Strong restriction on inflationary vacua from the local gauge invariance

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PTP **122** (2009) 779 [arXiv:0902.3209 [hep-th]].
PTP **122** (2010) 1207 [arXiv:0904.4415 [hep-th]].
PRD **82** (2010) 121301 [arXiv:1007.0468 [hep-th]].
PTP **125** (2011) 1067 [arXiv:1009.2947 [hep-th]].
JCAP **1105** (2011) 014 [arXiv:1103.1251 [astro-ph.CO]].
PTEP **2013** (2013) 8, 083E01 [arXiv:1209.1914 [hep-th]].
PTEP **2013** (2013) 6, 063E02 [arXiv:1301.3088 [hep-th]].
CQG **30** (2013) 233001 [arXiv:1306.4461 [hep-th]].
and the work that will soon appear.

Various IR issues

IR divergence coming from k-integral: $\approx \int_{k_{\min}} \frac{d^3k}{k^3}$

Secular growth in time $\propto (Ht)^n$

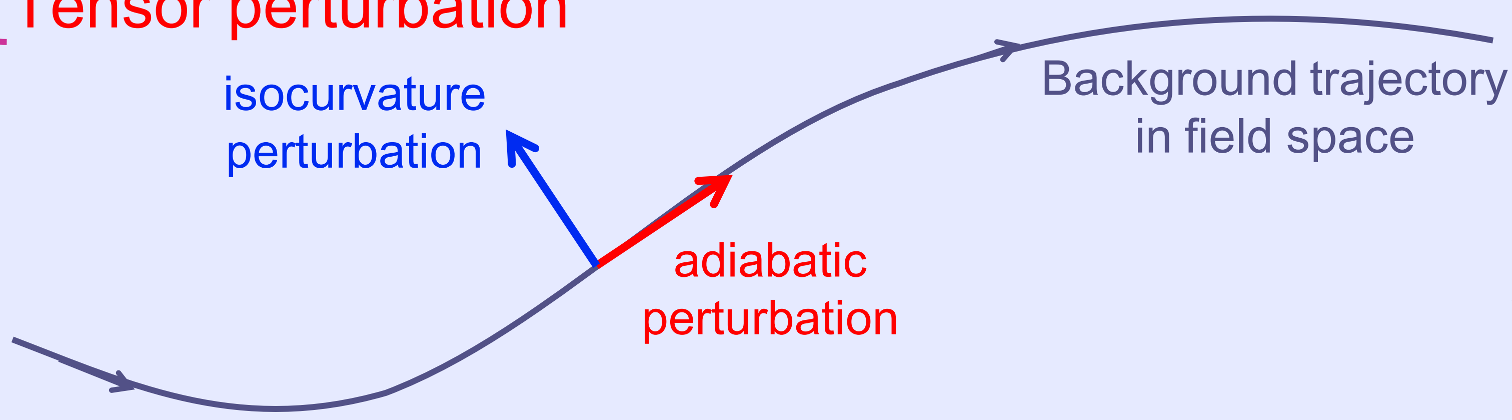
Adiabatic perturbation,

which can be locally absorbed by the choice of time slicing.

