A unifying description of Dark Energy (& modified gravity)

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Outline

1. Motivations
2. Main formalism
3. Illustration: Horndeski's theories
4. Link with observations

Based on J. Gleyzes, D.L., F. Piazza & F. Vernizzi, 1304.4840 [hep-th]

Introduction & motivations

- **Plethora of dark energy models**:
  - Dynamical dark energy: quintessence, K-essence
  - Modified gravity

- Large amount of data from future large scale cosmological surveys (DES, LSST, Euclid, etc…)

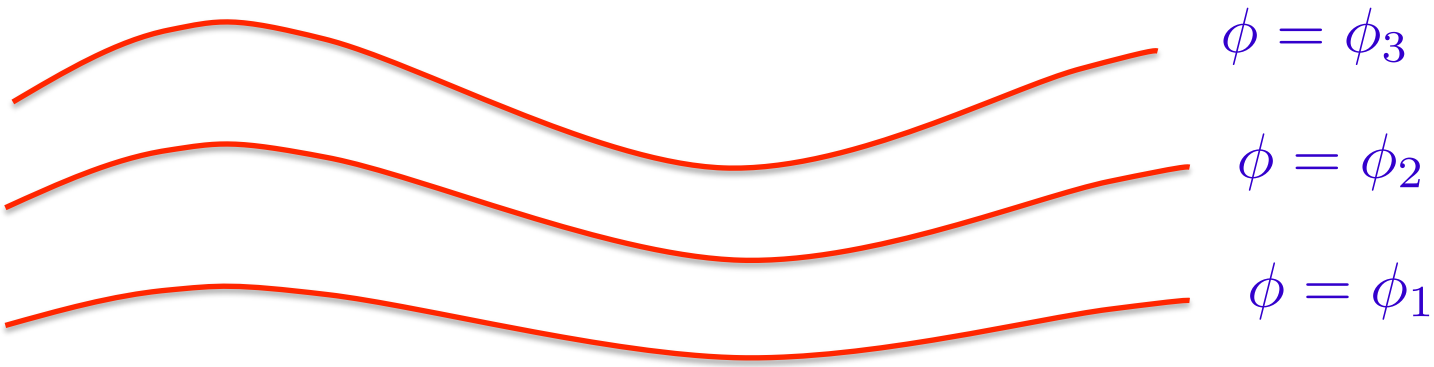
- Goal: **effective description** as a bridge between models and observations.

- Assumptions:
  - **Single scalar field** models
  - All matter fields minimally coupled to the same metric $g_{\mu\nu}$

ADM approach

- The scalar field defines a **preferred slicing**

  Constant time hypersurfaces = uniform field hypersurfaces



- **ADM decomposition** based on this preferred slicing

ADM approach

- **Basic ingredients:**

$$n_\mu = -\frac{\partial_\mu \phi}{\sqrt{-(\partial\phi)^2}} \qquad\qquad h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$$

  - Intrinsic curvature tensor $\quad {}^{(3)}R_{\mu\nu}$
  - Extrinsic curvature tensor $\quad K_{\mu\nu} = h_{\mu\sigma}\nabla^\sigma n_\nu \qquad K = \nabla_\mu n^\mu$

- **Generic action:** $S = \int d^4x \sqrt{-g}\, L(t; N, K, \mathcal{S}, \mathcal{R}, \mathcal{Z}, \dots)$

$$K \equiv K^\mu{}_\mu \,, \quad \mathcal{S} \equiv K_{\mu\nu}K^{\mu\nu}, \quad \mathcal{R} \equiv {}^{(3)}R \,, \quad \mathcal{Z} \equiv {}^{(3)}R_{\mu\nu}\,{}^{(3)}R^{\mu\nu}$$

Note: dependence on ${}^{(3)}R^{\mu\nu} K_{\mu\nu}$ can be reabsorbed.

