of d(or ρ)

- Thus the lesson from this toy model is that when squeezed the baryons do not seat anymore in the regular chain sites but instead pop into the holographic dimension
- The question of course is how is this picture modified once we discuss instantons.

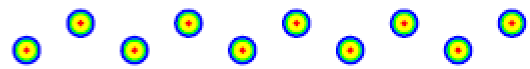
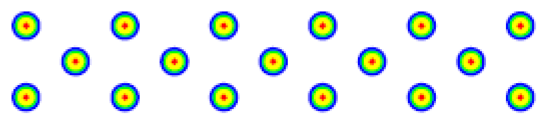
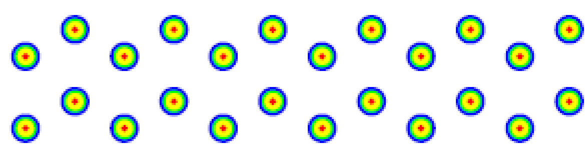
Energy as a function of ρ and the phase transitions



1d point charges phase diagram



Point charges transitions into multi-layer system

#layers	Z	configuration	energy per baryon
1	0		$\frac{\pi^2\rho^2}{12}$
2	$\pm\epsilon$		$\epsilon^2 + \frac{\pi^2\rho^2}{48} + \frac{\pi\rho}{16\epsilon}\tanh\frac{\pi\rho}{\epsilon}$
3	$+\epsilon, 0, -\epsilon$		$\frac{2}{3}\epsilon^2 + \frac{\pi^2\rho^2}{108} + \frac{\pi\rho}{9\epsilon}\tanh\frac{\pi\epsilon\rho}{3} + \frac{\pi\rho}{36\epsilon}\coth\frac{2\pi\epsilon\rho}{3}$
4	$\pm\epsilon\pm\delta$		$\epsilon^2 + \delta^2 + \frac{\pi^2\rho^2}{96} + \frac{\pi\rho}{32\epsilon}\coth\frac{\pi\epsilon\rho}{2} + \frac{\pi\rho}{32\delta}\coth\frac{\pi\epsilon\rho}{2} + \frac{\pi\rho}{64(\epsilon+\delta)}\tanh\frac{\pi\rho(\epsilon+\delta)}{2} + \frac{\pi\rho}{64(\epsilon-\delta)}\tanh\frac{\pi\rho(\epsilon-\delta)}{2}$

(ib) 3D lattice of point charges

- Repeating this analysis in 3D we find that the minimal energy is achieved for close packing.
- This means the largest inter-distance between nearest neighbors for a given density.
- In 3D it is the FCC lattice.
- Above a critical density the analog of the 1D zig-zag will turn the FCC into two sublattices with broken cubic symmetry.
- A structure of BCC will transform into two SC sub-lattices.
- For a 2d case the transition is like that a chessboard where the white and the black are displaced

3D lattice of point charges

- Now Ec diverges, however for the stability analysis we are interested only in the variation of the energy so constant infinity can be subtracted.
- The regularized energy per instanton is

$$E - E_0 = N_c \lambda M^3 \epsilon^2 + \frac{N_c}{\lambda M}\left(-\frac{\Delta\mu^2 \epsilon^2}{d^4} + \frac{4\ell\epsilon^4}{d^6} + O(\epsilon^6)\right)$$

where

$$\Delta\mu^2 = \sum_{\text{odd}} \frac{1}{\left(n_1^2 + n_2^2 + n_3^2\right)^2}, \qquad \ell = \sum_{\text{odd}} \frac{1}{\left(n_1^2 + n_2^2 + n_3^2\right)^3}$$

- This implies a critical spacing

$$d_c = \frac{1}{M}\sqrt{\frac{\Delta\mu}{\lambda}} \simeq \; 0.69\,\frac{1}{M\sqrt{\lambda}}$$

Exact ADHM chains of instantons

1d chain : The ADHM construction

- For the 1d chain of instantons, we first determine the ADHM data, namely solve the self duality condition subjected to the symmetries.
- We then compute the non-abelain and coulomb energies of the chain as a function of the geometrical arrangement and the SU(2) orientations.
- From this we determine the structure of the multi instanton configurations and the corresponding phase transitions.

The ADHM construction of a chain of instantons

- For instanton # N of SU(2) the ADHM data includes

$$X = \Gamma^\mu \tau^\mu \,, \qquad y = y^\mu \tau^\mu$$

4 NxN real matrices

4 real N vector

$\tau^i, i = 1, 2, 3$ Pauli matrices τ^4 unit matrix

- They have to fulfill the following ADHM equation

$$\boxed{\left([\Gamma^\mu, \Gamma^\nu] + y^\mu \otimes y^\nu - y^\nu \otimes y^\mu\right) = \frac{1}{2}\epsilon^{\mu\nu\kappa\lambda}\left([\Gamma^\kappa, \Gamma^\lambda] + y^\kappa \otimes y^\lambda - y^\lambda \otimes y^\kappa\right)}$$

The ADHM construction

- For a periodic 1D infinite chain, we impose translational symmetry

$$\mathbf{S} : x^\mu \to x^\mu + d^\mu$$

- The S tran. acts on Γ^μ_{mn} and y^μ_m as follows

$$\Gamma^\mu \to S^{-1}\Gamma^\mu S = \Gamma^\mu + d^\mu \mathbf{1} \qquad \langle\!\langle \text{to keep the } x^\mu\mathbf{1} - \Gamma^\mu \text{ invariant} \rangle\!\rangle$$

$$(y^\mu_n \tau^\mu) \to \sum G(y^\mu_m \tau^\mu) S_{mn} = (y^\mu_n \tau^\mu)$$

$$G = \exp(i\phi\tau_3/2) \qquad S_{mn} = \delta_{m,n+1} \qquad d^\mu = (0,0,0,d)$$

The ADHM construction

- Consequently translation symmetry requires

$$i\tau^\mu y_n^\mu = a \exp\left(i\tfrac{\phi}{2}\tau_3\right) \iff y_n^\mu = (0, 0, a\sin(n\phi/2), a\cos(n\phi/2))$$

$$\text{and} \quad \Gamma_{mn}^\mu = d\delta^{\mu 4} \times n\delta_{mn} + \hat{\Gamma}^\mu(m-n)$$

- The diagonal Γ_{nn}^μ are the 4D coordinates of the centers, y_n^μ combine the radii and SU(2) orientations
- Combining with the ADHM constraint we get

$$\begin{aligned}
y_m^\mu \otimes y_n^\mu &= a^2 \cos\left[(m-n)\phi/2\right], \\
\Gamma_{mn}^4 &= dn\,\delta_{mn}, \\
\Gamma_{mn}^1 &= \Gamma_{mn}^2 = 0, \\
\Gamma_{mn}^3 &= \frac{a^2}{d} \times \frac{\sin\left[(m-n)\phi/2\right]}{m-n} \quad \text{for } m \neq n \text{ but } 0 \text{ for } m = n.
\end{aligned}$$

Chain of instantons- The ADHM construction

- For our purposes we will need to know only the instanton density

$$\boxed{I(x) = \frac{1}{32\pi^2}\,\epsilon_{MNPQ}\,\mathrm{tr}\,F_{MN}F_{PQ}}$$

expressed in terms of the ADHM data.

$$\boxed{I(x) \;=\; -\frac{1}{16\pi^2}\,\Box\Box\,\log\det(L(x))}$$

where

$$\boxed{L(x) \;=\; (x^\mu\mathbf{1} - \Gamma^\mu)(x^\mu\mathbf{1} - \Gamma^\mu) \;+\; y^\mu \otimes y^\mu}$$

The ADHM construction

- To evaluate the determinant , it is natural to use Fourier transform from infinite matrices into linear operators acting on periodic functions of $\theta \pmod{2\pi}$
- A lengthy calculation yields the following determinant

$$
\begin{aligned}
\det(L) = & \left(\cosh\frac{\phi r_1}{d} + \frac{\pi a^2}{d r_1}\sinh\frac{\phi r_1}{d}\right)\left(\cosh\frac{(2\pi-\phi)r_2}{d} + \frac{\pi a^2}{d r_2}\sinh\frac{(2\pi-\phi)r_2}{d}\right) \\
& + \frac{r_1^2 + r_2^2 - (\pi a^2/d)^2}{2 r_1 r_2}\sinh\frac{\phi r_1}{d}\sinh\frac{(2\pi-\phi)r_2}{d} \\
& - \cos\frac{2\pi x_4}{d},
\end{aligned}
$$

$$
\begin{aligned}
r_1^2 &= x_1^2 + x_2^2 + \left(x_3 + \frac{a^2(\phi-2\pi)}{2d}\right)^2, \\
r_2^2 &= x_1^2 + x_2^2 + \left(x_3 + \frac{a^2\phi}{2d}\right)^2.
\end{aligned}
$$

The total energy of the spin chain

- The total energy is the sum of the non-abelian and coulomb energies.
- We first determine the spread $< {x_i}^2 >$

$$\langle x_i^2 \rangle \equiv \int_0^d dx_4 \int d^3x \, x_i^2 \times \left(I(x) = \frac{-1}{16\pi^2} \Box\Box \log\det(L) \right)$$

This gives us

$$\langle x_1^2 \rangle = \langle x_2 \rangle^2 = \frac{a^2}{2} \quad \langle x_3^2 \rangle = \frac{a^2}{2} + \frac{a^4}{4d^2} \times \phi(2\pi - \phi) \, ,$$

The non-abelian energy

- We add a ``potential" to constrain the multi-instanton configuration to a 1d by assuming a 5d guage coupling of the form

$$\boxed{\frac{8\pi^2}{g_{5D}^2(x)} = N_c \lambda M \left(1 + M^2(x_1^2 + x_2^2 + x_3^2) + O(M^4 x^4)\right)}$$

- For small instanton $a \ll M^{-1}$ the non-abelian

$$E_{NA} = N_c \lambda M \left(1 + M^2 \langle x_1^2 + x_2^2 + x_3^2 \rangle + O(M^4 a^4)\right)$$
$$= N_c \lambda M \left(1 + \tfrac{3}{2} M^2 a^2 + \frac{M^2 a^4}{4 d^2} \times \phi(2\pi - \phi) + O(M^4 a^4)\right)$$

Coulomb energy

- The abelian electric potential obeys

$$\hat{A}_0(x) = +\frac{N_c g^2}{32\pi^2}\Box \log\det(L(x)) + \text{const}$$

- Thus the Coulomb energy per instanton is given by

$$\boxed{\begin{aligned} E_C &= \frac{1}{2g^2}\int_0^d dx_4 \int d^3x\,(\partial_\mu \hat{A}_0)^2 \\ &= \frac{N_c}{256\pi^2\lambda M}\int_0^d dx_4 \int d^3x\,(\partial_\mu \Box \log\det(L))^2 \end{aligned}}$$

- For large lattice spacing d>>a

$$\boxed{E_C \approx \frac{N_c}{\lambda M}\left[\frac{1}{5a^2} + \frac{4\pi^2+3(\pi-\phi)^2}{30d^2} + O(a^2/d^4)\right]}$$

Minimum for overlapping instantons

- Combining the non-abelian and Coulomb energies and minimizing with respect to the instanton radius and twist angle we find

