A “HAIRY” BLACK HOLE

and its stability

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1. A HAIRY BLACK HOLE

found by Andrès Anabalon

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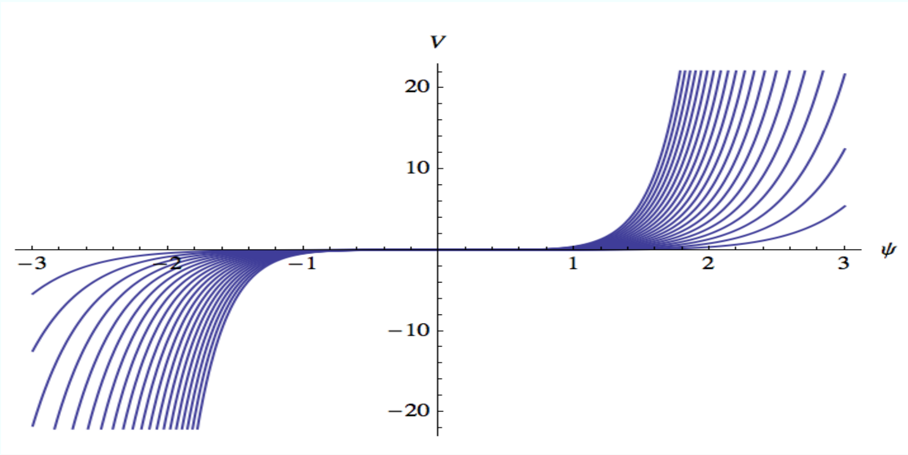
THE MODEL

$$G_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - g_{\mu\nu}\left(\tfrac{1}{2}g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi + V(\phi)\right) \quad , \quad \psi \equiv \sqrt{\frac{2}{\nu^2-1}}\,\phi\,,$$

$$V(\psi) = \frac{\alpha}{\nu^2}\left\{\frac{\nu-1}{\nu+2}\sinh[(1+\nu)\psi] + \frac{\nu+1}{2-\nu}\sinh[(\nu-1)\psi] - 4\frac{\nu^2-1}{4-\nu^2}\sinh\psi\right\}$$



α has dimensions L^{-2} ; $\nu > 1$; $V(\psi) \simeq \alpha(\nu^2-1)\psi^5/30$ near $\psi = 0$

$\nu = 1$: $G_{\mu\nu} = 0$, $\psi \neq 0$, "decoupling limit", see below

A BLACK HOLE SOLUTION

$$\psi = \ln x \qquad , \qquad ds^2 = \Omega(x)\left[-F(x)dt^2 + \frac{\eta^2\, dx^2}{F(x)} + d\theta^2 + \sin^2\theta d\phi^2\right]$$

with $\Omega(x) = \frac{\nu^2 x^{\nu-1}}{\eta^2(1-x^\nu)^2}$ η characterizes the solution

$$F(x) = \eta^2\,\frac{x^{2-\nu}(1-x^\nu)^2}{\nu^2} - \alpha\left[\frac{1}{4-\nu^2} + \frac{x^2}{\nu^2}\left(1 - \frac{x^{-\nu}}{2-\nu} - \frac{x^\nu}{2+\nu}\right)\right] ,$$

A 4D, static, spherically symmetric solution

Radial coordinate : $r = \frac{\nu}{\eta}\frac{x^{\frac{\nu-1}{2}}}{|1-x^\nu|}$.

There are two branches : $0 \le x \le 1$ and $x \ge 1$.

Spatial infinity is at $x = 1$; The solution is asymptotically flat

There is a curvature singularity at $x = 0$ or $x = \infty$.

The gravitational mass (read off $g_{tt} = 1 - 2m/r + \cdots$) and the inertial mass (Komar integral etc) are equal and given by

$$m = \pm \frac{\alpha + 3\eta^2}{6\eta^3} \quad \text{upper sign for } x > 1 \text{ branch}$$

The PPN parameters γ and β are both equal to 1

The function F has one and only one zero (that is, one Killing horizon) between infinity and the curvature singularity.

Therefore the solution is a black hole.