Expansion of the Lagrangian

- Separation between FLRW background and perturbations

$$\delta K \equiv K - 3H, \quad \delta K_{\mu\nu} \equiv K_{\mu\nu} - H\, h_{\mu\nu}$$

which implies $\mathcal{S} \equiv K_{\mu\nu}K^{\mu\nu} = 3H^2 + 2H\delta K + \delta K^{\mu}{}_{\nu}\, \delta K^{\nu}{}_{\mu}$

- **Lagrangian up to … first order**

$$L(N, K, \mathcal{S}, \mathcal{R}, \mathcal{Z}) \quad = \quad \bar{L} + L_N\, \delta N + \mathcal{F}\delta K + L_{\mathcal{R}}\, \delta\mathcal{R} + \dots$$

with $\mathcal{F} \equiv L_K + 2HL_S$

Using $\mathcal{F}\,\delta K = \mathcal{F}(K - 3H)$

and $\int d^4x\sqrt{-g}\,\mathcal{F}K = -\int d^4x\sqrt{-g}\, n^{\mu}\nabla_{\mu}\mathcal{F} = -\int d^4x\sqrt{-g}\frac{\dot{\mathcal{F}}}{N}$

$$L(N, K, \mathcal{S}, \mathcal{R}, \mathcal{Z}) = \bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F} + (\dot{\mathcal{F}} + L_N)\,\delta N + L_{\mathcal{R}}\,\delta\mathcal{R} + \dots$$

Expansion (up to second order)

Up to **quadratic order** in the perturbations, one finds

$$
\begin{aligned}
L(N, K, \mathcal{S}, \mathcal{R}, \mathcal{Z}) &= \bar{L} - \dot{\mathcal{F}} - 3H\mathcal{F} + (\dot{\mathcal{F}} + L_N)\,\delta N + L_{\mathcal{R}}\,\delta\mathcal{R} \\
&+ \frac{\mathcal{A}}{2}\,\delta K^2 + L_{\mathcal{S}}\,\delta K^\mu{}_\nu \delta K^\nu{}_\mu + \left(\frac{1}{2} L_{NN} - \dot{\mathcal{F}}\right)\delta N^2 \\
&+ \frac{1}{2} L_{\mathcal{R}\mathcal{R}}\,\delta\mathcal{R}^2 + \mathcal{B}\,\delta K \delta N + \mathcal{C}\,\delta K \delta\mathcal{R} + L_{N\mathcal{R}}\,\delta N \delta\mathcal{R} + L_{\mathcal{Z}} \delta\mathcal{Z}
\end{aligned}
$$

with the coefficients

$$
\begin{aligned}
\mathcal{A} &\equiv 4H^2 L_{\mathcal{S}\mathcal{S}} + 4H L_{\mathcal{S}K} + L_{KK} \\
\mathcal{B} &\equiv 2H L_{\mathcal{S}N} + L_{KN}\,, \\
\mathcal{C} &\equiv 2H L_{\mathcal{S}\mathcal{R}} + L_{K\mathcal{R}} \\
\mathcal{F} &\equiv 2H L_{\mathcal{S}} + L_K
\end{aligned}
$$

Background equations

- FLRW metric: $ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j$

- Homogeneous action:

$$S_0 = \int dt d^3x \, N \, a^3 \, L \left[N, K = \frac{3H}{N}, \mathcal{S} = \frac{3H^2}{N^2}, \mathcal{R} = 0, \mathcal{Z} = 0 \right]$$

- **Friedmann equations**

$$\delta S_0 = \int dt d^3x \left[a^3 \left(\bar{L} + L_N - 3H\mathcal{F} \right) \delta N + \left(\bar{L} - 3H\mathcal{F} - \dot{\mathcal{F}} \right) \delta(a^3) \right]$$

$$\bar{L} + L_N - 3H\mathcal{F} = 0 \, , \qquad \bar{L} - 3H\mathcal{F} - \dot{\mathcal{F}} = 0$$

- Including matter

$$\bar{L} + L_N - 3H\mathcal{F} = \rho_m \, , \qquad \bar{L} - 3H\mathcal{F} - \dot{\mathcal{F}} = -p_m$$

Quadratic Lagrangian

- Explicit gauge choice: $$h_{ij} = a^2(t)e^{2\zeta}\delta_{ij}\ , \qquad N_i = \partial_i\psi$$