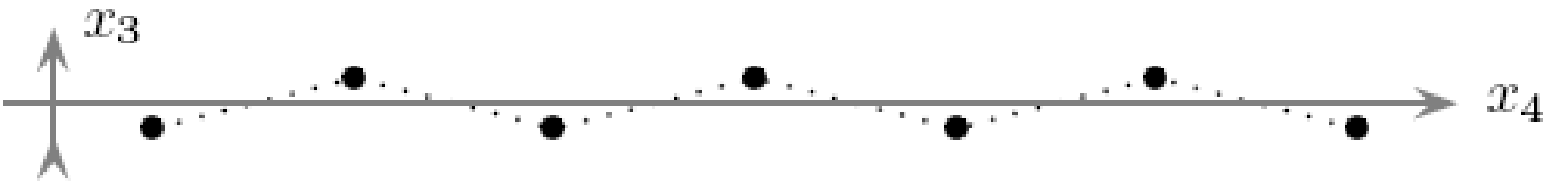
$$\boxed{a[@\mathrm{min}] = a_0 - \frac{\pi^2 a_0^3}{d^2} + O(a_0^4/d^2), \qquad \phi[@\mathrm{min}] = \pi}$$

ao is the equilibrium radius of a standalone instanton

$$\boxed{a_0 = \frac{(2/5)^{1/4}}{M\sqrt{\lambda}}, \qquad \left(\text{or } \frac{9\pi^{1/2}}{M\sqrt{\lambda}}\left(\frac{2}{40\zeta^3 - 25}\right)^{1/4} \text{ for original } \lambda, M_{KK}\,.\right)}$$

The zig-zag chain

- The gauge coupling keeps the centers lined up along the x_4 axis for low density.
- At high density, such alignment becomes unstable because the abelian Coulomb repulsion makes them move away from each other in other directions.
- Since the repulsion is strongest between the nearest neighbors, the leading instability should have adjacent instantons move in opposite ways forming a zigzag pattern



$$\delta X^3[n] = \epsilon \times (-1)^n$$

The Zig-Zag

- We study the instability against transverse motions.
- In particular we restrict the motion to z=x3 by making the instaton energies rise faster in x1 and x2

$$\frac{8\pi^2}{g_{5D}^2(x)} = N_c\lambda M\left(1 + M'^2(x_1^2 + x_2^2) + M^2 x_3^2 + O(x^4 M^4)\right), \qquad M' > M$$

- The ADHM data is based on keeping

$$y_n^\mu = (0, 0, a\sin(n\phi/2), a\cos(n\phi/2)), \qquad \Gamma_{mn}^4 = dn\delta_{mn}$$

- While changing

$$\delta\Gamma_{mn}^3 = \delta_{mn} \times \delta X^3[n]$$

The energies of the zigzag deformation

- The zigzag deformation changes the width

$$\langle x_1^2\rangle = \langle x_2^2\rangle = \frac{a^2}{2}, \qquad \langle x_3^2\rangle = \frac{a^2}{2} + \frac{\pi^2 a^4}{4d^2} + \epsilon^2.$$

- Hence the non abelian energy reads

$$= E_{NA}[\epsilon = 0] + N_c \lambda M^3 \times \epsilon^2$$

- The Coulomb energy

$$E_C = \frac{N_c}{\lambda M}\left[\frac{1}{5a^2} + \frac{3\pi^2}{80d^2} + \frac{\pi^2}{80d^2} \times \frac{\tanh(\pi\epsilon/d)}{\pi\epsilon/d} + O(a^4/d^6)\right]$$

- The net energy cost for small zigzag

$$\boxed{\Delta E_{\rm net} = \Delta E_{NA} + \Delta E_C = N_c\lambda M^3 \times \epsilon^2 + \frac{N_c}{\lambda M}\left[-\frac{\pi^4\epsilon^2}{240d^4} + \frac{\pi^6\epsilon^4}{600d^6} + O(\epsilon^4/d^6)\right]}$$

The zigzag phase transition

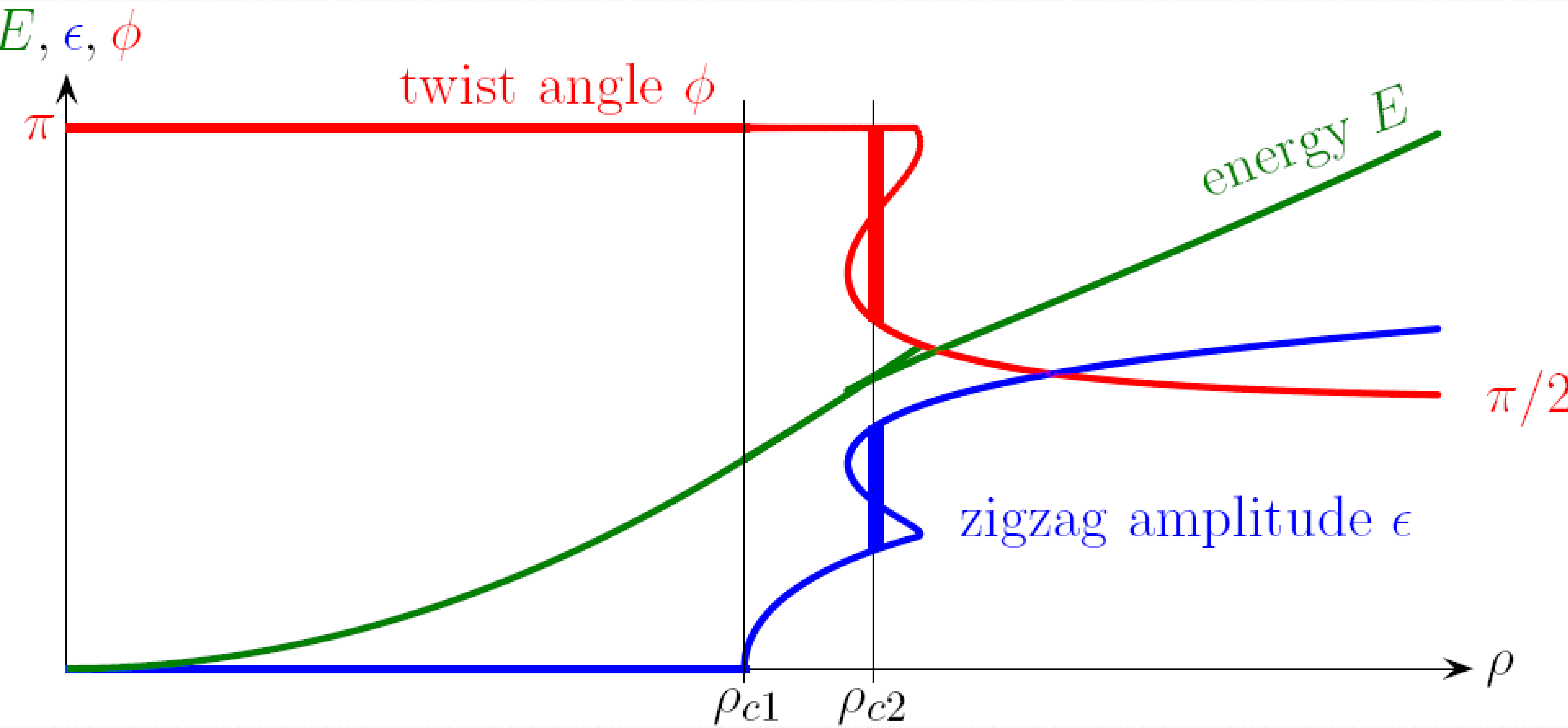
- For small lattice spacing $d < d_{crit}$, the energy function has a negative coefficient of ϵ^2 but positive coefficient of ϵ^4.
- Thus, for $d < d_{crit}$ the straight chain becomes unstable and there is a second-order phase transition to a zigzag configuration.
- The critical distance is

$$\boxed{d > d_{\text{crit}} = \frac{\pi}{\sqrt[4]{240}} \times \frac{1}{M\sqrt{\lambda}}}$$

$$\boxed{\langle \epsilon \rangle \approx \pm\frac{\sqrt{5}}{\pi} \times \sqrt{d_c(d_c - d)}}$$

The phase transitions

- Free energy, zig-zag parameter and phase as a function of the density



2. The two instanton approximation

Two –instanton interaction approximation

- In the low density regime the two body forces dominate the interactions. The multi-body forces are suppressed by $(a/D)^2$

- We sketch the proof of this statement and compute the corresponding two body energy.
- Recall the ADHM data

$$\boxed{\Gamma^{\mu}_{nn} = X^{\mu}_{n} \qquad \Gamma^{\mu}_{m\neq n} = \alpha^{\mu}_{mn} \qquad Y_n = a_n y_n}$$

$$\Im\left(\left(\Gamma^{\dagger}\Gamma\right)_{mn} + Y^{\dagger}_{m} Y_{n}\right) = 0$$

$$\boxed{\eta^{a}_{\mu\nu}\left[\Gamma^{\mu},\Gamma^{\nu}\right]_{mn} + a_m a_n \times \mathrm{tr}\left(y^{\dagger}_{m} y_{n}(-i\tau^{a})\right)} = 0$$

Perturbative solution of the ADHM equation

- We solve the ADHM equation and the constraints associated with the SO(N) symmetry of an N instantons chain in a power series of $(a/D)^2$

$$
\begin{aligned}
\alpha^{\mu}_{mn} &\equiv \Gamma^{\mu}_{m\neq n} = \alpha^{(1)}_{\mu mn} + \alpha^{(2)}_{\mu mn} + \alpha^{(3)}_{\mu mn} + \cdots, \\
\alpha^{(1)}_{\mu mn} &= \frac{\eta^{a}_{\mu\nu}(X_m - X_n)_\nu}{|X_m - X_n|^2} \times \overset{O(A)}{\tfrac{1}{2}} a_m a_n \operatorname{tr}\left(y^{\dagger}_m y_n(-i\tau^a)\right) = O(a^2/D), \\
\alpha^{(2)}_{\mu mn} &= -\frac{\eta^{a}_{\mu\nu}(X_m - X_n)_\nu}{|X_m - X_n|^2} \times \sum_{\ell\neq m,n} \eta^{a}_{\kappa\lambda}\alpha^{(1)}_{\kappa\ell m}\alpha^{(1)}_{\lambda\ell n} = O(a^4/D^3), \\
\alpha^{(3)}_{\mu mn} &= -2\frac{\eta^{a}_{\mu\nu}(X_m - X_n)_\nu}{|X_m - X_n|^2} \times \sum_{\ell\neq m,n} \eta^{a}_{\kappa\lambda}\alpha^{(1)}_{\kappa\ell m}\alpha^{(2)}_{\lambda\ell n} = O(a^6/D^5),
\end{aligned}
$$

- The leading term depends only on two instanton data

Perturbative solution of the ADHM equation

- Given the ADHM data the instanton number density is

$$\boxed{I(x) \; = \; -\frac{1}{16\pi^2} \Box\Box \log\det\big(L(x)\big)}$$

$$\boxed{L_{mn}(x) \; = \; \sum_\ell \big(\Gamma^\mu_{\ell m} - x^\mu \delta_{\ell m}\big)\big(\Gamma^\mu_{\ell n} - x^\mu \delta_{\ell n}\big) \; + \; \tfrac{1}{2} a_m a_n \, \mathrm{tr}\big(y_m^\dagger y_n\big)}$$