One can trade η for x_+. For $\alpha > 0$, $x_+ \in [0, 1]$; for $\alpha < 0$, $x_+ > 1$.

Large BH are Schwarzschild-like : $\lim_{x_+ \to 1} \frac{2m}{r_+} = 1$

$$\lim_{x_+ \to 0} m r_+^{\nu-2} = Cst \quad \text{for} \quad \alpha > 0 \quad \text{and} \quad \nu \in [1, 2]\,,$$
$$\lim_{x_+ \to 0} m = Cst \quad \text{for} \quad \alpha > 0 \quad \text{and} \quad \nu > 2\,,$$
$$\lim_{x_+ \to \infty} m = Cst \quad \text{for} \quad \alpha < 0\,.$$

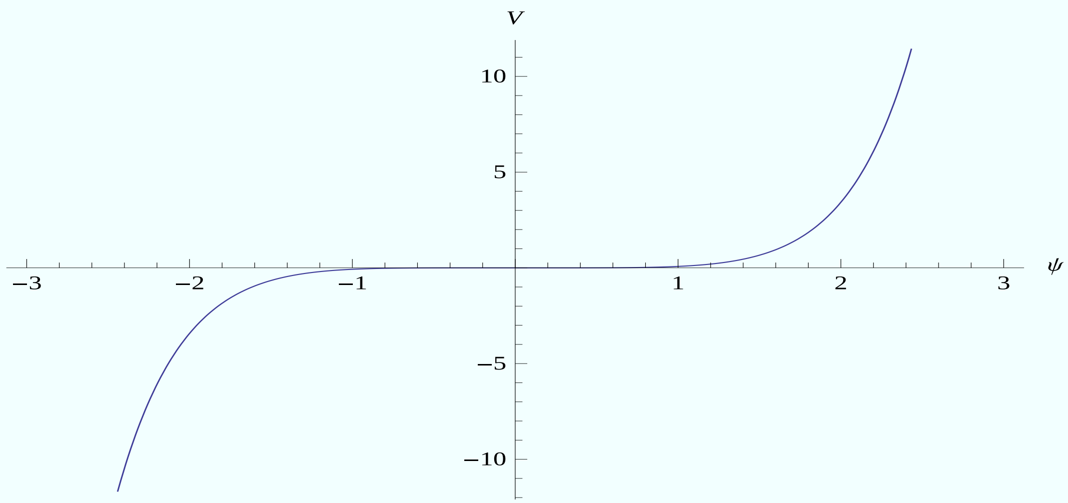
THE "DECOUPLING LIMIT" $\nu = 1$

Consider the Klein-Gordon equation :

$$D^2\psi - \frac{dV}{d\psi} = 0 \quad \text{with} \quad V = 2\alpha[\sinh\psi(\cosh\psi - 4) + 3\psi]$$

$$\psi = \ln[1 + 1/(\eta r)], \qquad \text{with } \eta \text{ such that} \qquad 2\eta^3 m + \eta^2 + \alpha = 0$$

solves the KG equation when metric is Schwarzschild's.
ψ is regular everywhere if $\alpha < -\eta^2$