- Quadratic Lagrangian: $$\mathcal{L}_2 = \mathcal{L}_2\left[\zeta, \delta N, \partial^2\psi\right]$$

$$\mathcal{L}_2 \supset \frac{1}{a}\left[\frac{1}{2}\left(\mathcal{A} + 2L_{\mathcal{S}}\right)(\partial^2\psi)^2 + 4\mathcal{C}\ \partial^2\psi\,\partial^2\zeta + 2\left(4L_{\mathcal{RR}} + 3L_{\mathcal{Z}}\right)(\partial^2\zeta)^2\right]$$

- One can get rid of higher spatial derivatives by imposing

$$\mathcal{A} + 2L_{\mathcal{S}} = 0\ , \quad \mathcal{C} = 0\ , \quad 4L_{\mathcal{RR}} + 3L_{\mathcal{Z}} = 0$$

The momentum constraint then reduces to: $$\delta N = \frac{4L_{\mathcal{S}}}{\mathcal{B} + 4HL_{\mathcal{S}}}\,\dot{\zeta}$$

Quadratic Lagrangian

- One finally obtains the quadratic Lagrangian in the form

$$\mathcal{L}_2 = \frac{a^3}{2}\left[\mathcal{L}_{\dot\zeta\dot\zeta}\,\dot\zeta^2 + \mathcal{L}_{\partial_i\zeta\partial_i\zeta}\,\frac{(\partial_i\zeta)^2}{a^2}\right]$$

with the coefficients

$$\mathcal{L}_{\dot\zeta\dot\zeta} \equiv \left(L_{NN} + 2L_N - 6H\mathcal{B} - 12H^2 L_{\mathcal{S}}\right)\mathcal{D}^2 + 12L_{\mathcal{S}}$$

$$\mathcal{L}_{\partial_i\zeta\partial_i\zeta} \equiv 4\left[L_{\mathcal{R}} - \frac{1}{a}\frac{d}{dt}\left(a\mathcal{D}(L_{\mathcal{R}} + L_{N\mathcal{R}})\right)\right] \qquad \mathcal{D} \equiv \frac{4L_{\mathcal{S}}}{\mathcal{B} + 4HL_{\mathcal{S}}}$$

- Absence of ghosts: $\mathcal{L}_{\dot\zeta\dot\zeta} > 0$

- Effective speed of sound: $c_s^2 = -\dfrac{\mathcal{L}_{\partial_i\zeta\partial_i\zeta}}{\mathcal{L}_{\dot\zeta\dot\zeta}}$

Simple example: K-essence

Armendariz-Picon et al. 00

- K-essence Lagrangian

$$L = P(\phi, X)\,, \qquad X \equiv \nabla_\mu \phi \nabla^\mu \phi$$

- Total Lagrangian in the ADM formulation

$$L = \frac{1}{2}\left(\mathcal{S} - K^2 + \mathcal{R}\right) + P(X, \phi) \qquad X = -\frac{\dot{\phi}^2}{N^2}$$

- Background Lagrangian and its derivatives:

$$\bar{L} = P - 3H^2,\ L_N = -2XP_X,\ L_K = -3H,\ L_\mathcal{S} = \frac{1}{2},\ L_\mathcal{R} = \frac{1}{2}$$

$$\mathcal{F} = 2HL_\mathcal{S} + L_K = -2H,\ \mathcal{A} = -1,\ \mathcal{B} = 0,\ \mathcal{C} = 0$$

- Speed of sound: $$c_s^2 = \frac{P_X}{P_X + 2XP_{XX}}$$

Link with the EFT formalism

Creminelli et al. 06, Cheung et al. 07; Creminelli et al. 08; Gubitosi et al. 12

- Action up to quadratic order (with no higher derivatives)

$$
\begin{aligned}
L \;=\; & \frac{M_*^2}{2} f(t) R - \Lambda(t) - c(t) g^{00} + \frac{M_2^4(t)}{2} (\delta g^{00})^2 - \frac{m_3^3(t)}{2} \delta K \delta g^{00} \\
& - m_4^2(t) \left(\delta K^2 - \delta K^\mu{}_\nu \, \delta K^\nu{}_\mu \right) + \frac{\tilde{m}_4^2(t)}{2} \, {}^{(3)}\!R \, \delta g^{00}
\end{aligned}
$$