- Using integration by parts we can compute several moments of the instanton density

$$\int d^4x \, I(x) \; = \; A,$$

$$\int d^4x \, I(x) \times x^\nu \; = \; \mathrm{tr}\big(\Gamma^\nu\big),$$

$$\int d^4x \, I(x) \times x^\mu x^\nu \; = \; \mathrm{tr}\big(\Gamma^\mu \Gamma^\nu\big) \; + \; \tfrac{1}{2}\delta^{\mu\nu} \, \mathrm{tr}\big(T\big)$$

$$\text{where } T_{mn} \; \equiv \; \tfrac{1}{2} a_m a_n \; \mathrm{tr}\big(y_m^\dagger y_n\big),$$

$$\int d^4x \, I(x) \times x^\lambda x^\mu x^\nu \; = \; \tfrac{1}{2}\big(\Gamma^\lambda \{\Gamma^\mu, \Gamma^\nu\}\big) \; + \; \tfrac{1}{2}\delta^{\lambda\mu} \, \mathrm{tr}\big(\Gamma^\nu T\big) \; + \; \tfrac{1}{2}\delta^{\lambda\nu} \, \mathrm{tr}\big(\Gamma^\mu T\big) \; + \; \tfrac{1}{2}\delta^{\mu\nu} \, \mathrm{tr}$$

The non-abelian energy

- The non-abelian energy is given by quadratic moment with $\mu = \nu = 4$

$$\begin{aligned}
\mathcal{E}_{NA} &= N_c \lambda M^3 \times \int d^4x\, I(x) \times \left(x^4\right)^2 \\
&= N_c \lambda M^3 \times \left(\mathrm{tr}\left(\Gamma^4 \Gamma^4\right) + \tfrac{1}{2}\,\mathrm{tr}\left(T\right) \right) \\
&= N_c \lambda M^3 \times \left(\sum_{i=n}^{A} \left(\left(\Gamma^4_{nn}\right)^2 + \tfrac{1}{2} T_{nn} \right) + \sum_{m \neq n} \left(\Gamma^4_{mn}\right)^2 \right) \\
&= N_c \lambda M^3 \sum_n \left(\left(X^4_n\right)^2 + \tfrac{1}{2} a^2_n \right) + N_c \lambda M^3 \sum_{m \neq n} \left(\alpha^4_{mn}\right)^2 .
\end{aligned}$$

Individual potential energy

Two-body interactions

The non-abelian energy

- Thus to leading order of the non abelian energy only the two intanton interaction are relevant

$$\mathcal{E}_{NA}^{\text{net interaction}} = N_c \lambda M^3 \sum_{m\neq n} \left(\alpha_{mn}^4\right)^2$$

$$\mathcal{E}_{NA}^{\text{interaction}} = \frac{1}{2} \sum_{m\neq n} \mathcal{E}_{NA}^{2\,\text{body}}(m,n) + O(N_c\lambda M^3 a^6/D^4) \text{ multi-body terms,}$$

$$\boxed{\begin{aligned}\mathcal{E}_{NA}^{2\,\text{body}}(m,n) &= 2N_c\lambda M^3 \times \left(a_m a_n \times \frac{\eta^a_{4\nu}(X_m - X_n)_\nu}{|X_m - X_n|^2_{4D}} \times \tfrac{1}{2}\operatorname{tr}\left(y_m^\dagger y_n(-i\tau^a)\right)\right)^2 \\ &= \frac{N_c\lambda M^3 a_m^2 a_n^2}{2|X_m - X_n|_{4D}} \times \operatorname{tr}^2\left(y_m^\dagger y_n\,(-i\vec{N}_{mn}\cdot\vec{\tau})_{3D}\right)\end{aligned}}$$

$$= O(N_c\lambda M^3 a^4/D^2).$$

where

$$N_{mn}^\mu \equiv (\vec{N}_{mn}, N_{mn}^4) = \frac{X_n^\mu - X_m^\mu}{|X_n - X_m|}$$

The Coulomb energy

- The Coulomb energy of the multi-instanton system

$$\boxed{\mathcal{E}_C = \frac{N_c}{4\lambda M}\iint d^4x_1\, d^4x_2\, \frac{I(x_1)I(x_2)}{|x_1 - x_2|^2}}$$

- The diagonal terms of L(x) are much larger than the off-diagonal so we take a power series of the ratio

$$\begin{aligned} I(x) &= \frac{-1}{16\pi^2}\Box\Box \log\det(L) \\ &= \sum_n \mathcal{I}_n^{(1)}(x) + \frac{1}{2}\sum_{m\neq n}\mathcal{I}_{mn}^{(2)}(x) + \frac{1}{6}\sum_{\substack{\text{different} \\ \ell,m,n}}\mathcal{I}_{\ell mn}^{(3)}(x) + \cdots \end{aligned}$$

where

$$\begin{aligned} \mathcal{I}_n^{(1)}(x) &= \frac{-1}{16\pi^2}\Box\Box \log\big(L_{nn}(x)\big), \\ \mathcal{I}_{mn}^{(2)}(x) &= \frac{+1}{16\pi^2}\Box\Box \frac{L_{mn}L_{nm}}{L_{mm}L_{nn}}, \\ \mathcal{I}_{\ell mn}^{(3)}(x) &= \frac{-2}{16\pi^2}\Box\Box \frac{L_{\ell m}L_{mn}L_{n\ell}}{L_{\ell\ell}L_{mm}L_{nn}}, \end{aligned}$$

etc., etc.

The Coulomb energy

- The net Coulomb energy

$$\mathcal{E}_C^{\rm net} = \frac{N_c}{4\lambda M} \times \left\{ \sum_n \frac{1}{\rho_n^2} + \sum_{m \neq n} \frac{1}{|X_m^\mu - X_n^\mu|^2} + O(a^4/D^6) \right\}$$

Self interaction including inteference

Point charge Coulomb repulsion

Density in inter instanon space

Note that the self interaction terms dominate the net Coulomb energy

The Coulomb energy

- The contribution of the two instanton interference terms is compareble to the direct repulsion. The three or more body interactions are negligable
- The final expression for the Coulomb energy

$$\mathcal{E}_C^{\rm net} = \frac{N_c}{4\lambda M}\left(\sum_n \frac{4/5}{a_n^2} + \sum_{m\neq n}\frac{1}{|X_m - X_n|^2}\times\left(1+\frac{1}{5}\left(\frac{a_m^2}{a_n^2}+\frac{a_n^2}{a_m^2}\right)\times\left(\mathrm{tr}^2(y_m^\dagger y_n)-2\right)\right)\right.$$
$$\left. + \; O(a^2/D^4)\right)$$

The total tow body total energy

- Combining the non-abelain and Coulomb energies

$$\mathcal{E}^{\text{total}} = \sum_n \mathcal{E}^{1\,\text{body}}(n) + \frac{1}{2}\sum_{m\neq n} \mathcal{E}^{2\,\text{body}}(m,n) + \frac{1}{6}\sum_{\substack{\text{different}\\ \ell,m,n}} \mathcal{E}^{3\,\text{body}}(\ell,m,n) + \cdots$$

$$\mathcal{E}^{1\,\text{body}}(n) = N_c M \left(\lambda M^2 \times (X_n^4)^2 + \frac{\lambda M^2}{2} \times a_n^2 + \frac{1}{5\lambda M^2} \times \frac{1}{a_n^2}\right),$$

$$\boxed{\mathcal{E}^{2\,\text{body}}(m,n) = \frac{N_c}{2\lambda M} \times \frac{1}{|X_m - X_n|^2_{4D}} \left(\begin{array}{c} \lambda^2 M^4 \times a_m^2 a_n^2 \times \mathrm{tr}^2\left(y_m^\dagger y_n(-i\vec{N}_{mn}\cdot\vec{\tau})\right) \\ + 1 + \frac{1}{5}\left(\frac{a_m^2}{a_n^2} + \frac{a_n^2}{a_m^2}\right) \times \left(\mathrm{tr}^2(y_m^\dagger y_n)\right) \\ + O\left(\frac{a^2}{D^2} \sim \frac{1}{\lambda M^2 D^2}\right) \end{array}\right)}$$

$$\mathcal{E}^{3\,\text{body}}(\ell,m,n) = O\left(\frac{N_c a^2}{\lambda M^2 D^4} \sim \frac{N_c}{\lambda^2 M^3 D^4}\right), \quad etc., etc.$$

The total tow body total energy

- We plug the equilibrium radii

$$a_n = a_0 + O(a^3/D^2)$$

$$a_0 = \frac{\sqrt[4]{2/5}}{\sqrt{\lambda}M}$$

- We finally get the two instanton interaction energy

$$\boxed{\mathcal{E}^{2\,\mathrm{body}}(m,n) = \frac{2N_c}{5\lambda M} \times \frac{1}{|X_m - X_n|^2_{4D}} \times \left[\frac{1}{2} + \mathrm{tr}^2\left(y_m^\dagger y_n\right) + \mathrm{tr}^2\left(y_m^\dagger y_n\,(-i\vec{N}_{mn}\cdot\vec{\tau})\right)\right]}$$

Linear chains of instantons

- For 1D lattice geometry $X^\mu_n = (nD, 0, 0, 0), \quad n \in \mathbb{Z}$

$$|X_m - X_n|^2 = D^2 \times (m-n)^2 \text{ while } \vec{N}_{mn} = (\pm 1, 0, 0)$$

- This 1d structure is enforced by a 5D gauge coupling

$$\boxed{\frac{8\pi^2}{g_5^2(x_2, x_3, x_4)} = N_c \lambda M \Big(1 + M^2 x_4^2 + M_3^2 x_3^2 + M_2^2 x_2^2 + O(M^4 x^4)\Big)}$$

- Let's us consider first the regime where

$$M_2^2, M_3^3 \ll M^2.$$

- So the impact on the instanton size of m_2, m_3 is negligble

Linear chains of instantons

- The net energy as a function of the orientations

$$\mathcal{E}^{\rm int} = \frac{N_c}{5\lambda M D^2} \times \sum_{m\neq n} \frac{1}{(m-n)^2} \times \left[\tfrac{1}{2} + \mathrm{tr}^2\left(y_m^\dagger y_n\right) + \mathrm{tr}^2\left(y_m^\dagger y_n(-i\tau_1)\right)\right]$$

- We minimize the energy with respect to the orientations of nearest neighbors pairs.

$$\forall m: \quad y_m^\dagger y_{m+1} = \cos\psi_m \times (i\tau_3) + \sin\psi_m \times (i\tau_2) \quad \text{for some angle } \psi_m$$

- The most general solution of these equations

$$y_n = \exp\left(i\phi_n\tau_i\right) \times (i\tau_3)^n = \begin{cases} \pm[\cos\phi_n \times 1 + \sin\phi_n \times (i\tau_1)] & \text{for even } n, \\ \pm[\cos\phi_n \times (i\tau_3) + \sin\phi_n \times (i\tau_2)] & \text{for odd } n, \end{cases}$$

Linear chains

- All these configurations have the same energy, thus there is a huge degeneracy of chains with

$$
\begin{aligned}
\mathcal{E}^{\text{interaction}}_{\text{per instanton}} &= \frac{N_c}{5\lambda M D^2} \times \sum_{m\neq n}^{\text{fixed}\,n} \frac{1}{(m-n)^2} \times \left[\tfrac{1}{2} + \operatorname{tr}^2(y_m^\dagger y_n) + \operatorname{tr}^2(y_m^\dagger y_n(-i\tau_1))\right] \\
&= \frac{N_c}{5\lambda M D^2} \times \sum_{\ell=m-n\neq 0} \frac{1}{\ell^2} \times \begin{cases} \frac{1}{2} & \text{for odd } \ell, \\ \frac{9}{2} & \text{for even } \ell, \end{cases} \\
&= \frac{N_c}{5\lambda M D^2} \times \left(\frac{1}{2}\sum_{\text{odd}\,\ell} \frac{1}{\ell^2} + \frac{9}{2}\sum_{\text{even}\,\ell\neq 0} \frac{1}{\ell^2}\right) \\
&= \frac{N_c}{5\lambda M D^2} \times \left(\frac{1}{2} \times \frac{\pi^2}{4} + \frac{9}{2} \times \frac{\pi^2}{12} = \frac{\pi^2}{2}\right).
\end{aligned}
$$

Regular patterns

- Among the configurations are certain regular patterns

- The *antiferromagnetic chain*, with 2 alternating instanton orientations:

$$y_{\text{even}\,n} = \pm 1, \quad y_{\text{odd}\,n} = \pm i\tau_3 . \tag{4.12}$$

In this configuration — which obtains for $\phi_n \equiv 0$ — the y_n (modulo sign) span a $\mathbb{Z}_2$ subgroup of the $SO(2) \times \mathbb{Z}_2$.