Isocurvature perturbation

≈ field theory on a fixed curved background

Tensor perturbation



IR problem for isocurvature perturbation

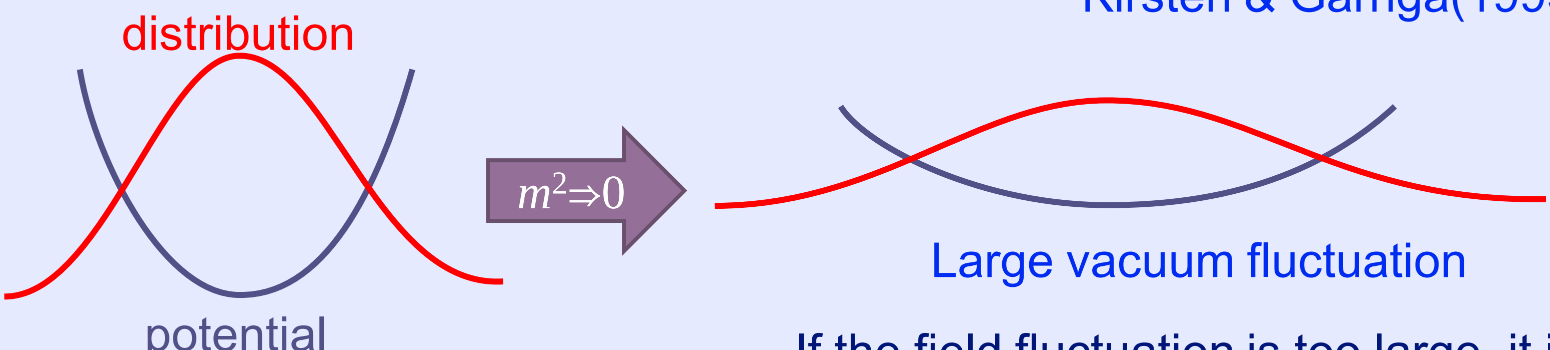
ϕ : a minimally coupled scalar field with a small mass ($m^2 \ll H^2$) in dS.

$$\left\langle \phi^2 \right\rangle^{reg} \approx \int_0^{aH} d^3k \, \frac{H^2}{k^3} \left(\frac{k}{aH} \right)^{\frac{2m^2}{3H^2}} \approx \frac{H^4}{m^2}$$

summing up only long wavelength modes beyond the Horizon scale

De Sitter inv. vac. state does not exist in the massless limit.

Allen & Folacci(1987)
Kirsten & Garriga(1993)



Large vacuum fluctuation

If the field fluctuation is too large, it is easy to imagine that a naïve perturbative analysis will break down once interaction is introduced.

Stochastic interpretation

- Let's consider local average of ϕ : (Starobinsky & Yokoyama (1994))

$$\bar{\phi} = \int_0^{aH} d^3k\, \phi_k e^{i\mathbf{k}\mathbf{x}}$$

More and more short wavelength modes participate in $\bar{\phi}$ as time goes on.

Equation of motion for $\bar{\phi}$:

$$\frac{d\bar{\phi}}{dN} = -\frac{V'(\bar{\phi})}{3H^2} + \frac{f}{H}$$

in slow roll approximation

Newly participating modes act as random fluctuation

$$\langle \phi_k \phi_{-k} \rangle \approx H^2/k^3$$

$$\Rightarrow \langle f(N) f(N') \rangle \approx H^4 \delta(N - N')$$

In the case of massless $\lambda\phi^4$: $\langle \bar{\phi}^2 \rangle \to \dfrac{H^2}{\sqrt{\lambda}}$

Namely, in the end, thermal equilibrium is realized : $V \approx T^4$

Wave function of the universe ~parallel universes

- Distant universe is quite different from ours.

- Each small region in the above picture
  gives one representation of many parallel universes.
- However: wave function of the universe
  = "a superposition of all the possible parallel universes"
  must be so to keep translational invariance of the wave fn. of the universe
- Question is "simple expectation values are really observables for us?"

“Are simple expectation values really observables for us?”

Answer will be No!

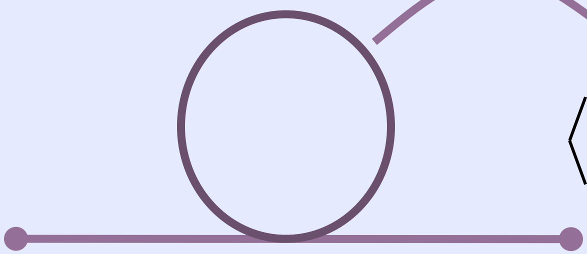
So, we need a new formulation

But the situation is quite different in single field inflation

§ IR divergence in single field inflation

Setup: 4D Einstein gravity + minimally coupled scalar field

Broadening of averaged field can be absorbed by the proper choice of time coordinate.