- Background evolution described by only three functions $f(t), \Lambda(t), c(t)$

$$
c + \Lambda = 3 M_*^2 \left(f H^2 + \dot{f} H \right) , \quad \Lambda - c = M_*^2 \left(2 f \dot{H} + 3 f H^2 + 2 \dot{f} H + \ddot{f} \right)
$$

- Link between the ADM & EFT formulations

$$
R = {}^{(3)}\!R + K_{\mu\nu} K^{\mu\nu} - K^2 + 2 \nabla_\nu (n^\nu \nabla_\mu n^\mu - n^\mu \nabla_\mu n^\nu) \qquad g^{00} = -\frac{1}{N^2}
$$

Explicit dictionary for the EFT coefficients

Generalized Galileons

Horndeski 74; Nicolis et al. 08; Deffayet et al. 09 & 11

- Most general action for a scalar field leading to at most second order equations of motion (Horndeski '74)

  Combination of the following four Lagrangians

$$
\begin{aligned}
L_2 &= G_2(\phi, X)\,, \qquad \text{with} \quad X \equiv \phi^{;\mu}\phi_{;\mu} \\
L_3 &= G_3(\phi, X)\Box\phi\,, \\
L_4 &= G_4(\phi, X)R - 2G_{4X}(\phi, X)(\Box\phi^2 - \phi^{;\mu\nu}\phi_{;\mu\nu})\,, \\
L_5 &= G_5(\phi, X)G_{\mu\nu}\phi^{;\mu\nu} + \frac{1}{3}G_{5X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{;\mu\nu}\phi^{;\mu\nu} + 2\,\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}{}_{;\sigma})
\end{aligned}
$$

- **ADM formulation** ?

- Which operators appear in the quadratic Lagrangian ?

Generalized Galileons: ADM form

Uniform scalar field slicing

– Unit normal vector $n_\mu = -\gamma\,\phi_{;\mu} \quad \text{with} \quad \gamma = \frac{1}{\sqrt{-X}}$

– Induced metric $h_{\mu\nu} = n_\mu n_\nu + g_{\mu\nu}$

– Using $K_{\mu\nu} = h^\sigma_\mu\, n_{\nu;\sigma} \qquad \dot{n}_\mu = n^\nu\, n_{\mu;\nu}$

one can write $n_{\nu;\mu} = K_{\mu\nu} - n_\mu \dot{n}_\nu$

$$\phi_{;\mu\nu} = -\gamma^{-1}(K_{\mu\nu} - n_\mu\dot{n}_\nu - n_\nu\dot{n}_\mu) + \frac{\gamma^2}{2}\phi^{;\lambda}X_{;\lambda}n_\mu n_\nu$$

– The Gauss-Codazzi relations are also useful.

Generalized Galileons: ADM form

- Lagrangian $L_3 = G_3(\phi, X)\Box\phi$

  Introducing the auxiliary function $F_3(\phi, X)$ such that

  $$G_3 \equiv F_3 + 2XF_{3X}$$

  one finds

  $$L_3 = 2\,(-X)^{3/2}F_{3X}K - XF_{3\phi}$$

- EFT coefficients:

$$L = \frac{M_*^2}{2}f(t)R - \Lambda(t) - c(t)g^{00} + \frac{M_2^4(t)}{2}(\delta g^{00})^2 - \frac{m_3^3(t)}{2}\,\delta K\delta g^{00}$$
$$- m_4^2(t)\left(\delta K^2 - \delta K^{\mu}{}_{\nu}\,\delta K^{\nu}{}_{\mu}\right) + \frac{\tilde{m}_4^2(t)}{2}\,{}^{(3)}R\,\delta g^{00}$$

Generalized Galileons: ADM form

- Lagrangian $L_4 = G_4(\phi, X)R - 2G_{4X}(\phi, X)(\Box\phi^2 - \phi^{;\mu\nu}\phi_{;\mu\nu})$

  Using the Gauss-Codazzi equations, one finally gets

$$L_4 = G_4{}^{(3)}R + (2XG_{4X} - G_4)(K^2 - K_{\mu\nu}K^{\mu\nu}) - 2\sqrt{-X}\,G_{4\phi}K$$

