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Modeling scalar fields consistent with positive mass

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Nozawa and Shiromizu, Physical Review D89, 023011(2014)

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1. Introduction

Positive mass theorem

~ Positive mass theorem

Schoen&Yau 1981, Witten 1981, Gibbons et al 1983,…

$$M \geqq 0$$

$$M = 0 \Leftrightarrow \text{ Minkowski/anti-deSitter}$$

for GR, SUGRA, regular spacetimes, energy condition, …

The existence of ground state

Restriction on theories

Scalar potentials consistent with positive mass

Boucher 1984, Townsend 1985

$$L = R - \frac{1}{2}(\nabla\phi)^2 - U(\phi)$$

$$\Rightarrow \quad U(\phi) = 8\left(\frac{dW(\phi)}{d\phi}\right)^2 - 12\big(W(\phi)\big)^2$$

Cf) SUGRA ,W superpotential

Summary of our work

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action

$$S = \int d^4x\sqrt{-g}\left[R + 2K(\phi, X) + L_{matter}\right] \qquad X = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$$

The cases consistent with positive mass are

(i)

$$K = X - U(\phi) = X - 8\left(\frac{dW(\phi)}{d\phi}\right)^2 + 12(W(\phi))^2$$

Canonical form with “superpotential”

(ii)

$$K = 4\sqrt{2}\,\frac{dW(\phi)}{d\phi}(-X)^{1/2} + 12(W(\phi))^2$$

No cosmological solution

- strong restriction
- classical stability is automatically guaranteed

2. Positive mass theorem

Back to Witten 1981

Positivity: essence

$$\gamma^i \nabla_i \varepsilon = 0 \qquad\qquad \varepsilon : \text{spinor}$$

$$M \sim \int_{S_\infty} \varepsilon^+ \nabla_i \varepsilon dS^i$$

$$= \int_\Sigma \nabla^i (\varepsilon^+ \nabla_i \varepsilon) d\Sigma$$

$$= \int_\Sigma \left(|\nabla \varepsilon|^2 + \varepsilon^+ \nabla^2 \varepsilon \right) d\Sigma$$

$$\sim \int_\Sigma \left(|\nabla \varepsilon|^2 + T_{00} |\varepsilon|^2 \right) d\Sigma \geq 0$$

If the energy-momentum tensor satisfies the energy condition, we can prove the positivity of mass.

Rigidity

$$M \sim \int_{\Sigma} \left(|\nabla \varepsilon|^2 + T_{00} |\varepsilon|^2 \right) d\Sigma = 0$$

$$\Rightarrow \quad \nabla \varepsilon = 0$$

$$\Rightarrow \quad R_{abcd} = 0$$

Minkowski spacetime

Precisely

$$N^{\mu\nu} := -i\left(\bar{\varepsilon}\gamma^{\mu\nu\rho}\nabla_\rho\varepsilon - \overline{\nabla_\rho\varepsilon}\gamma^{\mu\nu\rho}\varepsilon\right) \qquad \bar{\varepsilon} = i\varepsilon^+\gamma^0$$

$$\frac{1}{2}\int_{\partial\Sigma} N_{\mu\nu}dS^{\mu\nu} = -\int_\Sigma \nabla_\nu N^{\mu\nu}u_\mu d\Sigma$$

u^μ : future directed unit normal vector to Σ

$$\nabla_\nu N^{\mu\nu} = 2i\nabla_\rho\varepsilon\gamma^{\mu\nu\rho}\nabla_\nu\varepsilon - G^\mu_\nu V^\nu$$

$$V^\mu := i\bar{\varepsilon}\gamma^\mu\varepsilon$$

$$\gamma^i\nabla_i\varepsilon = 0$$

≧0 (energy condition)

$$\Rightarrow 8\pi GM = \frac{1}{2}\int_{\partial\Sigma} N_{\mu\nu}dS^{\mu\nu} = \int_\Sigma \left(2|\nabla_i\varepsilon|^2 - 8\pi G T^0{}_\mu V^\mu\right)d\Sigma \geq 0$$