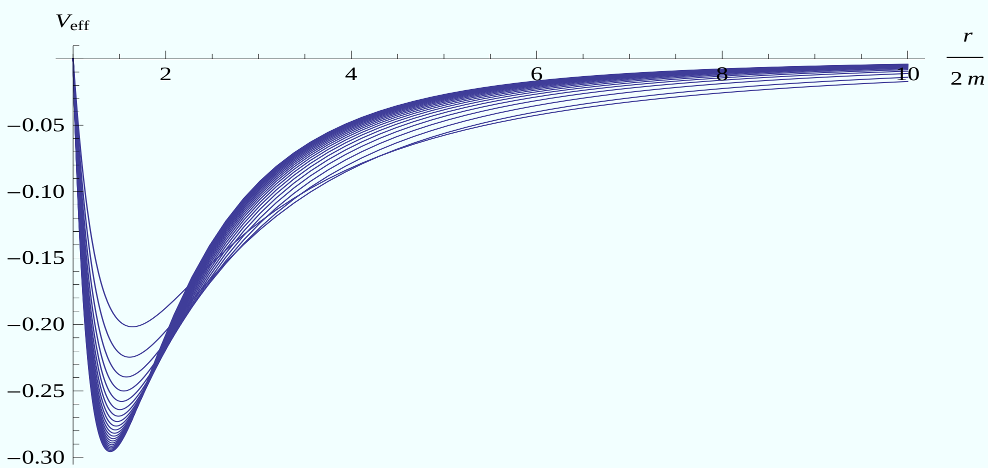
2. STABILITY ANALYSIS

The decoupling limit case

$$D^2\delta\psi - \frac{d^2V}{d\psi^2}\bigg|_b \delta\psi = 0\,, \qquad \text{where} \qquad \frac{d^2V}{d\psi^2}\bigg|_b = -\frac{2(1+2\eta m)(1+2\eta r)}{r^2(1+\eta r)^2}$$

$$\delta\psi = e^{-\mathrm{i}Et}\,Y^l_m(\theta,\phi)\frac{u(r)}{r} \qquad , \qquad \rho = r + 2m\ln(r/2m-1)$$

$$\frac{d^2u}{d\rho^2} = (V_{\rm eff} - E^2)u \qquad \text{with} \qquad V_{\rm eff} = \left(1-\frac{2m}{r}\right)\left(\frac{d^2V}{d\psi^2}\bigg|_b + \frac{l(l+1)}{r^2} + \frac{2m}{r^3}\right)$$



$l = 0$ case.

Bound states with negative E^2 exist; hence modes blow up in time.

The general case

$$\psi = \ln x \qquad , \qquad ds^2 = \Omega(x)\left[-F(x)dt^2 + \frac{\eta^2\, dx^2}{F(x)} + d\theta^2 + \sin^2\theta d\phi^2\right]$$

with $\quad \Omega(x) = \frac{\nu^2 x^{\nu-1}}{\eta^2(1-x^\nu)^2}$ $\qquad \eta$ characterizes the solution

$$F(x) = \eta^2\,\frac{x^{2-\nu}(1-x^\nu)^2}{\nu^2} - \alpha\left[\frac{1}{4-\nu^2} + \frac{x^2}{\nu^2}\left(1 - \frac{x^{-\nu}}{2-\nu} - \frac{x^\nu}{2+\nu}\right)\right]\,,$$

The Bronnikov *et al.* equations for linear radial perturbations
arXiv:1109.6576 [gr-qc] , arXiv:1205.2224 [gr-qc]

$$ds^2 = -e^{2[\gamma_0(x)+\delta\gamma(t,x)]}dt^2 + e^{2[\alpha_0(x)+\delta\alpha(t,x)]}dx^2 + e^{2\lambda_0(x)}(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Linearize Einstein's equations coupled to a scalar field $\phi = \phi_0 + \delta\phi$

Get two constraints :

$$\delta\alpha = \frac{\phi_0'}{2\lambda_0'}\,\delta\phi \quad , \quad \delta\gamma' = \frac{1}{2\lambda_0'^2}\left[(\phi_0' e^{-2\lambda_0} - \phi_0' V - \lambda_0' V_\phi)e^{2\alpha_0}\delta\phi + \lambda_0'\phi_0'\,\delta\phi'\right]$$

and an equation of propagation for $\delta\phi$. Set $\delta\phi \equiv e^{\mathrm{i}Et} e^{-\lambda_0} u(x)$, so that