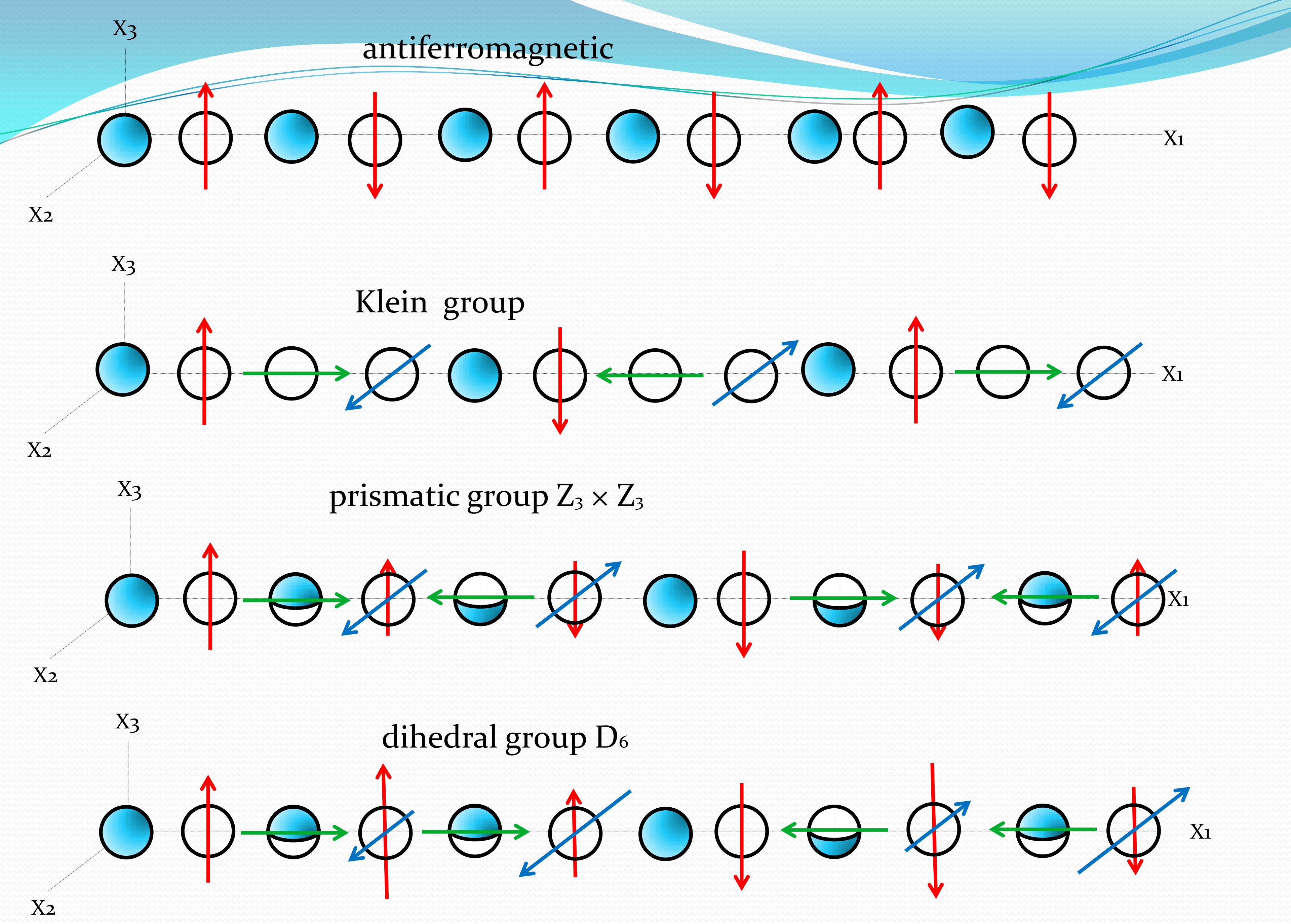
- Cartesian axes[1]: e 3

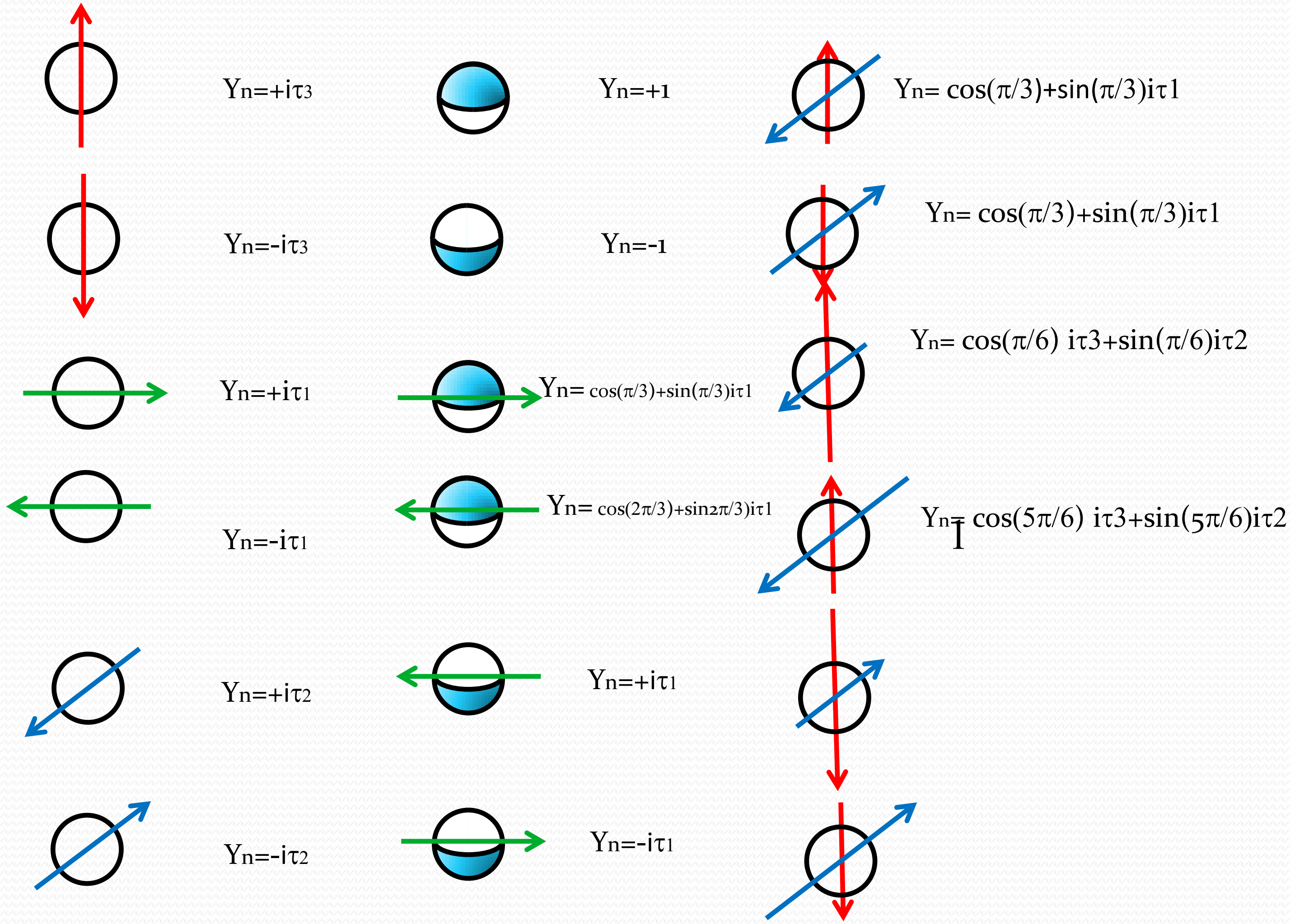
$$y_{n\equiv 0 \pmod 4} = \pm 1, \quad y_{n\equiv 1 \pmod 4} = \pm \tau_3, \quad y_{n\equiv 2 \pmod 4} = \pm \tau_1, \quad y_{n\equiv 3 \pmod 4} = \pm \tau_2. \tag{4.13}$$

- Period $= 2k = 6, 8, 10, \ldots$ configurations spanning *prismatic groups* $\mathbb{Z}_k \times \mathbb{Z}_2$:

$$y_{\text{even}\,n} = \cos\frac{\pi n}{2k} \times 1 + \sin\frac{\pi n}{2k} \times (i\tau_1), \quad y_{\text{odd}\,n} = \cos\frac{\pi(n-1)}{2k} \times (i\tau_3) + \sin\frac{\pi(n-1)}{2k} \times (i\tau_2) \tag{4.14}$$

- Period $= 2k = 6, 8, 10, \ldots$ configurations spanning *dihedral groups* D_{2k}, which obtain for $\phi_n = n \times (\pi/2k)$, *i.e.*


antiferromagnetic
X3
X2
X1
Klein group
X3
X2
X1
prismatic group Z3 × Z3
X3
X2
X1
dihedral group D6
X3
X2
X1



Yn=+iτ3
Yn=+1
Yn= cos(π/3)+sin(π/3)iτ1
Yn=-iτ3
Yn=-1
Yn= cos(π/3)+sin(π/3)iτ1
Yn= cos(π/6) iτ3+sin(π/6)iτ2
Yn=+iτ1
Yn= cos(π/3)+sin(π/3)iτ1
Yn= cos(2π/3)+sin2π/3)iτ1
Yn= cos(5π/6) iτ3+sin(5π/6)iτ2
Yn=-iτ1
Yn=+iτ2
Yn=+iτ1
Yn=-iτ2
Yn=-iτ1

The general case of linear chain

- For the regime $M_2, M_3 \sim M$ the huge degeneracy is lifted and the net energy is

$$\boxed{\mathcal{E}^{\rm int}_{\rm net} \equiv \frac{1}{2}\sum_{n\neq m}\mathcal{E}^{2)}(n,m) = \frac{N_c}{5\lambda M D^2}\times\sum_{m\neq n}\frac{Q(m,n)}{(m-n)^2}}$$

where

$$Q(m,n) \stackrel{\rm def}{=} \tfrac{1}{2} + \mathrm{tr}^2\big(y_m^\dagger y_n\big) + C_4\,\mathrm{tr}^2\big(y_m^\dagger y_n(-i\tau^1)\big) + C_3\,\mathrm{tr}^2\big(y_m^\dagger y_n(-i\tau^2)\big) + C_2\,\mathrm{tr}^2\big(y_m^\dagger y_n(-i\tau^3)\big)$$

and with

$$C_\mu \stackrel{\rm def}{=} \frac{M_\mu^2}{M_4^2+M_3^2+M_2^2}, \quad C_4 + C_3 + C_2 = 1.$$