3. Einstein-scalar system

Nozawa & Shiromizu 2014

Model

action

$$S = \int d^4x \sqrt{-g}\left[R + 2K(\phi, X) + 2L_{matter}\right]$$

$$X = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$$

$$G_{\mu\nu} = T^{(\phi)}_{\mu\nu} + T^{(matter)}_{\mu\nu}$$

$$\boxed{T^{(\phi)}_{\mu\nu} = \partial_X K \nabla_\mu \phi \nabla_\nu \phi + K g_{\mu\nu}}$$

does not satisfy the (dominant) energy condition in general

Mass expression

$$\hat{\nabla}_\mu \varepsilon = (\nabla_\mu + A_\mu)\varepsilon, \;\; \gamma^{\mathrm{i}} \hat{\nabla}_i \varepsilon = 0$$

$$8\pi GM = \int_\Sigma d\Sigma \left[2i\overline{\hat{\nabla}_\rho \varepsilon} \gamma^{\mu\nu\rho} \hat{\nabla}_\nu \varepsilon - G^\mu_\nu V^\nu + S^\mu \right] u_\mu$$

$$\begin{cases} G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \\ V^\mu = i\bar{\varepsilon}\gamma^\mu \varepsilon \\ S^\mu := -i\bar{\varepsilon}\gamma^{\mu\nu\rho} F_{\mu\nu} \varepsilon \\ F_{\mu\nu} = 2(\partial_{[\mu} A_{\nu]} + A_{[\mu} A_{\nu]}) \end{cases}$$

Required condition

We imposed

$$\overline{A}_\nu \gamma^{\mu\nu\rho} = \gamma^{\mu\nu\rho} A_\nu$$

Otherwise, non-controllable terms appear

$$\nabla_\nu \hat{N}^{\mu\nu} = 2i\overline{\hat{\nabla}_\rho \varepsilon}\gamma^{\mu\nu\rho}\hat{\nabla}_\nu \varepsilon - G^\mu_\nu V^\nu - \frac{i}{2}\bar{\varepsilon}(\overline{F}_{\nu\rho}\gamma^{\mu\nu\rho} + \gamma^{\mu\nu\rho}F_{\nu\rho})\varepsilon$$

$$\boxed{-i\bar{\varepsilon}(\overline{A}_\nu\gamma^{\mu\nu\rho} - \gamma^{\mu\nu\rho}A_\nu)\hat{\nabla}_\rho\varepsilon + i\hat{\nabla}_\rho\varepsilon(\overline{A}_\nu\gamma^{\mu\nu\rho} - \gamma^{\mu\nu\rho}A_\nu)\varepsilon}$$

$$F_{\mu\nu} = 2(\partial_{[\mu}A_{\nu]} + A_{[\mu}A_{\nu]})$$

Strategy

$$\gamma^i \hat{\nabla}_i \varepsilon = 0 \quad \varepsilon : \mathrm{spinor}$$

$$M \sim \int_{S_\infty} dS_i \varepsilon^+ \hat{\nabla}^i \varepsilon = \int_\Sigma \nabla_i (\varepsilon^+ \hat{\nabla}^i \varepsilon) d\Sigma$$

$$T^{(\phi)}_{\mu\nu} = \partial_X K \nabla_\mu \phi \nabla_\nu \phi + K g_{\mu\nu}$$

$$= \int_\Sigma \left(|\hat{\nabla}\varepsilon|^2 + T^{matter}_{00} + T^{(\phi)}_{00} + S^0 \right) d\Sigma$$

Einstein eq.

Look for the theory for scalar field to have the form $|\delta\lambda|^2$ for

Look at detail more

$$A_\mu = W(\phi)\gamma_\mu$$

$$\hat{\nabla}_\mu \varepsilon = (\nabla_\mu + A_\mu)\varepsilon$$

$$8\pi GM = \int_\Sigma d\Sigma \left[2i\overline{\hat{\nabla}_\rho \varepsilon} \gamma^{\mu\nu\rho} \hat{\nabla}_\nu \varepsilon - G^\mu_\nu V^\nu + S^\mu \right] u_\mu$$

$$F_{\mu\nu} = 2(\partial_{[\mu} A_{\nu]} + A_{[\mu} A_{\nu]})$$

$$\begin{aligned} S^\mu &= -i\bar{\varepsilon}\gamma^{\mu\nu\rho} F_{\mu\nu}\varepsilon \\ &= -4i\bar{\varepsilon}\gamma^{\mu\nu}\varepsilon \nabla_\nu \phi \partial_\phi W + 12V^\mu W^2 \\ &= i\overline{\delta\lambda}\gamma^\mu \delta\lambda + V^\nu \left[f^2 \nabla^\mu \phi \nabla_\nu \phi + \left(-\frac{1}{2} f^2 (\nabla\phi)^2 - 8 f^{-2} (\partial_\phi W)^2 + 12W^2 \right) \right] \end{aligned}$$

$$\delta\lambda := \frac{1}{\sqrt{2}} \left(f(\phi, X)\gamma^\mu \nabla_\mu \phi - 4 f^{-1}(\phi, X) \frac{dW(\phi)}{d\phi} \right) \varepsilon$$