Factor coming from this loop:

$$\langle \zeta(y)\zeta(y)\rangle \approx \int d^3k\, \underbrace{P(k)}_{\boxed{\propto 1/k^3}} \approx \log(aH/k_{\min})$$

curvature perturbation in co-moving gauge.

scale invariant spectrum
- no typical mass scale

$$\begin{cases} \gamma_{ij} = e^{2\rho+2\zeta}\exp(h)_{ij} \\ \delta\phi = 0 \end{cases}$$

Transverse traceless

Gauge issue in single field inflation

Yuko Urakawa and T.T., PTP122: 779 arXiv:0902.3209

- In conventional cosmological perturbation theory, gauge is not completely fixed.

Time slicing can be uniquely specified: $\delta\phi = 0$ OK!

but spatial coordinates are not.

$$h_j^j = 0 = h_{i\ ,j}^{\ j}$$

Residual gauge:

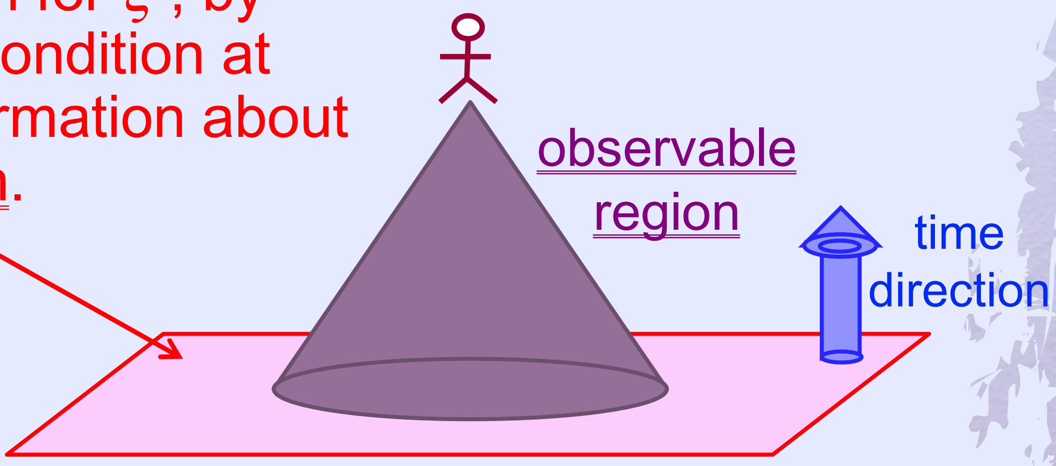
$$\delta_g h_{ij} = \xi_{i,j} + \xi_{j,i}$$

Elliptic-type differential equation for ξ^i.

$$\Delta \xi^i = \cdots$$

Not unique locally!

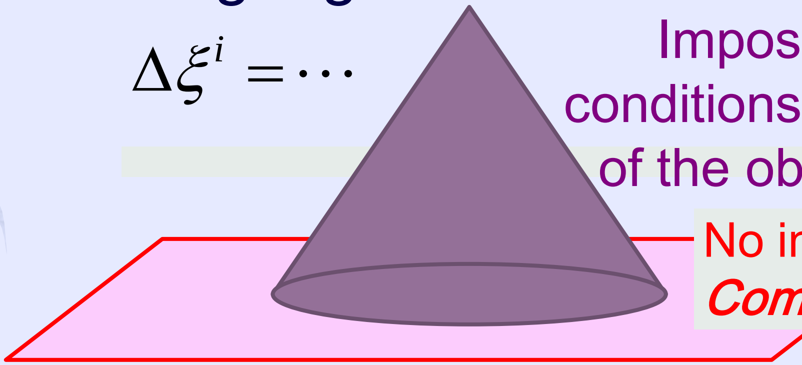
- To solve the equation for ξ^i, by imposing boundary condition at infinity, we need information about un-observable region.



Complete gauge fixing vs. Genuine gauge-invariant quantities

- Local gauge conditions.

$\Delta\xi^i = \cdots$



Imposing boundary conditions on the boundary of the observable region ⇒ But unsatisfactory? The results depend on the choice of boundary conditions. Translation invariance is lost.