Relaxation method

- We now minimize the energy with respect to the orientation using a computer relaxation method.
- We take a lattice of 200 SU(2) matrices y_n .
- In each run we started with random elements of SU(2)

y_n

- We let the y_n relax to the minimum energy via

$$\boxed{\frac{dy_n(t)}{dt} = -K \times \frac{\delta \mathcal{E}^{\text{int}}_{\text{net}}}{\delta y_n}}$$

A mobility constant

$$\frac{\delta \mathcal{E}}{\delta y_n} \stackrel{\text{def}}{=} y_n \left(-i\vec{\tau}\right) \cdot \nabla_s \mathcal{E}\left(y_n \to y_n(1 + i\vec{s} \cdot \vec{\tau})\right)\Big|_{\vec{s}=0}$$

Link-periodic chains

- We find that the lattices are link-periodic with

$$\text{even } n-m, \quad y_m^\dagger y_n = \cos((n-m)\varphi) \times i^{n-m} + \sin((n-m)\varphi) \times i^{n+m+1}\tau^1$$

$$\text{odd } n-m, \quad y_m^\dagger y_n = \cos((n-m)\varphi \pm \theta) \times i^{n-m}\tau^3 + \sin((n-m)\varphi \pm \theta) \times i^{n+m}\tau^2$$

- The average interaction energy per instanton is

$$\mathcal{E}^{\text{interaction}}_{\text{per instanton}} = \frac{N_c}{5\lambda M D^2} \times \left\langle \sum_{m\neq n}^{\text{fixed } n} \frac{Q(m,n)}{(n-m)^2} \right\rangle^{\text{average}}_{\text{over } n} = \frac{N_c}{5\lambda M D^2} \times \sum_{\ell \neq 0} \frac{\overline{Q}(\ell)}{\ell^2}$$

$$= \frac{N_c}{5\lambda M D^2} \times \left(\begin{array}{c} \frac{\pi^2}{2} \times \Big(1 + C_3 + C_2 - (C_3 - C_2)\cos(2\theta)\Big) \\ -\, 2\pi|\varphi| \times \Big(C_3 + C_2 - (C_3 - C_2)\cos(2\theta)\Big) \\ +\, 4\varphi^2 \times \Big(C_3 + C_2\Big) \end{array} \right)$$

Link-periodic chains

- Minimizing the energy with respect to φ and θ

$$
\begin{aligned}
(1) \quad \varphi &= +\frac{\pi}{2} \times \frac{C_2}{C_2 + C_3}, & \theta &= 0, \\
(2) \quad \varphi &= -\frac{\pi}{2} \times \frac{C_2}{C_2 + C_3}, & \theta &= 0, \\
(3) \quad \varphi &= -\frac{\pi}{2} \times \frac{C_3}{C_2 + C_3}, & \theta &= \frac{\pi}{2}, \\
(4) \quad \varphi &= +\frac{\pi}{2} \times \frac{C_3}{C_2 + C_3}, & \theta &= \frac{\pi}{2},
\end{aligned}
$$

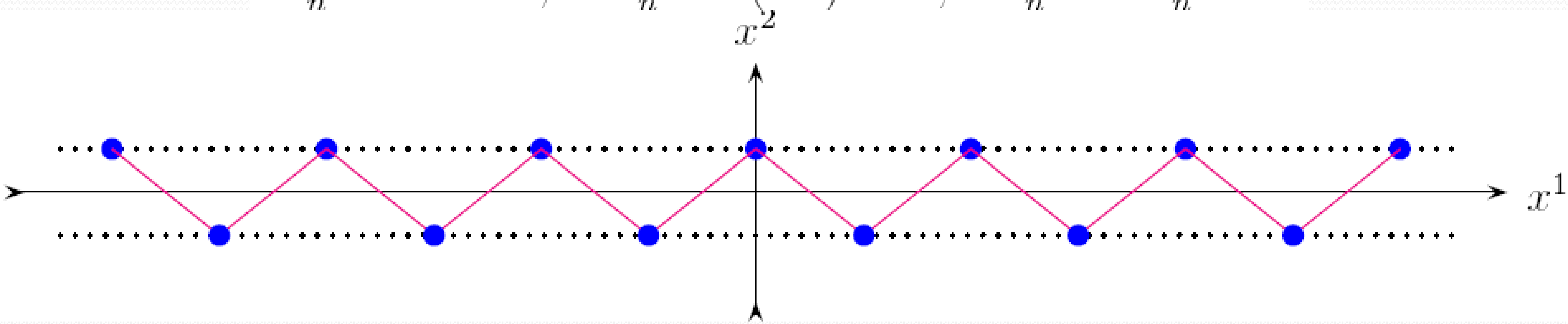
- Thus there are two degenerate ground states related by

$$\varphi \rightarrow -\varphi$$

Instanton zigzags

- The zigzag chains analyzed previously using the point charge approximation and the exact ADHM solution can be determined also using the two body approximation.
- Recall the zigzag data

$$X^1_n = n \times D, \quad X^2_n = (-1)^n \times \epsilon, \quad x^3_n = X^4_n = 0$$



- We work in the general 5d gauge coupling

$$\boxed{\frac{8\pi^2}{g_5^2(x_2, x_3, x_4)} = N_c \lambda M \Big(1 + M^2 x_4^2 + M_3^2 x_3^2 + M_2^2 x_2^2 + O(M^4 x^4)\Big)}$$

The phase diagram

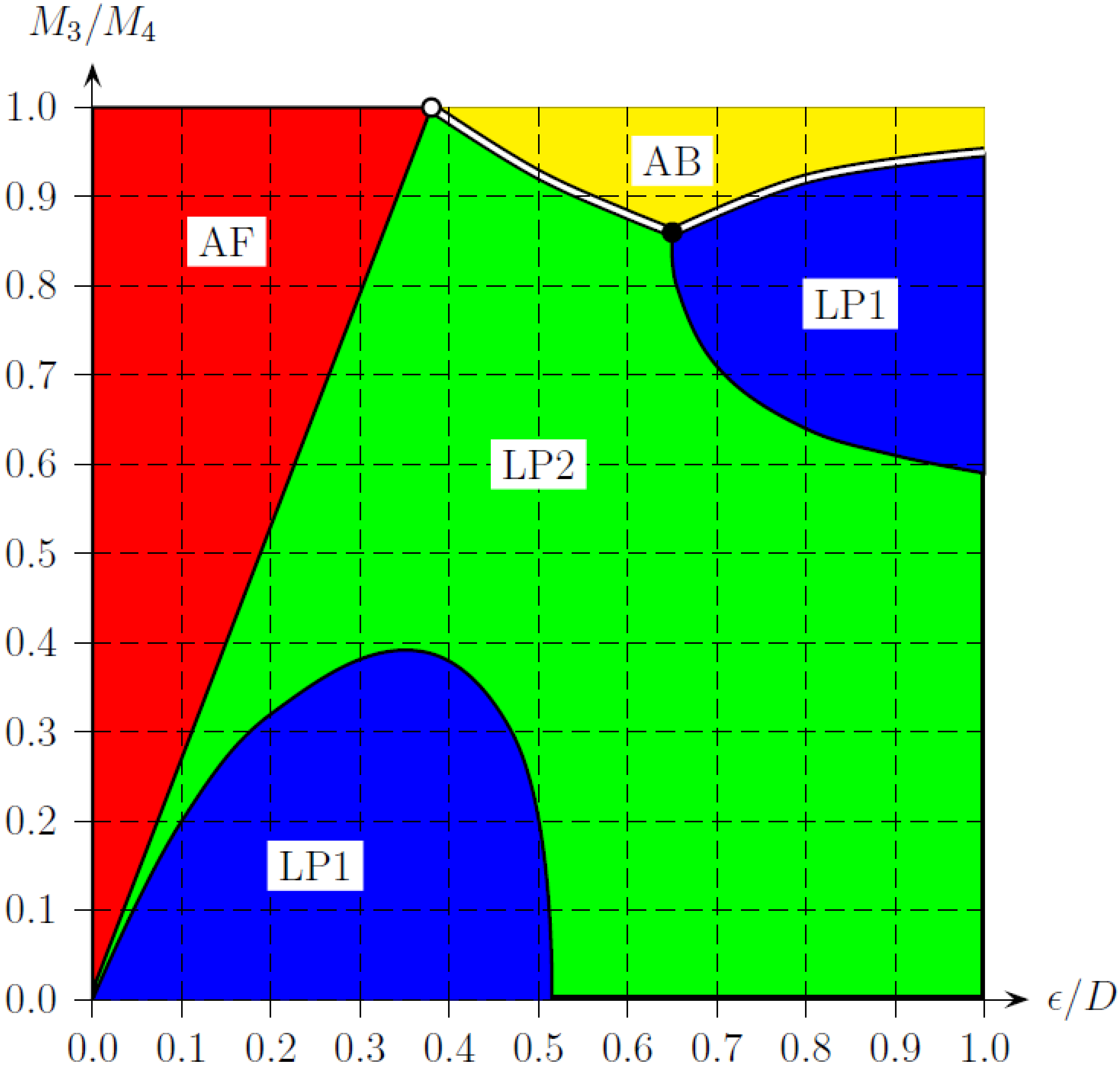
- The two body energy for the zigzag

$$\mathcal{E}^{\mathrm{int}}_{\mathrm{net}}[\mathrm{zigzag}] = \frac{N_c}{5\lambda M}\sum_{m\neq n}\frac{Q_z(m,n)}{|X_m - X_n|^2}$$

$$\text{where} \quad Q_z(m,n) = \tfrac{1}{2} + \mathrm{tr}^2\left(y_m^\dagger y_n\right) + C_3\sum_{a=1,2}\mathrm{tr}^2\left(y_m^\dagger y_n(-i\tau^a)\right) + (1-2C_3)\times\mathrm{tr}^2\left(y_m^\dagger y_n(-i\vec{\tau}\cdot\vec{N}_{mn})\right).$$

- We compute numerically the lowest energy configuration of the orientations

y_n as a function of ϵ/D and M_3/M_4



M_3/M_4
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
AF
AB
LP1
LP2
LP1
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
ϵ/D

The different phases

- The red dots on this diagram denote the antiferromagnetic pattern (AF) of instanton orientations in which the the nearest neighbors always differ by a 180° rotation around the third axis,

$$y_n = \begin{Bmatrix} \pm 1 & \text{for even } n, \\ \pm i\tau_3 & \text{for odd } n, \end{Bmatrix}, \qquad \text{same } y_n^\dagger y_{n+1} = i\tau_3 \text{ for all } n. \tag{5.11}$$

- The yellow dots denote another abelian pattern (AB) in which all nearest neighbors differ by the same $U(1) \subset SU(2)$ rotation, but now the rotation angle is $< 180°$,

$$\text{same } y_n^\dagger y_{n+1} = \exp\left(\tfrac{i}{2}\phi\tau_3\right) \text{ for all } n, \quad 0 < \phi < \pi. \tag{5.12}$$

- The blue dots denote a non-abelian link-periodic pattern (LP1) in which the relative rotation between nearest neighbors is always through a 180° angle, but the direction

The different phases

of rotation alternates between two different axes in the (12) plane, one axis for the odd-numbered instantons and the other for the even-numbered. In $SU(2)$ terms,

$$
\begin{aligned}
y^\dagger_{2k} y_{2k+1} &= \exp\left(\tfrac{i\pi}{2}\,\vec{n}_e \cdot \vec{\tau}\right) = i\vec{n}_e \cdot \vec{\tau} = +iA\tau_1 + iB\tau_2\,, \\
y^\dagger_{2k+1} y_{2k+2} &= \exp\left(\tfrac{i\pi}{2}\,\vec{n}_o \cdot \vec{\tau}\right) = i\vec{n}_o \cdot \vec{\tau} = +iA\tau_i - iB\tau_2\,,
\end{aligned}
\tag{5.13}
$$

for some $A, B \neq 0$ $(A^2 + B^2 = 1)$.