If

$$\begin{cases} \partial_X K = f^2 \\ K = f^2 X - 8 f^{-2} \left(\dfrac{dW(\phi)}{d\phi} \right)^2 + 12 (W(\phi))^2 \end{cases}$$

$$\Rightarrow \quad \boxed{S^\mu = i\overline{\delta\lambda}\gamma^\mu \delta\lambda + T^{(\phi)\mu}_\nu V^\nu}$$

Then

$$\delta\lambda := \frac{1}{\sqrt{2}}\left(f(\phi, X)\gamma^{\mu}\nabla_{\mu}\phi - 4 f^{-1}(\phi, X)\frac{dW(\phi)}{d\phi}\right)\varepsilon$$

$$\begin{cases} \partial_X K = f^2 \\ K = f^2 X - 8 f^{-2}\left(\dfrac{dW(\phi)}{d\phi}\right)^2 + 12\big(W(\phi)\big)^2 \end{cases}$$

$$XK_X - K - \frac{8W_{\phi}^2}{K_X} = -12W(\phi)^2$$

$$\partial_X\left(XK_X - K - \frac{8W_{\phi}^2}{K_X}\right) = K_{XX}\left(X + \frac{8W_{\phi}^2}{K_X^2}\right) = 0$$

$$(i)K_{XX} = 0 \quad K = X - U(\phi) = X - 8\left(\frac{dW(\phi)}{d\phi}\right)^2 + 12\big(W(\phi)\big)^2$$

$$(ii)X + \frac{8W_{\phi}^2}{K_X^2} = 0 \quad K = 4\sqrt{2}\frac{dW(\phi)}{d\phi}(-X)^{1/2} + 12\big(W(\phi)\big)^2$$

Case (ii)

$$K = 4\sqrt{2}\,\frac{dW(\phi)}{d\phi}(-X)^{1/2} + 12\big(W(\phi)\big)^2$$

For homogeneous-isotropic spacetimes,

$$\phi = \phi(t) \quad \Rightarrow X = \dot{\phi}^2/2 > 0$$

Due to the factor of $(-X)^{1/2}$, the case (ii) does not work

Summary

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action

$$S = \int d^4x\sqrt{-g}\left[R + 2K(\phi, X) + L_{matter}\right] \qquad X = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$$

(i)

$$K = X - U(\phi) = X - 8\left(\frac{dW(\phi)}{d\phi}\right)^2 + 12\left(W(\phi)\right)^2$$

Canonical form with “superpotential”

(ii)

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No cosmological solution

4. Future issues

Future issues

$$\left.\begin{aligned} &\overline{A}_\nu \gamma^{\mu\nu\rho} = \gamma^{\mu\nu\rho} A_\nu \\ &A_\mu = W(\phi)\gamma_\mu \end{aligned}\right\} \text{general enough?}$$

Extension to more general cases/modified gravity?

Some basics

Covariant derivative

Local Lorentz transformation $\varepsilon^{\hat{\mu}\hat{\nu}}$

$$\psi \to \psi + \delta\psi, \quad \delta\psi = \frac{1}{4}\varepsilon^{\hat{\mu}\hat{\nu}}\gamma_{\hat{\mu}\hat{\nu}}\psi$$

$$\nabla_\mu \psi = \left(\partial_\mu + \frac{1}{4}\omega^{\hat{\alpha}\hat{\beta}}{}_\mu \gamma_{\hat{\alpha}\hat{\beta}}\right)\psi$$

$$\omega^{\hat{\alpha}\hat{\beta}}{}_\mu \to \omega^{\hat{\alpha}\hat{\beta}}{}_\mu + \delta\omega^{\hat{\alpha}\hat{\beta}}{}_\mu$$

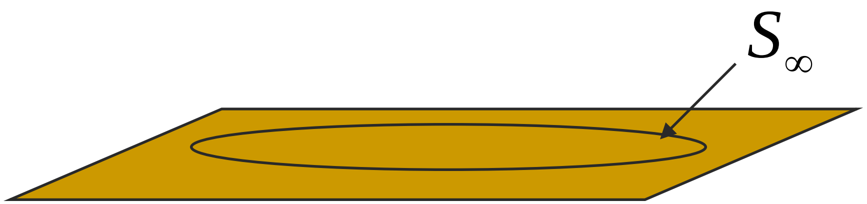
$$\delta\omega^{\hat{\alpha}\hat{\beta}}{}_\mu = \varepsilon^{\hat{\alpha}}{}_{\hat{\gamma}}\omega^{\hat{\gamma}\hat{\beta}}{}_\mu + \varepsilon^{\hat{\beta}}{}_{\hat{\gamma}}\omega^{\hat{\alpha}\hat{\gamma}}{}_\mu - \partial_\mu \varepsilon^{\hat{\alpha}\hat{\beta}}$$