$\frac{d^2u}{d\rho^2} + (E^2 - V_{\text{eff}})u = 0$ where ρ is the "tortoise coordinate" $\rho' = e^{\alpha_0 - \gamma_0}$

and where

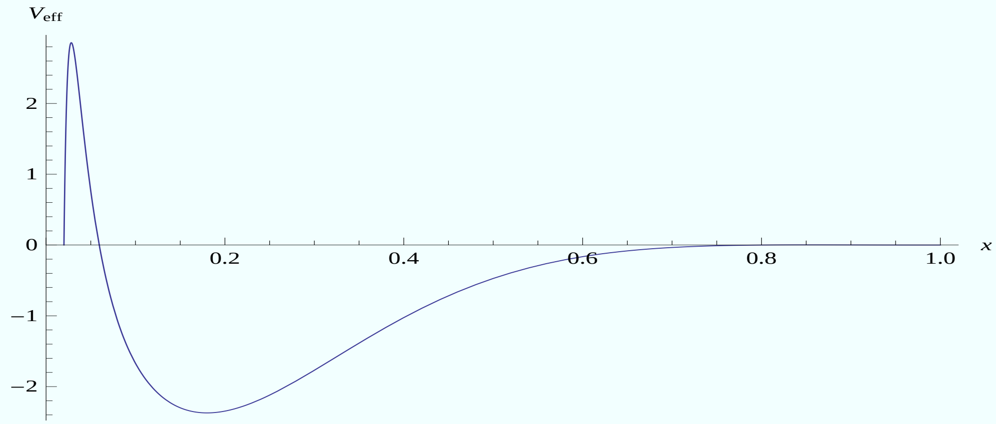
$$V_{\text{eff}} = \frac{e^{2\gamma_0}}{\lambda_0'^2}\left[2\lambda_0'\phi_0'V_\phi - e_0^{-2\alpha_0}\lambda_0'^4 + \phi_0'^2(V - e^{-2\lambda_0}) + \lambda_0'^2(e^{-2\lambda_0} - V + V_{\phi\phi})\right]$$

In the case of Andrès' solution:

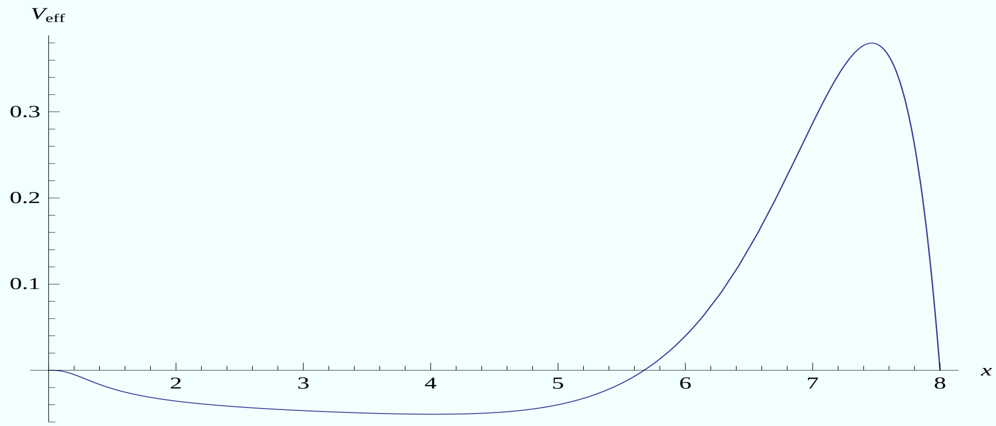
$$\gamma_0 = \ln\sqrt{\Omega F} \quad , \quad \alpha_0 = \ln\eta\sqrt{\frac{\Omega}{F}} \quad , \quad \lambda_0 = \ln\sqrt{\Omega}, \quad , \quad \phi_0 = \sqrt{\frac{\nu^2-1}{2}}\psi$$

V_{eff} is "horribile visu"

However has simple shapes: it is either negative everywhere, OUT!
or negative and edged with one, or two, barriers : hope is permitted



