Geometries of D1-D5-P states

Stefano Giusto

Università di Padova

Exotic structures of spacetime, YITP, March 2013

Based on

Introduction: the D1-D5-P system

Paradigmatic example of BPS black hole in string theory
(Strominger, Vafa)

Type II B

\[ \mathbb{R} \times \mathbb{R}^4 \times S^1 \times T^4 \]

\[ t \quad x^i \quad y \quad z^a \]

\[ D1 : \quad - \quad \circ \quad - \quad \circ \]

\[ D5 : \quad - \quad \circ \quad - \quad - \]

\[ P : \quad - \quad \circ \quad - \quad \circ \]
The “naive” D1-D5-P geometry

\[ ds^2 = -\frac{2}{\sqrt{Z_1 Z_2}} \, dv \left( du + \frac{\mathcal{F}}{2} \, dv \right) + \sqrt{Z_1 Z_2} \, ds_4^2 + \sqrt{\frac{Z_1}{Z_2}} \, ds^2_{T_4} \]

\[ v = \frac{t + y}{\sqrt{2}} \quad , \quad u = \frac{t - y}{\sqrt{2}} \quad , \quad ds_4^2 = dx^i dx^i \]

\[ Z_i = 1 + \frac{Q_i}{r^2} \quad , \quad \mathcal{F} = -\frac{Q_p}{r^2} \]

- \( Q_1 \): D1 charge  
- \( Q_2 \): D5 charge  
- \( Q_p \): P charge

- The geometry has an “extremal” horizon of finite area at \( r = 0 \).
- The geometry only depends on the charges: “no-hair theorem”.
The near-horizon decoupling limit

In the limit

\[ r^2, Q_p \ll Q_1, Q_2 \Rightarrow Z_i \rightarrow Z_i - 1 \]

one obtains a geometry that is \( \text{AdS}_3 \times S^3 \times T^4 \) asymptotically \((r^2 \gg Q_p)\)

\[
\begin{align*}
    ds^2 &\approx -\frac{r^2}{\sqrt{Q_1 Q_2}} 2 \, du \, dv + \frac{\sqrt{Q_1 Q_2}}{r^2} \, dr^2 + \sqrt{Q_1 Q_2} \, d\Omega_3^2 + \sqrt{\frac{Q_1}{Q_2}} \, ds_{T^4}^2
\end{align*}
\]

This limit is described by a CFT:

(4,4) sigma model with target space (a deformation of) \( (T^4)^{n_1 n_5} / S_{n_1 n_5} \)

\[ \uparrow \]

D1-D5 CFT
Microstate geometries

- There exist solutions of supergravity that reduce to the “naive” D1-D5-P geometry for large $r$ up to $O\left(\frac{1}{r^2}\right)$ (and hence carry the same charges as the D1-D5-P black hole) but are completely smooth and have no horizon.

- In the near-horizon limit these geometries are dual to states of the D1-D5 CFT.

Conjecture: these geometries account for (a finite fraction of) the entropy of the D1-D5-P black hole.

(Mathur, Bena, Warner, . . .)
Our goal

To develop systematic methods to construct geometries that

- carry D1,D5,P charges and preserve 4 supersymmetries;
- are dual to states of the D1-D5 CFT.

Note: for geometries carrying only two charges (e.g., D1-D5) this problem has been solved in full generality.

(Lunin, Mathur; Kanitscheider, Skenderis, Taylor)

We will follow the steps:

- find the general supergravity ansatz that preserves the same susy as D1-D5-P;
- select solutions dual to known states of the D1-D5 CFT: apply symmetries of the CFT to known microstates.
We apply methods of **Generalized Geometry** in 10D (Tomasiello)

- $\exists \epsilon_1, \epsilon_2$ Majorana-Weyl Killing spinors;
- the spinor bilinear $K \equiv -\frac{1}{2}(\bar{\epsilon}_1 \Gamma^M \epsilon_1 + \bar{\epsilon}_2 \Gamma^M \epsilon_2) \partial_M$ is a Killing vector;
- we restrict to the case $K$ is null $\Rightarrow K = \frac{\partial}{\partial u}$;
- other spinor bilinears

\[(i) \quad \chi \equiv -\frac{1}{2}(\bar{\epsilon}_1 \Gamma^M \epsilon_1 - \bar{\epsilon}_2 \Gamma^M \epsilon_2) \, dx^M \quad 1 - \text{form} \]
\[(ii) \quad \psi \equiv -32 \, \epsilon_1 \otimes \bar{\epsilon}_2 \, \Gamma_{(10)} \quad \text{polyform} \]

satisfy

\[d\chi = i_K H \quad , \quad (d - H) \wedge (e^{-\phi} \psi) = i_K F + \chi \wedge F \]