- In the quadratic Lagrangian, all terms are present, but with $m_4^2 = \tilde{m}_4^2$

$$\begin{aligned} L \;=\; & \frac{M_*^2}{2} f(t)R - \Lambda(t) - c(t)g^{00} + \frac{M_2^4(t)}{2}(\delta g^{00})^2 - \frac{m_3^3(t)}{2}\,\delta K \delta g^{00} \\ & - m_4^2(t)\left(\delta K^2 - \delta K^\mu{}_\nu\, \delta K^\nu{}_\mu\right) + \frac{\tilde{m}_4^2(t)}{2}\,{}^{(3)}R\,\delta g^{00} \end{aligned}$$

Generalized Galileons: ADM form

$$L_5 = \frac{1}{3} G_{5X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{;\mu\nu}\phi^{;\mu\nu} + 2\,\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}{}_{;\sigma}) + G_5(\phi, X) G_{\mu\nu}\phi^{;\mu\nu}$$

Using the auxiliary function $F_5(\phi, X)$ such that $G_{5X} = F_{5X} + \frac{F_5}{2X}$
one finally gets

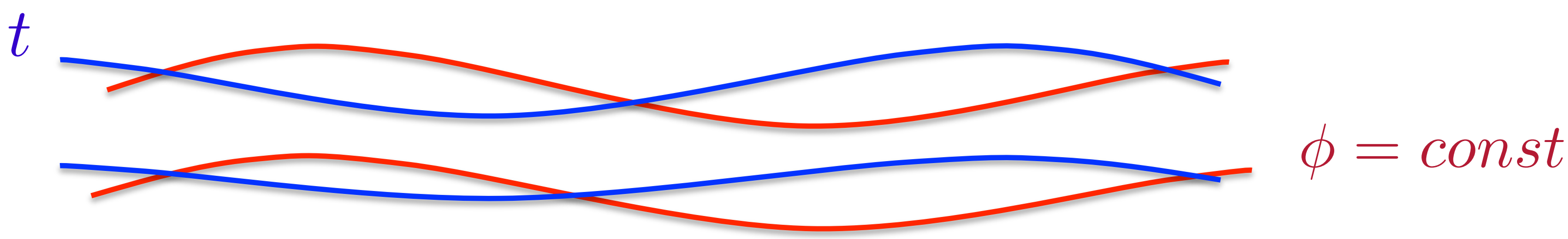
$$L_5 = \frac{1}{2} X (G_{5\phi} - F_{5\phi})^{(3)}R + \frac{1}{2} X\, G_{5\phi}(K^2 - K_{\mu\nu}K^{\mu\nu}) - \sqrt{-X}\, F_5 \left(K^{\mu\nu\,(3)}R_{\mu\nu} - \frac{1}{2} K^{(3)}R \right) - \frac{1}{3}(-X)^{3/2} G_{5X}\mathcal{K}$$

with
$$\mathcal{K} \equiv K^3 - 3K K_{\mu\nu}K^{\mu\nu} + 2K_{\mu\nu}K^{\mu\sigma}K^{\nu}{}_{\sigma} = 6H^3 - 6H^2K + 3HK^2 - 3H\mathcal{S} + \mathcal{O}(3)$$

- All operators are present, again with the restriction $m_4^2 = \tilde{m}_4^2$

Perturbations in an arbitrary gauge

- Description in an arbitrary slicing ?



- Transformation $t \to t + \pi(t, \vec{x})$

Perturbations in an arbitrary gauge

- Stueckelberg trick: $t \to t + \pi(t, \vec{x})$

- The new quadratic action can be derived via the substitutions:

$$
\begin{aligned}
f &\to f + \dot{f}\pi + \frac{1}{2}\ddot{f}\pi^2 \, , \\
g^{00} &\to g^{00} + 2g^{0\mu}\partial_\mu \pi + g^{\mu\nu}\partial_\mu \pi \partial_\nu \pi \, , \\
\delta K_{ij} &\to \delta K_{ij} - \dot{H}\pi h_{ij} - \partial_i \partial_j \pi \, , \\
\delta K &\to \delta K - 3\dot{H}\pi - \frac{1}{a^2}\partial^2 \pi \, , \\
{}^{(3)}R_{ij} &\to {}^{(3)}R_{ij} + H(\partial_i \partial_j \pi + \delta_{ij}\partial^2 \pi) \, , \\
{}^{(3)}R &\to {}^{(3)}R + \frac{4}{a^2} H \partial^2 \pi \, .
\end{aligned}
$$

Note: the 3-dim quantities on the right are defined with respect to the new time hypersurfaces.