$\alpha > 0$, $\nu = 1.95$ $x_+ = 0.02$



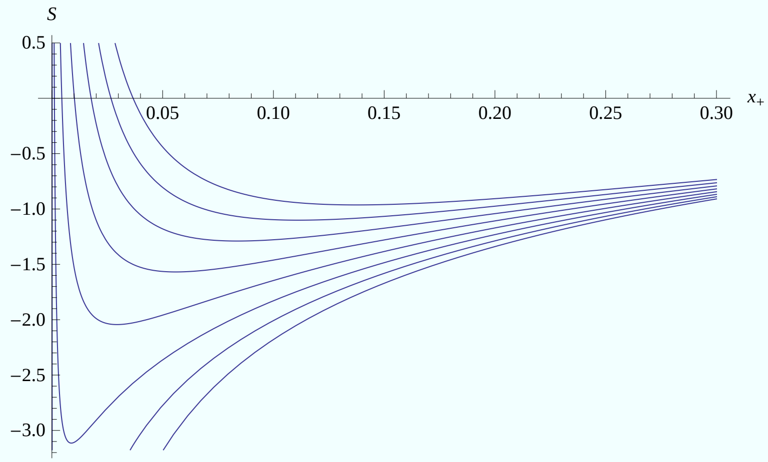
$\alpha < 0$, $\nu = 8$, $x_+ = 8$.

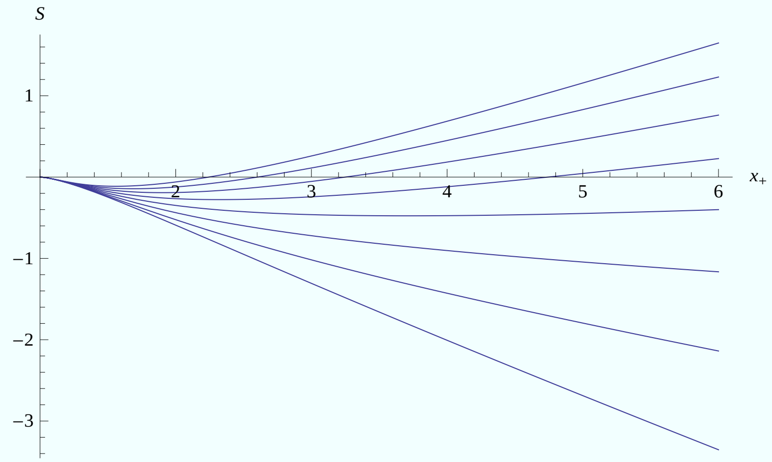
Question : does $\frac{d^2u}{d\rho^2} + (E^2 - V_{\rm eff})u = 0$ have bound state solutions ?

Simon's criterion (1976)

When $V_{\rm eff}$ is bounded and fall off faster than $|\rho|^{-2}$, a necessary (but not sufficient) condition for the absence of bound states of negative E^2 is

$$S \equiv \int_{-\infty}^{+\infty} V_{\rm eff}\, d\rho > 0$$

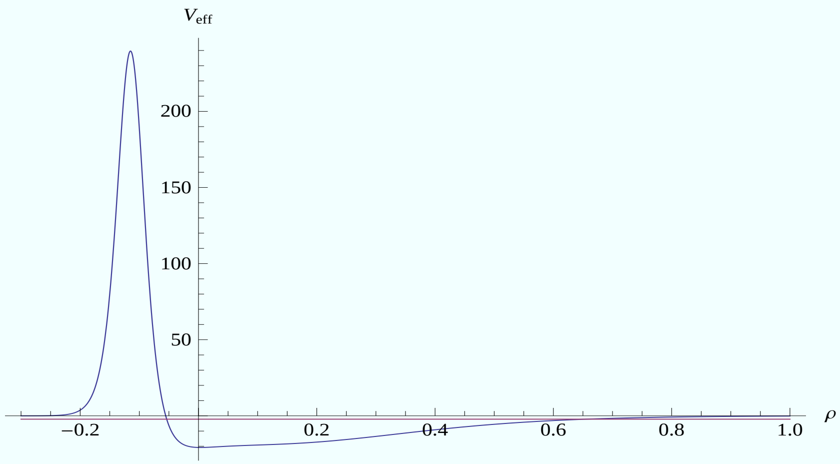




$\alpha > 0$: $S > 0\ \forall \nu$ if x_+ small enough ;