No influence from outside
Complete gauge fixing ☺

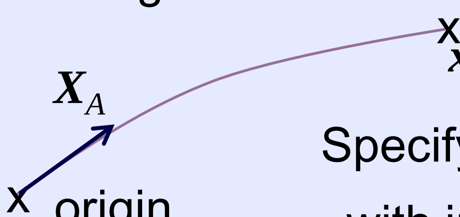
- ***Genuine*** coordinate-independent quantities.

Correlation functions for 3-d scalar curvature on ϕ=constant slice.

$\langle R(\boldsymbol{x}_1)R(\boldsymbol{x}_2)\rangle$ Coordinates do not have gauge invariant meaning.

(Giddings & Sloth 1005.1056)
(Byrnes et al. 1005.33307)

Use of geodesic coordinates:



$\boldsymbol{x}(\boldsymbol{X}_A, \lambda=1) = \boldsymbol{X}_A + \delta\boldsymbol{x}_A$

Specify the position by solving geodesic eq. $D^2x^i/d\lambda^2 = 0$

with initial condition $Dx^i/d\lambda\big|_{\lambda=0} = X^i$

$${}^gR(\boldsymbol{X}_A) := R(\boldsymbol{x}(\boldsymbol{X}_A, \lambda=1)) = R(\boldsymbol{X}_A) + \delta\boldsymbol{x}_A \nabla R(\boldsymbol{X}_A) + \ldots$$

$\langle {}^gR(\boldsymbol{X}_1)\, {}^gR(\boldsymbol{X}_2)\rangle$ should be truly coordinate independent.

Extra requirement for IR regularity

In $\delta\phi$ =0 gauge, EOM is very simple

$$\left[\partial_t^2 + (3+\varepsilon_2)\dot{\rho}\,\partial_t - e^{-2(\rho+\zeta)}\Delta\right]\zeta \approx 0$$

Only relevant terms in the IR limit were kept.

Non-linearity is concentrated on this term.

Formal solution in IR limit can be obtained as

$$\zeta = \zeta_I - 2\zeta_I \mathcal{L}^{-1} e^{-2\rho}\Delta\zeta_I + \cdots$$

$$\varepsilon_2 = -\frac{d^2}{d\rho^2}\log H$$

with $\mathcal{L}^{-1}$ being the formal inverse of $\mathcal{L} = \partial_t^2 + (3+\varepsilon_2)\dot{\rho}\,\partial_t - e^{-2\rho}\Delta$

$${}^gR \approx -4e^{-2\rho}\Delta\left[\zeta_I - \zeta_I\left(2\mathcal{L}^{-1}e^{-2\rho}\Delta + \boldsymbol{x}\cdot\partial_{\boldsymbol{x}}\right)\zeta_I + \cdots\right]$$

$$\left\langle {}^gR(\boldsymbol{x}_1)\,{}^gR(\boldsymbol{x}_2)\right\rangle \ni \underline{\left\langle \zeta_I^2 \right\rangle}\left\langle \Delta\left(2\mathcal{L}^{-1}e^{-2\rho}\Delta + \boldsymbol{x}\cdot\partial_{\boldsymbol{x}}\right)\zeta_I(\boldsymbol{x}_1) \times \Delta\left(2\mathcal{L}^{-1}e^{-2\rho}\Delta + \boldsymbol{x}\cdot\partial_{\boldsymbol{x}}\right)\zeta_I(\boldsymbol{x}_1)\right\rangle$$

IR divergent factor

IR regularity is not guaranteed for a generic quantum state!

It looks quite non-trivial to find consistent IR regular states.

However, the Euclidean vacuum state (that requires the regularity of n-point fns at $t_0 \to \pm i\,\infty$) satisfies this condition.

I will show just the outline of the proof.

Although the proof can be extended to graviton loops, we are restricted to the adiabatic scalar perturbation, for simplicity.

We consider time-dependent scale transformation: $x^i = e^{-s(t)}\tilde{x}^i$

Then, the curvature perturbation transforms as

$$\tilde{\zeta}(\tilde{x}) = \zeta(x) \qquad d\tilde{s}^2_{(3)} \equiv e^{2(\rho+\tilde{\zeta}-s)}$$

The Hamiltonian after transformation is not identical to the original one but very similar.