$$\Rightarrow \quad \delta(\nabla_\mu \psi) = \frac{1}{4}\varepsilon^{\hat{\alpha}\hat{\beta}}\gamma_{\hat{\alpha}\hat{\beta}}\nabla_\mu \psi$$

$$D_\mu e_{\hat{\alpha}\nu} := \partial_\mu e_{\hat{\alpha}\nu} + \omega_{\hat{\alpha}\hat{\beta}\mu} e^{\hat{\beta}}_\nu - \Gamma^\rho_{\mu\nu} e_{\hat{\alpha}\rho} = \nabla_\mu e_{\hat{\alpha}\nu} + \omega_{\hat{\alpha}\hat{\beta}\mu} e^{\hat{\beta}}_\nu = 0$$

$$\Rightarrow \quad D_\mu g_{\alpha\beta} = 0$$

Witten spinor



(Σ, q): (n - 1) - dim. spacelike hypersurface

$$\gamma^i D_i \varepsilon = 0$$ (Witten equation)

$$D_i\varepsilon = (\partial_i + {}^{(n-1)}\Gamma_i)\varepsilon, \qquad {}^{(n-1)}\Gamma_i = -\frac{1}{8}(e^{\hat{k}})^j D_i (e^{\hat{l}})_j [\gamma_{\hat{l}}, \gamma_{\hat{k}}]$$

$$g_{kl}(e_{\hat{i}})^k (e_{\hat{j}})^l = \delta_{\hat{i}\hat{j}}$$

We have solutions which are asymptotically approaches a constant spinor

$$\varepsilon \xrightarrow[r\to\infty]{} \varepsilon_0$$

Proof

$$\frac{1}{2}D^iD_i|\varepsilon|^2 = |D\varepsilon|^2 + \frac{1}{4}{}^{(n-1)}R|\varepsilon|^2$$

$$8\pi M_{ADM}|\varepsilon_0|^2 = \frac{1}{2}\int dS_i(\varepsilon^+ D^i\varepsilon + c.c.) = \int_\Sigma \left[|D\varepsilon|^2 + \frac{1}{4}{}^{(n-1)}R|\varepsilon|^2\right]$$

- ${}^{(n-1)}R \geq 0 \quad \Rightarrow \quad M_{ADM} \geq 0$
- $M_{ADM} = 0 \Rightarrow D_i\varepsilon = 0 \Rightarrow [D_i, D_j]\varepsilon \propto {}^{(n-1)}R_{ijkl}[\gamma^k, \gamma^l]\varepsilon = 0$
- Σ is flat space

Surface integral

$$\varepsilon^+ D_1\varepsilon = \varepsilon_0^+\partial_1\varepsilon + \varepsilon_0^{+\,(n-1)}\Gamma_1\varepsilon_0$$

1

$$\boxed{\begin{aligned} \gamma^i D_i\varepsilon = 0 \quad &\Rightarrow \quad \gamma^i(\partial_i\varepsilon + {}^{(n-1)}\Gamma_i\varepsilon_0) \approx 0 \\ &\Rightarrow \varepsilon_0^+\gamma_1\,\gamma^i(\partial_i\varepsilon + {}^{(n-1)}\Gamma_i\varepsilon_0) \approx 0 \\ &\Rightarrow \varepsilon_0^+\partial_1\varepsilon = -\varepsilon_0^+\gamma_1\gamma^A\partial_A\varepsilon - \varepsilon_0^+\gamma_1\gamma^{i\,(n-1)}\Gamma_i\varepsilon_0 \end{aligned}}$$

$$\frac{1}{2}\int dS_1(\varepsilon^+ D^1\varepsilon + c.c.) = \int dS^1\varepsilon_0^+({}^{(n-1)}\Gamma_1 - \gamma_1\gamma^{j\,(n-1)}\Gamma_j)\varepsilon_0 - \cancel{\int dS^i\varepsilon_0\gamma_i\gamma^A\partial_A\varepsilon}$$

$$= \frac{1}{4}\int dS^i\varepsilon_0^+(\partial_j h_i^j - \partial_i h_j^j)\varepsilon_0$$

$${}^{(n-1)}\Gamma_i = \frac{1}{16}(\partial^j h_i^k - \partial^k h_i^j)[\gamma_k,\gamma_j] + O\left(\frac{1}{r^{n-1}}\right)$$

$$g_{ij} = \delta_{ij} + h_{ij},\quad (e^{\hat{i}})_j = ({}^{(0)}e^{\hat{i}})_j + \frac{1}{2}h_{kj}({}^{(0)}e^{\hat{i}})^k$$