Perturbations in the Newtonian gauge

- Perturbed metric

$$ds^2 = -(1+2\Phi)dt^2 + a^2(t)\,(1-2\Psi)\,\delta_{ij}\,dx^i dx^j$$

- Einstein's equations
  - Generalized Poisson equation

$$-\frac{k^2}{a^2}\left[(2fM_*^2+4\tilde{m}_4^2)\Psi-(\dot{f}M_*^2-m_3^3+4Hm_4^2-4H\tilde{m}_4^2)\pi\right]+(6M_*^2H^2\dot{f}-6Hc-\dot{c}-\dot{\Lambda}+3m_3^3\dot{H})\pi$$
$$-(2c+4M_2^4)\dot{\pi}+(3M_*^2H\dot{f}+2c+4M_2^4)\Phi-3M_*^2\dot{f}\dot{\Psi}+3m_3^3(\dot{\Psi}+H\Phi)=\rho_m\Delta_m$$

  - Anisotropic equation

$$M_*^2\left[f(\Phi-\Psi)+\dot{f}\pi\right]+2\left[m_4^2\dot{\pi}+m_4^2H\pi+(m_4^2)^{\cdot}\,\pi\right]+2\tilde{m}_4^2(\Phi-\dot{\pi})=0$$

- Equation for π

Deviations from GR

- **Quasi-static approximation** for sub-horizon scales (time derivatives are neglected)

$$\mathcal{M}(t,k)\begin{pmatrix}\Phi\\ \Psi\\ \pi\end{pmatrix} \equiv \begin{pmatrix} D_\Phi & D_\Psi & D_\pi \\ E_\Phi & E_\Psi & E_\pi \\ F_\Phi & F_\Psi & F_\pi \end{pmatrix}\begin{pmatrix}\Phi\\ \Psi\\ \pi\end{pmatrix} = \begin{pmatrix}0\\ 0\\ \rho_m\Delta_m\end{pmatrix}$$

- **Effective Newton's constant**

$$-\frac{k^2}{a^2}\Phi \equiv 4\pi\, G_{\rm eff}(t,k)\,\rho_m\Delta_m \qquad\qquad 4\pi G_{\rm eff}(t,k) = -\frac{k^2}{a^2}\left[\mathcal{M}^{-1}\right]_{13}$$

- Ratio between the two gravitational potentials

$$\Psi \equiv \gamma(t,k)\,\Phi \qquad\qquad \gamma = [{\rm com}(\mathcal{M})]_{32}/[{\rm com}(\mathcal{M})]_{31}$$

Conclusions

- Unified treatment of single-field models of dark energy/modified gravity

- Models leading to *second-order* equations of motion for *linear* perturbations can be parametrized by 7 time-dependent functions (3 needed for the background)

$$
\begin{aligned}
L \; = \; & \frac{M_*^2}{2} f(t) R - \Lambda(t) - c(t) g^{00} + \frac{M_2^4(t)}{2} (\delta g^{00})^2 - \frac{m_3^3(t)}{2} \delta K \delta g^{00} \\
& - m_4^2(t) \left(\delta K^2 - \delta K^\mu{}_\nu \, \delta K^\nu{}_\mu \right) + \frac{\tilde{m}_4^2(t)}{2} \, {}^{(3)}R \, \delta g^{00}
\end{aligned}
$$

- Horndeski's theories: $m_4^2(t) = \tilde{m}_4^2(t)$

- Link with observations (effective Newton constant, ratio of the two gravitational potentials)

Theoretical models ⟺ **Effective description** ⟺ **Observations**