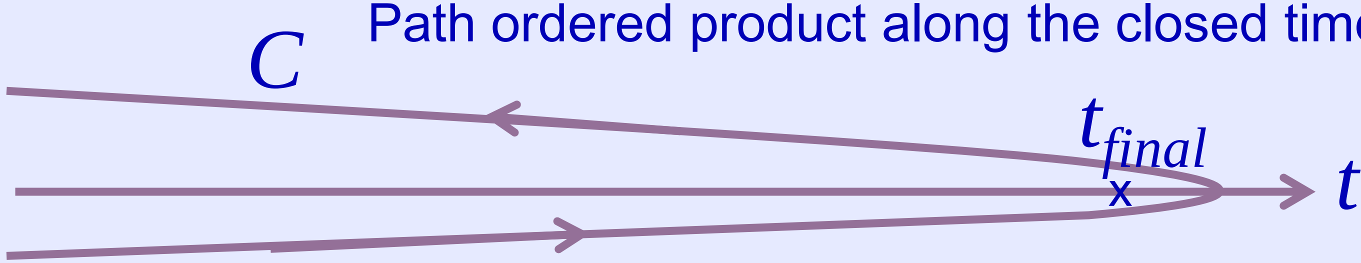
$$H[\zeta,\pi] \longrightarrow \tilde{H} = H[\tilde{\zeta}-s,\pi] + \dot{s}(\cdots)$$

due to time-dependent canonical transform

Define the unitary operator of the time evolution

$$U(t,t') = T_C \exp\left(-i\int_{t'}^{t} H\right)$$

Path ordered product along the closed time path



n-point fn: $\langle 0|U(-\infty+i\varepsilon,t_f)^g\zeta\cdots{}^g\zeta U(t_f,-\infty-i\varepsilon)|0\rangle$

Discretizing the closed time path, we insert decomposition of unity by eigenstates of the average value of ζ at each time step.

$$ {}^{g}\bar{\zeta}\left|s\right\rangle = s\left|s\right\rangle $$

$$\left\langle 0\right| U\left(-\infty+i\varepsilon, t_f\right) {}^{g}\zeta \cdots {}^{g}\zeta U\left(t_f, -\infty - i\varepsilon\right)\left|0\right\rangle$$

$$= \left\langle 0\right|\left(\prod_i U\left(t_{i+1}, t_i\right)\int ds\left|s\right\rangle\left\langle s\right|\right) {}^{g}\zeta \cdots {}^{g}\zeta \left(\prod_i U\left(t_{i+1}, t_i\right)\int ds\left|s\right\rangle\left\langle s\right|\right)\left|0\right\rangle$$

Picking up a representative value of s from each time step, we have a path of s.

Using this function $s(t)$, we move to the tilder system.

$$\supset \left\langle 0\right|\left(\prod_i \tilde{U}\left(t_{i+1}, t_i\right)\left|s(t_i)\right\rangle\left\langle s(t_i)\right|\right)\tilde{\zeta} \cdots \tilde{\zeta}\left(\prod_i \tilde{U}\left(t_{i+1}, t_i\right)\left|s(t_i)\right\rangle\left\langle s(t_i)\right|\right)\left|0\right\rangle$$

The Euclidean vacuum state is defined in the same way irrespective of the choice of $s(t)$.

$\tilde{U}$ contains $s(t)$, which can be erased by replacing it with $\bar{\tilde{\zeta}}$.

Then, one can remove $\int ds\left|s\right\rangle\left\langle s\right|$, and $\tilde{\zeta}$ in the vertices appear with differentiation or in the combination $\tilde{\zeta} - \bar{\tilde{\zeta}}$.

Summary

Quantum state should satisfy some conditions for the absence of IR divergences due to adiabatic scalar perturbation and tensor perturbation..

"Wave function must be homogeneous in the direction of background scale transformation"

Euclidean vacuum and its excited states satisfy the IR regular condition.

It requires further investigation whether there are other (non-trivial and natural) quantum states compatible with the IR regularity.