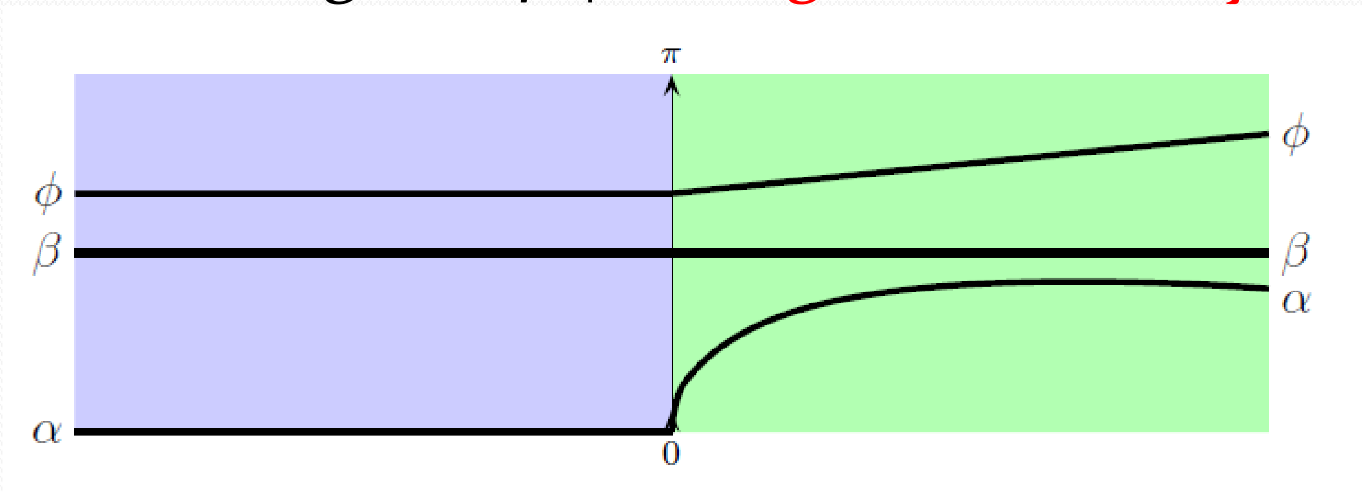
- The green dots denote another non-abelian link-periodic pattern (LP2). Again, the relative rotation between nearest neighbors is always through a 180° angle, but the direction of rotation alternates between two different axes. However, this time the two axes no longer lie withing the (12) plane, thus

$$
\begin{aligned}
y^\dagger_{2k} y_{2k+1} &= iA\tau_1 + iB\tau_2 + iC\tau_3\,, \\
y^\dagger_{2k+1} y_{2k+2} &= iA\tau_1 - iB\tau_2 - iC\tau_3\,,
\end{aligned}
\tag{5.14}
$$

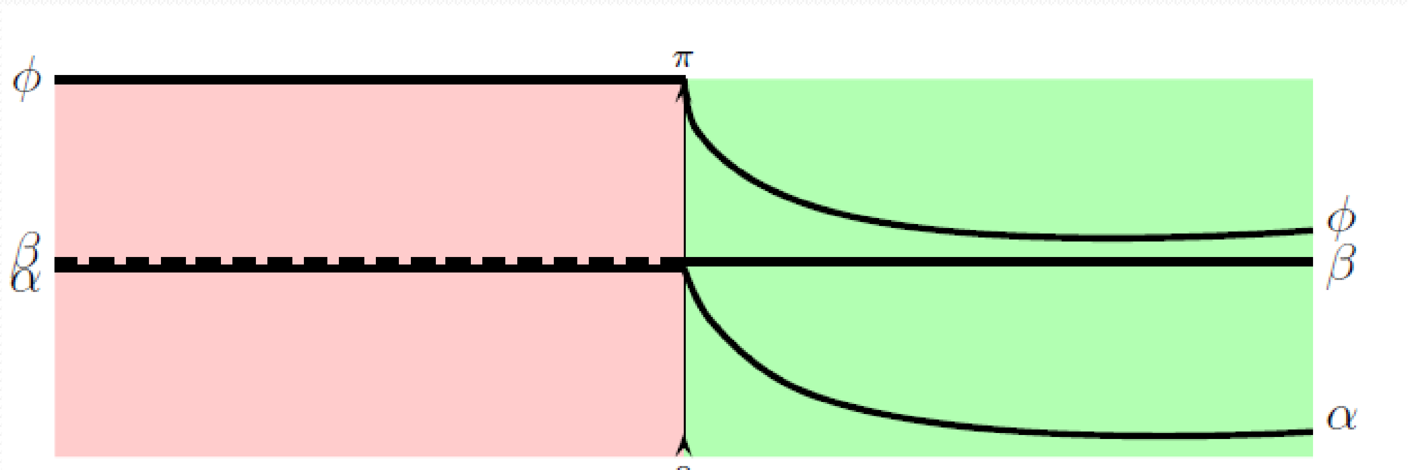
where A, B, C all $\neq 0$ $(A^2 + B^2 + C^2 = 1)$.

The phase transitions

- Both transitions from LP 1 to LP2 are second order all three angles α,β,ϕ change continuously