$\alpha < 0$: $S > 0$ for $\nu > 7$ if x_+ big enough ; hope is still permitted

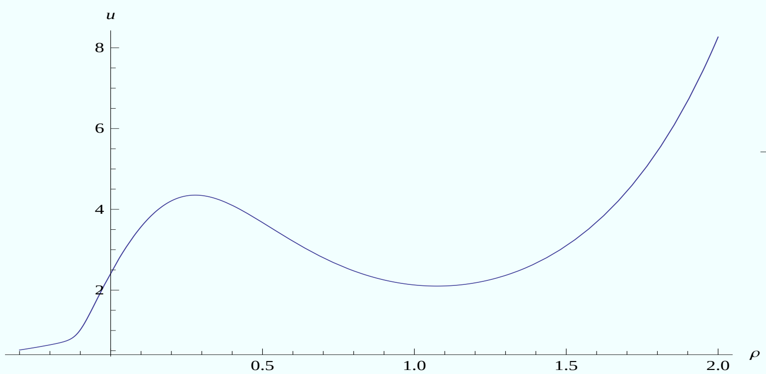
Lasciate ogni speranza...

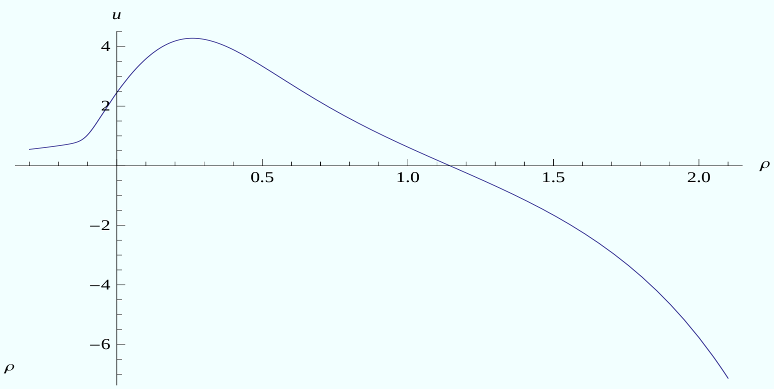


$$\frac{d^2u}{d\rho^2} + (E^2 - V_{\rm eff})u = 0$$
$$\alpha = 1 \ , \ \nu = 1.7 \ , \ x_+ = 0.001$$

Positive Simon integral : $S = 5$

... but $\exists$ a (SINGLE) bound state
$$-2.2 < E^2 < -2.1 ...$$





... We did not find ANY values for α, ν and x_+ with NO bound state...

Now, $V_{\rm eff}$ is proportional to α (which sets the scale of the potential $V(\phi)$).

Therefore write $E^2 = |\alpha|\bar{E}^2$. $\bar{E}^2$ is a number, of order unity if the dimensionless parameters ν, x_+ and $\eta^2\alpha^{-1}$ are all taken of order unity.

The unstable mode $\delta\phi \propto e^{\mathrm{i}Et} = e^{+\sqrt{-\bar{E}^2}\,\sqrt{\alpha}\,t}$ grows on a time scale t of the order of $1/\sqrt{\alpha}$.

Now, the characteristic time scale of gravitational effects is set by the mass m of the black hole .

Hence the growth of the (single) unstable mode is tamed if $\sqrt{\alpha}\,m \ll 1$.

This condition is met if $\alpha > 0$ and $\nu \in [1, 2]$ in which case the mass and size of the black hole are related by

$$2\nu\sqrt{\alpha}\,m \approx (4-\nu^2)^{(\nu-1)/2}(\sqrt{\alpha}\,r_+)^{2-\nu} \ll 1\,.$$

Such black holes are therefore quasi-stable, that is, “long-lived”.

Summary and conclusion

- 4D BH solutions of Einstein's equations with minimally coupled scalar hair, which are asymptotically flat and Schwarzschild's up to PN order, are rare objects, which are worth studying.

- The no-hair theorem (Sudarsky 1995) states that the potential cannot be positive everywhere.

- Andrès Anabalon's BH is an elegant, analytical and fairly simple solution.

- The potential is unbounded from below, so that one is tempted to predict gross instabilities.

- we found that indeed instabilities exist but are surprisingly mild.

- Techniques learned : Bronnikov's equations for radial modes ; Simon's integral criterion.