$(H, F$ NSNS and RR field strengths);
General D1-D5-P ansatz II

- the missing susy equations come from the $\nu$-component of the gravitino variation $\Rightarrow$ extra constraints;

- the previous equations plus the Bianchi identities for $H$ and $F$ imply all the equations of motion apart from the $\nu\nu$-component of Einstein equations $\Rightarrow R_{\nu\nu} + \ldots = 0$;

- split the metric as $10 \rightarrow (u,\nu) x^i z^a$ and restrict (for simplicity) to geometries that are homogeneous and isotropic with respect to $T^4 \Rightarrow$ non-trivial dependence on $(\nu, x^i)$;

- impose that the susy preserved is of D1-D5 type:

$$\Gamma_{D_1} \epsilon_2 = \epsilon_1 \quad , \quad \Gamma_{D_5} \epsilon_2 = \epsilon_1$$
the various conditions constrain the metric to be of the form

\[ ds^2 = -\frac{2\alpha}{\sqrt{Z_1 Z_2}} (dv + \beta) \left[ du + \omega + \frac{\mathcal{F}}{2} (dv + \beta) \right] + \sqrt{Z_1 Z_2} \, ds_4^2 + \sqrt{\frac{Z_1}{Z_2}} \, ds_{T4}^2 \]

with \[\alpha = \frac{Z_1 Z_2}{Z_1 Z_2 - Z_4^2}\]

\[e^{2\phi} = \ldots\] , \[B = \ldots\] , \[F = \ldots\]

all fields of type IIB sugra can be excited;
\[ ds^2 = -\frac{2\alpha}{\sqrt{Z_1Z_2}}(dv + \beta) \left[ du + \omega + \frac{\mathcal{F}}{2}(dv + \beta) \right] + \sqrt{Z_1Z_2} \, ds_4^2 + \sqrt{\frac{Z_1}{Z_2}} \, ds_{T^4}^2 \]

- The sugra equations can be organized in such a way that the problem reduces to an almost linear one \cite{Bena:2010vy, Shigemori:2012ah, Warner:2012rs}.
- The non-linear part involves \( ds_4^2, \beta \):
  - \( ds_4^2 \) admits an almost hyperkähler structure
  \[ J_A \wedge J_B = -2 \delta_{AB} \, \text{vol}_4 \quad , \quad J_A = -\ast_4 J_A \quad , \quad dJ_A = \partial_v (\beta \wedge J_A) \]
  - \( D\beta = \ast_4 D\beta \) with \( D \equiv d - \beta \wedge \partial_v \);
- The equations for \( Z_1, Z_2, Z_4, \omega, \mathcal{F} \ldots \) are linear (but inhomogeneous), if solved in the right order.
The sugra equations simplify for \( \nu \)-independent geometries:

- \( ds_4^2 \) is hyperkähler;
- \( d\beta = \ast_4 d\beta \).

On general grounds, geometries are expected to be \( \nu \)-independent if they are dual to eigenstates of momentum.

Typically classical geometries are dual to coherent states (e.g., \( e^{\alpha a_{-n}}|\psi_0\rangle \) with \( a_{-n} \) carrying momentum \( n \))

\( \Rightarrow \) generic 3-charge microstates are \( \nu \)-dependent.
**D1-D5 geometries** (Lunin, Mathur; Kanitscheider, Skenderis, Taylor)

- D1-D5 microstates are $\nu$-independent.
- They have a flat 4D base: $ds_4^2 = dx^i dx^i$.
- The ($T^4$-isometric) geometries are encoded in a profile in $\mathbb{R}^5$ $(F^i(\nu'), \mathcal{F}(\nu'))$:

$$
\beta = \frac{-A + B}{\sqrt{2}}, \quad A = -\frac{Q_2}{L} \int_0^L \frac{dv' \dot{F}^i}{|x - F|^2} dx^i, \quad dB = -*_4 dA
$$

$$
Z_2 = 1 + \frac{Q_2}{L} \int_0^L \frac{dv'}{|x - F|^2}, \quad Z_1 = 1 + \frac{Q_2}{L} \int_0^L \frac{dv' |\dot{F}|^2}{|x - F|^2}
$$

$$
Z_4 = -\frac{Q_2}{L} \int_0^L \frac{dv' \ddot{F}}{|x - F|^2}, \quad \omega = -\frac{A + B}{\sqrt{2}}
$$

- $\beta \neq 0$: KK-monopole dipole charge from binding D1 and D5.
Adding momentum to D1-D5 I

- Start from a D1-D5 geometry and take its near-horizon limit:
  \[ Z_1 \rightarrow Z_1 - 1 \quad , \quad Z_2 \rightarrow Z_2 - 1 \Rightarrow \text{asymptotically AdS geometry} \]

- Act with an operator of the CFT chiral algebra that preserves susy and adds momentum:
  - \( L_{-n} : \text{Virasoro} \);
  - \( J_{-n}^\alpha : SU(2)_L \text{ R-symmetry} \);
  - \( J_{-n}^z^a : U(1)^4 \text{ T}^4 \text{ translations} \).