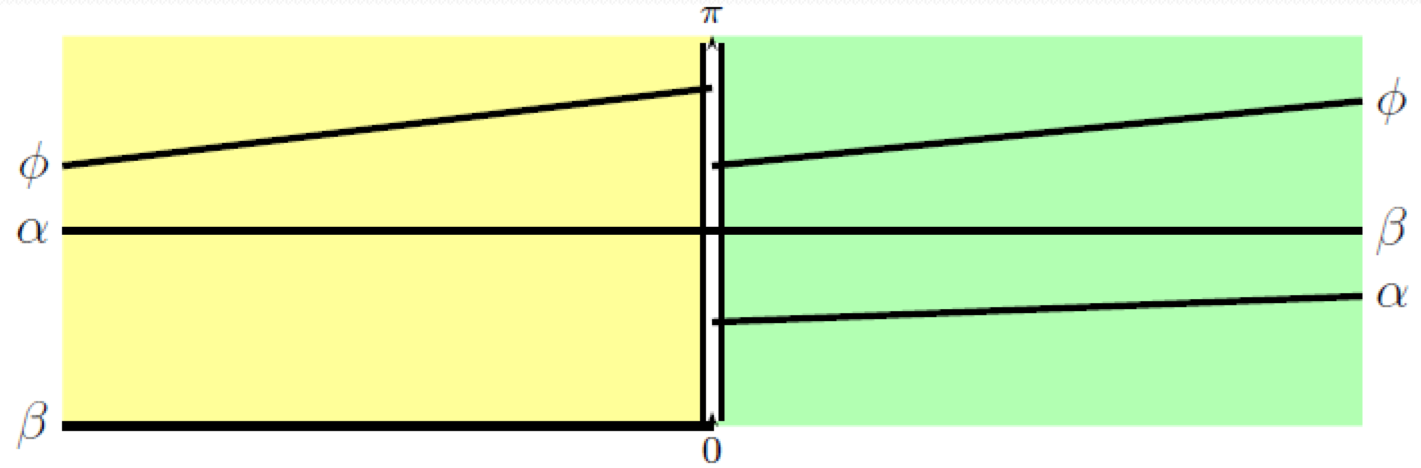


- Likewise the transition between LP2 and AF

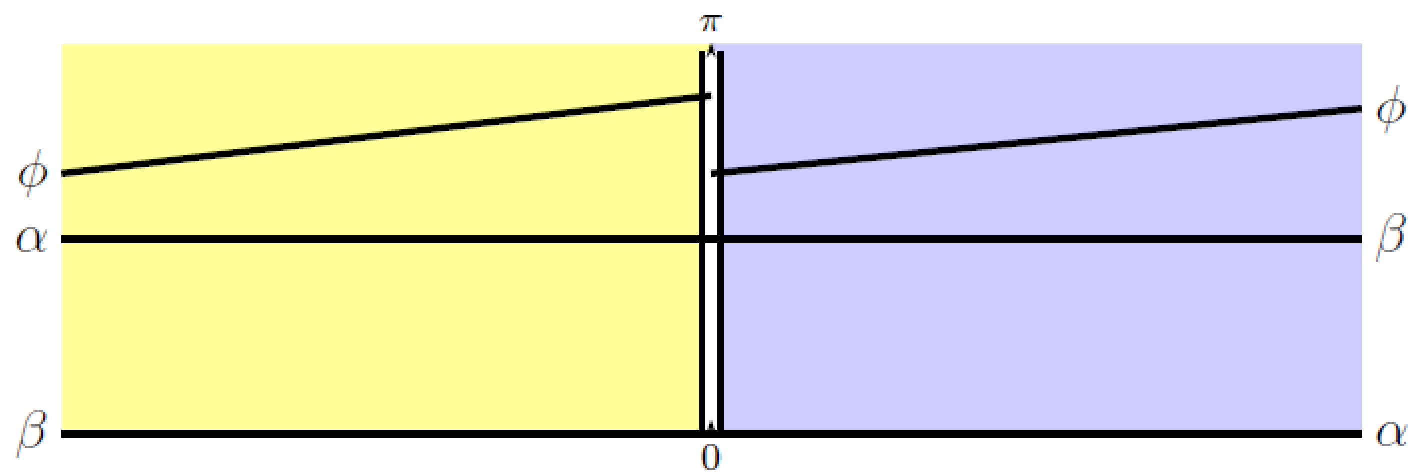


The phase transitions

- The transition between AB and LP1 and LP2 are first order

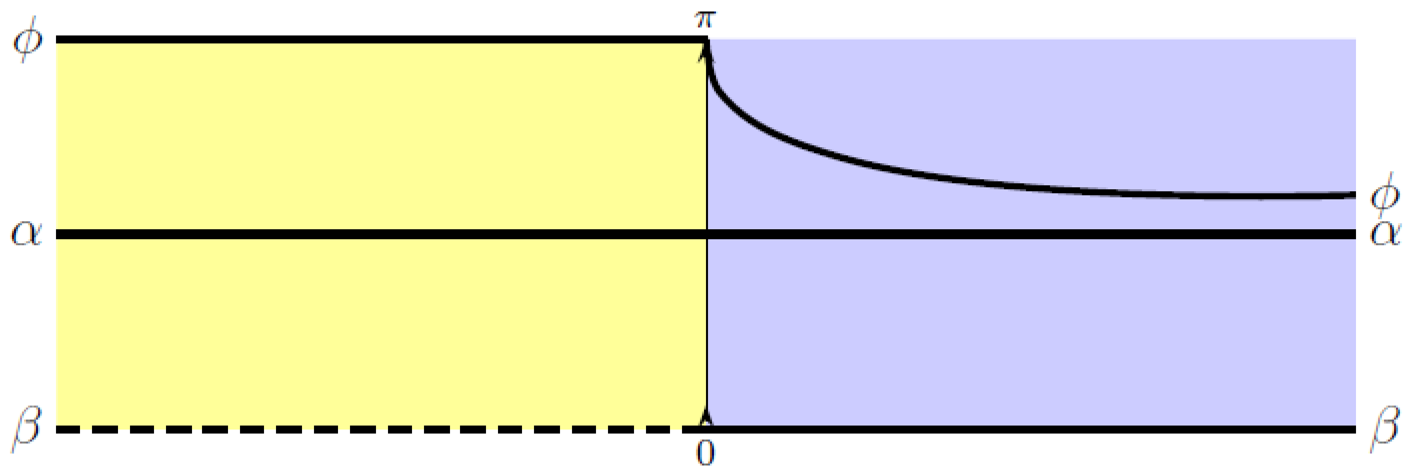


or



The phase transitions

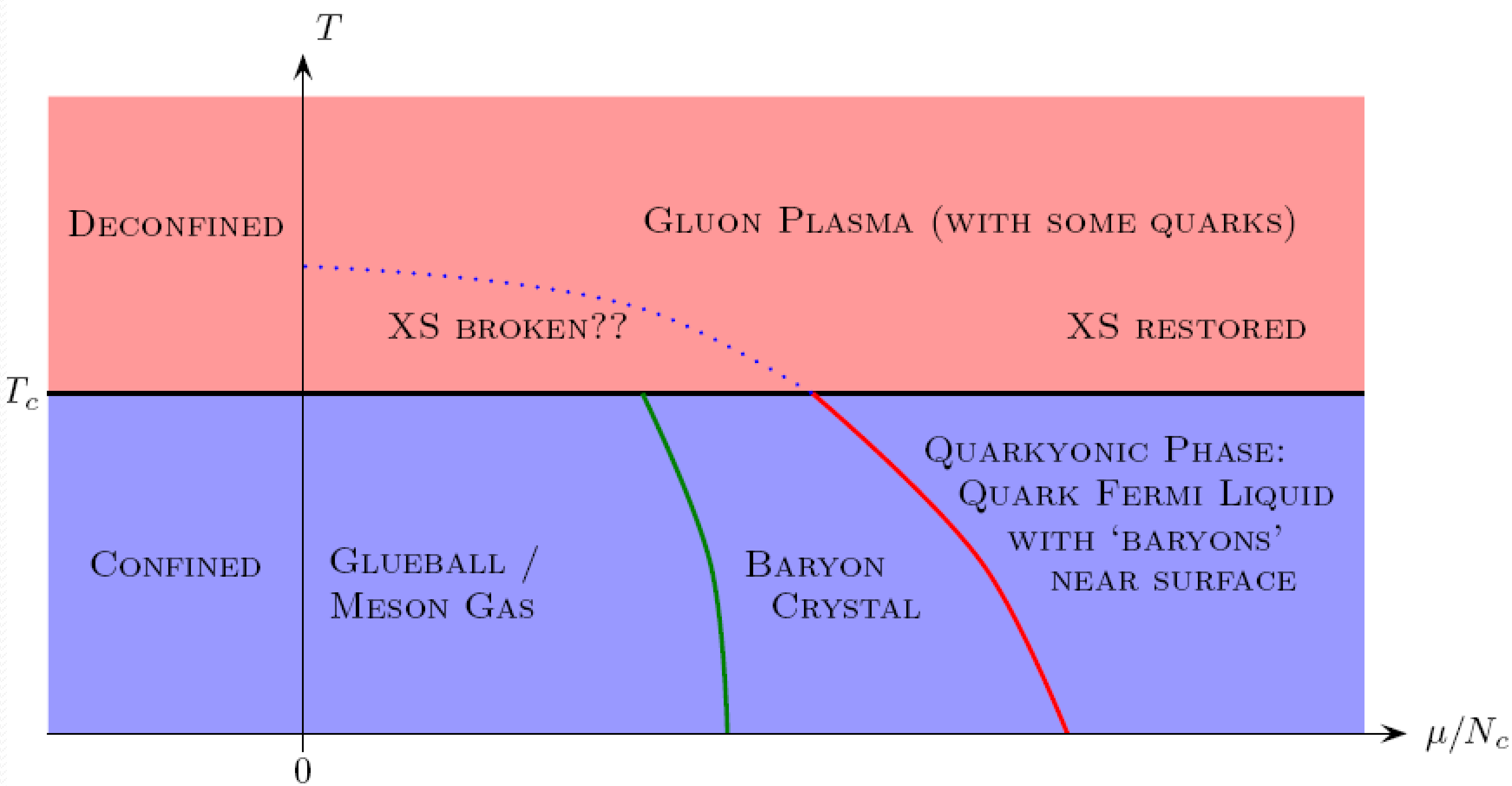
- There are two triple points of the phase diagram
- At the origin there is no triple point
- The black dot at $(\epsilon/D) \approx 0.65,\ (M_3/M_4) \approx 0.86$ is an ordinary triple point between AB and LP1 LP2
- The white circle a $(M_3/M_4) = 1,\ (\epsilon/D) \approx 0.38$ is a critical triple point between AF and AB of second order



The holographic 2CD phase diagram

Large N Phase diagram

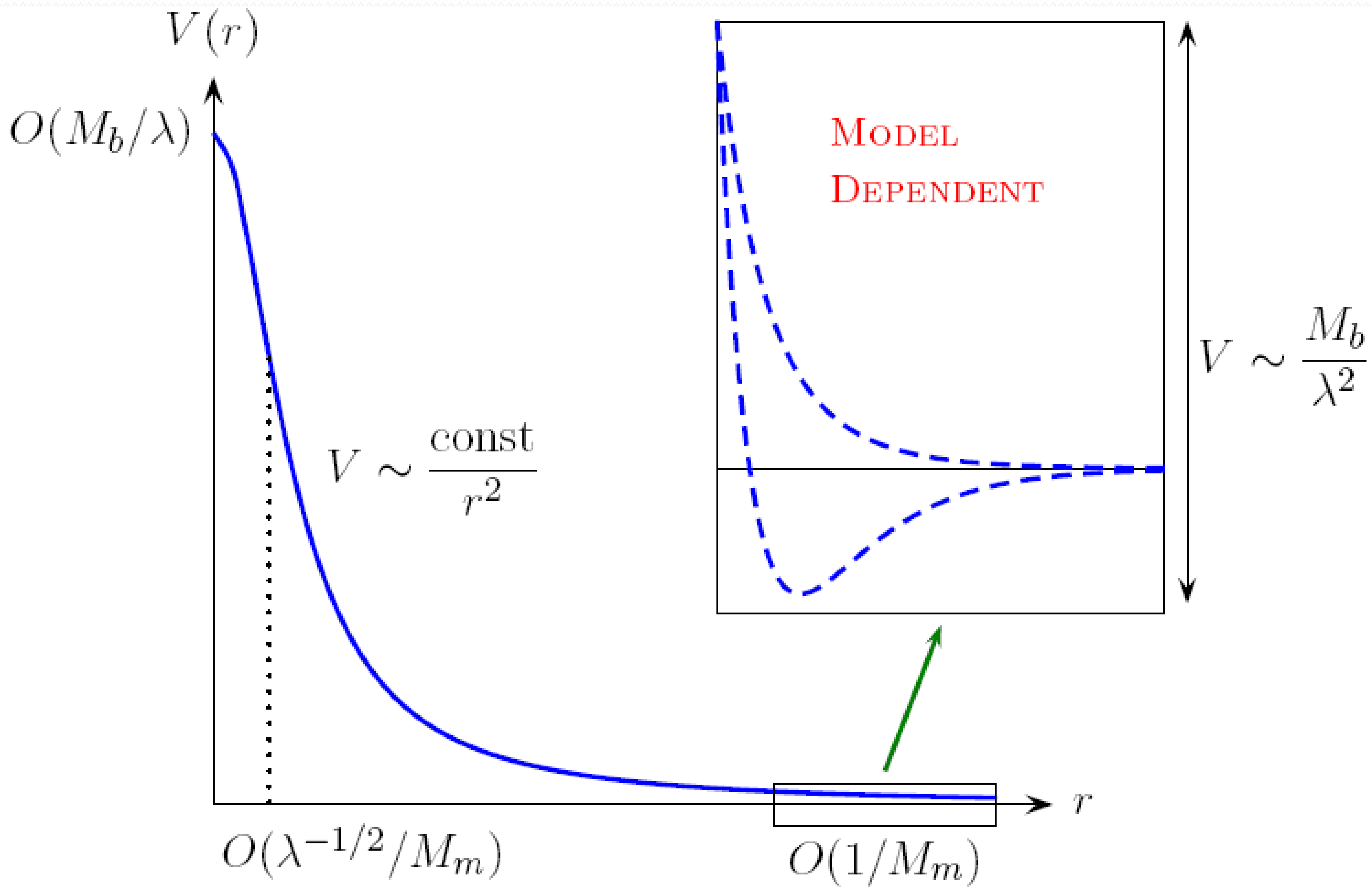
- We can summarize in terms of the holographic QCD phase diagram in the (temperature, chemical potential plane)