On the gravity side this is equivalent to performing a change of coordinates that does not vanish at the boundary of AdS.

At the perturbative level, this method was applied by Mathur, Saxena, Srivastava; Mathur, Turton; Shigemori.
Adding momentum to D1-D5 II

- Rewrite metric and gauge fields in the form of the general ansatz and read-off $ds^2, \beta, Z_1, \omega, F$.

One generates a $v$-dependent solution of the sugra equations that is asymptotically $AdS_3 \times S^3 \times T^4$.

- Extend the metric back to the asymptotically flat region, where the metric reduces to $\mathbb{R}^{4,1} \times S^1 \times T^4$.

Note: the replacement $Z_1 \rightarrow Z_1 + 1, Z_2 \rightarrow Z_2 + 1$, leaving the remaining metric functions unchanged, does not produce a solution of the sugra constraints.

This is a non-trivial problem that can be solved exactly in some cases.
Case 1: $\nu$-independent $ds_4^2$

- Assume that $ds_4^2$, $\beta$ do not depend on $\nu$ after the change of coordinates.
- Assume also that $ds_4^2$, $\beta$ have a tri-holomorphic isometry $\frac{\partial}{\partial \tau} \Rightarrow$

$$ds_4^2 = V^{-1} (d\tau + A)^2 + V ds_3^2 \quad , \quad *_3 dA = dV$$

$$\beta = \frac{K_3}{V} (d\tau + A) + \xi \quad , \quad *_3 d\xi = -dK_3$$

Egs: $ds_4^2 = \text{flat } \mathbb{R}^4 \Rightarrow \tau = \phi + \psi \quad , \quad V = \frac{1}{r}$

- The remaining metric coefficients may depend on $\nu$ and $\tau$. 
Case 1: $\nu$-independent $ds_4^2$ - An example

- Start from the D1-D5 geometry associated with the profile

$$F^1 + iF^2 = a e^{\frac{2\pi i \nu'}{L}}, \quad F^3 = F^4 = 0, \quad \mathcal{F} = -b \sin\left(\frac{2\pi \nu'}{L}\right)$$

The geometry depends on the angle of the $(1, 2)$ plane but $\beta$ does not.

- Apply the operator

$$e^{\chi(J_+^1 - J_1^-)} = e^\Sigma e^{\chi(J_0^+ - J_0^-)} e^{-\Sigma}$$

$$\phi \rightarrow \phi + \frac{t}{R}, \quad \psi \rightarrow \psi + \frac{\psi}{R} \quad \text{spectral flow \ rotation in } \mathbb{R}^4$$

Note: when $b = 0$ the action of this operator is trivial.

- One can see that $ds_4^2$ and $\beta$ are left invariant, while the other metric coefficients transform non-trivially under this operator.
Niehoff, Warner found a general way to solve the sugra constraints in Case 1:

(almost) all metric coefficients can be expressed algebraically in terms of harmonic functions for the covariant Laplacian $*_4 D *_4 D$.

Egs:

$\begin{align*}
Z_1 &= L_1 + \frac{K_2 K_3}{V}, \\
Z_2 &= L_2 + \frac{K_1 K_3}{V} \\
\partial_\tau K_{2,1} + \partial_v L_{1,2} &= 0, \\
*_4 D *_4 D K_{1,2} &= *_4 D *_4 D L_{1,2} = 0
\end{align*}$

To find $\omega$ one has to solve a system of partial differential equations.
Case 1: $\nu$-independent $ds_4^2$ - The solution II

- To construct an asymptotically flat 3-charge geometry one starts from the geometric data generated by the chiral algebra transformation, extracts the corresponding harmonic functions, and replaces

\[ L_1 \rightarrow 1 + L_1, \quad L_2 \rightarrow 1 + L_2 \]

- The coefficients of other harmonic functions have to be modified to preserve regularity and absence of Dirac-Misner strings.

- There is a unique asymptotically flat solution that solves all the regularity constraints and matches the near-horizon solution.

- The asymptotic charges of the solution match the values expected from the CFT (more on this in Rodolfo’s talk).
Case 2: $v$-dependent $ds_4^2$

- After the change of coordinates $ds_4^2$ and $\beta$ are $v$-dependent.
- An example:
  - Start from the 2-charge metric (in the near-horizon limit) with
    \[
    F^1 + iF^2 = a e^{2\pi i v'}, F^3 = F^4 = F = 0;
    \]
  - Apply the transformation $e^{\epsilon (J^3_n - J^3_n)}$ which is equivalent to
    \[
    (*) \quad \psi \to \psi - \epsilon \sin \left( \frac{n\sqrt{2} v}{R} \right), \quad \phi \to \phi - \epsilon \sin \left( \frac{n\sqrt{2} v}{R} \right) \Leftrightarrow x^i \to x^i - f^i(v)
    \]
    - The coordinate transformation generates the 3-charge near-horizon geometry; $ds_4^2, \beta$ depend on $v$.

- The Niehoff, Warner method does not apply directly: how to construct the asymptotically flat solution?

- Note: the transformation $(*)$ cannot be applied to the asymptotically flat solution because it screws up the asymptotic structure.
Case 2: $v$-dependent $ds^2_4$ - The solution

- Strategy: apply the Niehoff, Warner method before the change of coordinates (when $ds^2_4$ and $\beta$ are $v$-independent) to generate a new non-explicitly asymptotically flat solution that becomes explicitly asymptotically flat after the transformation ($\ast$).

- The new solution needs to have the following asymptotic limit

$$Z_{1,2} \to 1 \ , \ \omega \to -\dot{f}(v)dx^i \ , \ \mathcal{F} \to -|\dot{f}(v)|^2$$

Note: these boundary conditions force all metric coefficients (apart from $ds^2_4, \beta$) to be $v$-dependent.

- Result: there is a unique solution with the correct asymptotic behavior that is regular.
Case 2: relation with "Vachaspati transform"

- The solution generating technique we employ to add momentum to D1-D5 geometries is analogous to the one used by Dabholkar et al. and Callan at al. to construct D1-P geometries:
  - start from the “naive” D1 solution;
  - turn on $\omega = -\dot{f_i}(v)dx^i$, $F = -|\dot{f}(v)|^2$;
  - perform the coordinate shift $x^i \rightarrow x^i - f^i(v)$.

- One important technical difference: the D1 solution has $\beta = 0 \Rightarrow \frac{\partial}{\partial u}$ is hypersurface orthogonal

$\Rightarrow$ constant $\omega$ and $F$ solve the equations of motion.

- Moreover, for D1-P solutions all equations are linear and homogeneous $\Rightarrow$ one can superpose solutions
Multiply-wound profiles

- One can also consider profiles that are multiply-wound along $S^1$:

$$f^i(v + 2\pi w \frac{R}{\sqrt{2}}) = f^i(v), \quad w > 1$$

- This is crucial to obtain the full D1-P entropy.

- The shift $x^i \rightarrow x^i - f^i(v)$ is not defined globally on spacetime.

- An equivalent description: $f^i(v) \rightarrow f^i_\alpha(v), \quad \alpha = 1, \ldots, w$ with

$$f^i_\alpha(v + 2\pi \frac{R}{\sqrt{2}}) = f^i_{\alpha+1}(v)$$

- For D1-P: use linearity to superpose solutions for different $\alpha$.

- For D1-D5-P: the problem is non-linear.

How to construct geometries for multiply-wound profiles?
Summary

- We have determined the general class of susy solutions carrying D1,D5,P charges.
- Within this class, we have constructed geometries dual to BPS states of the D1-D5 CFT with momentum.
- The class of microstates we can construct is very particular: chiral algebra descendants of RR ground states.
- It is non-trivial that a unique, regular and asymptotically flat solution exists with the quantum numbers predicted by the CFT.
Outlook

- Main open problems:
  - can one extend the construction of microstate geometries to more generic states?
    Note: constructing geometries corresponding to multi-wound profiles would be a big step in this direction;
  - can microstate geometries account for a finite fraction of the 3-charge entropy?

- Extract more informations from the geometries we have:
  - compute holographically 1-point functions of chiral operators in the 3-charge microstates and compare with the CFT;
    (Kanitscheider, Skenderis, Taylor)
  - compute holographically the entanglement entropy in the microstate geometries.
    (Ryu, Takayanagi)
Entanglement entropy in black hole microstates

Most computations of entanglement entropy are done in the vacuum or the thermal state $\Rightarrow \text{AdS}_3 \times S^3 \times T^4$.

$T^4$-isotropic microstates can be easily reduced on $T^4$ $\Rightarrow ds^2$ 6D Einstein metric.

In the near-horizon limit $ds^2$ is asymptotically $\text{AdS}_3 \times S^3$ but depends non-trivially on $S^3$.

The Ryu, Takayanagi holographic recipe for computing entanglement entropy applies to asymptotically $\text{AdS}_3$ spaces.

A small generalization: consider a 1D domain $A$ and a geodesic (with respect to $ds^2$) 4-manifold $\Gamma_A = \gamma_A \times S^3$, where $\gamma_A$ is a curve at $t = \text{const.}$ such that $\partial \gamma_A = \partial A$; the entanglement entropy is

$$S_A = \frac{\text{Area}(\Gamma_A)}{4G_6}$$

Can one compare with the D1-D5 CFT prediction? (work in progress with R. Russo)