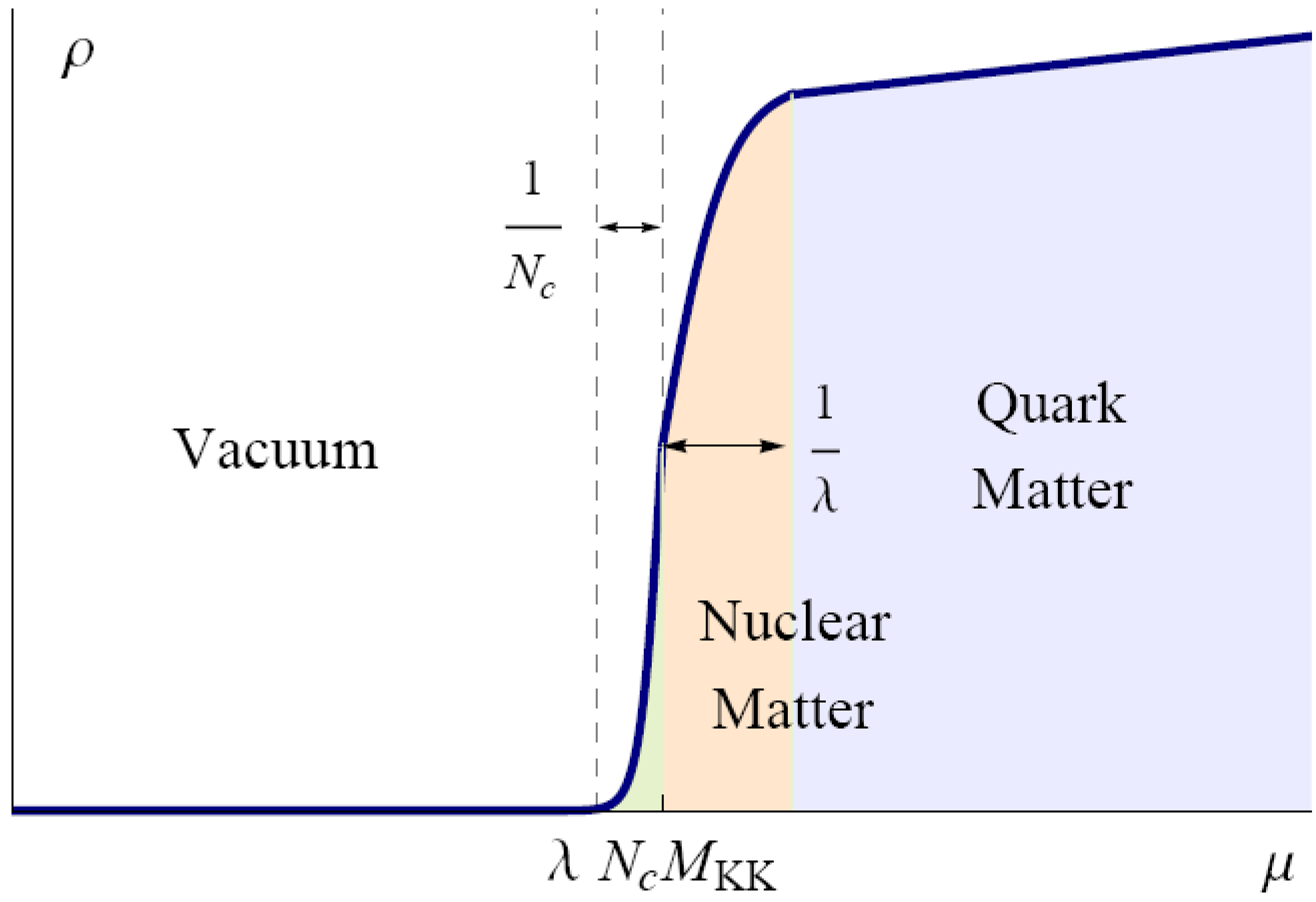
Large N Phase diagram

- Large $N_c \to \infty$ but fixed $N_f = 2$ or 3.
- Gluons dominate QGP.
- Sharp confinement-deconfinement transition at T_c almost independent on μ.
- No color SC or CFL in a quark liquid at high $\mu_q = \mu/N_c$.
- For $\mu_q >> \Lambda_{QCD}$ the quarks form a weakly coupled Fermi liquid. But near the Fermi surface, the quarks and the holes combine into meson-like and baryon-like excitations $\Longrightarrow$ the quarkyonic phase.
- $M_{baryon} \propto N_c \to \infty$ while M_{meson} and $M_{glueball}$ stay finite.
- No baryons in glueball/meson gas for $T < T_c$, $\mu < M_{baryon}$.

Generic effects of large λ

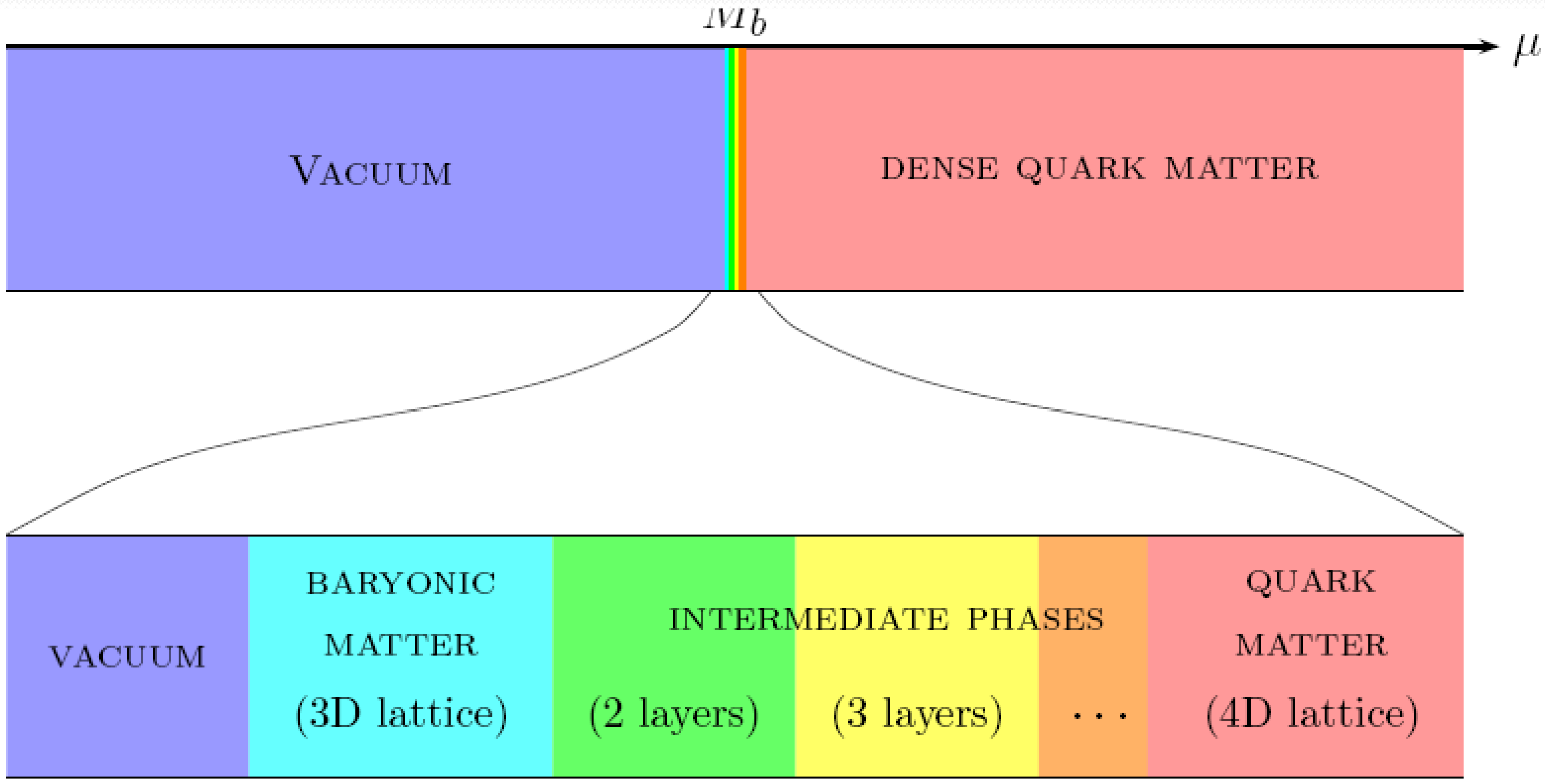


Transitions at large Nc and large λ



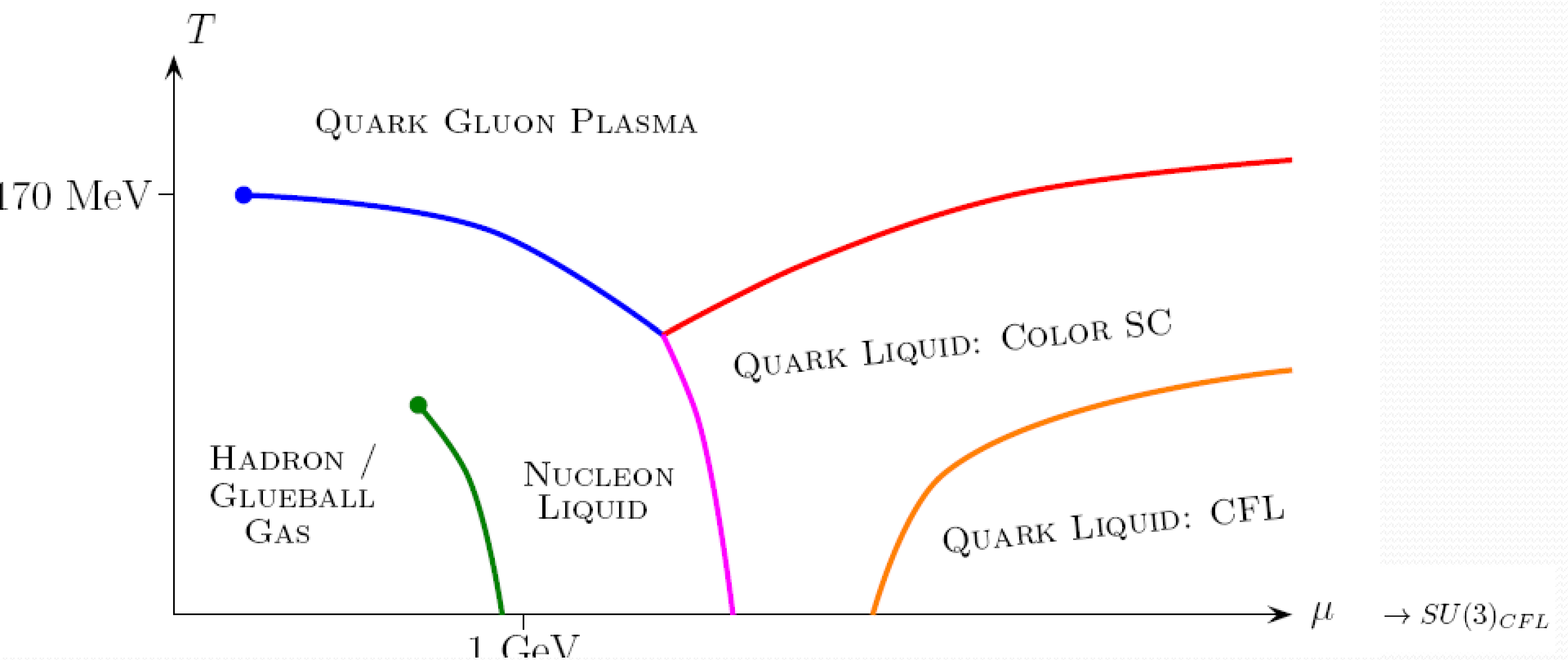
Generic effects of large λ

- $V \ll M_b \Longrightarrow$ transitions between different phases of cold nuclear matter happen very close to $\mu = M_b \Longrightarrow$ need to zoom into the $\mu \sim M_b$ region of the phase diagram to see all the phases. At $T = 0$



QCD Phase diagram

- This is to be compared with the "lore" of QCD phase diagram at finte Nc



Summary

- The holographic stringy picture for a baryon favors a baryonic vertex that is immersed in the flavor brane
- Baryons as instantons lead to a picture that is similar to the Skyrme model.
- We showed that on top of the repulsive hard core due to the abelian field there is an attraction potential due the scalar interaction in the generalized Sakai Sugimoto model.

Summary

- The is no `` nuclear physics” in the gSS model
- We showed that in the DKS model one may be able to get an attractive interaction at the far zone with an almost cancelation which will resolve the binding energy puzzle.
- We showed that the holographic nuclear matter takes the form of a lattice of instantons
- We found that there is a second order phase + a first order transitions that drives a chain of instantons into a zigzag structure namely to split into two sub-lattices separated along the holographic direction
- Using 2-instanton approximation we found a rich phase structure of nuclear matter

The phase transitions

- At large densities the straight chain of instantons is unstable against formation of a zigzag ($\varepsilon \neq 0$) in the holographic dimension.

- There is a second order phase transition, which takes the straight chain to the zigzag.

- For small amplitude of the zigzag the neighboring instantons remain antiparallel as in the ($\varepsilon = 0$) case.

- At some larger density (zigzag amplitude), the relative orientation of instantons changes from $\phi = \pi$ to $\phi \simeq 117°$. This occurs in a first order transition.

The phase transitions

- For densities larger than the one of the first order phase transition orientation changes smoothly to asymptotical value $\pi/2$.

- That it is the neighboring instantons in each of the two layers prefer to orient themselves in an antiferromagnetic way, $\phi = \pi$.

- Notice that the orientation twist between instantons never becomes non-abelian.