Classifying Supersymmetric Solutions in 3D Maximal Supergravity

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- J. de Boer, D. R. Mayerson, M. Shigemori [arXiv:1403:4600] (Today!)

1. Introduction

2. Classifying SUSY Solutions

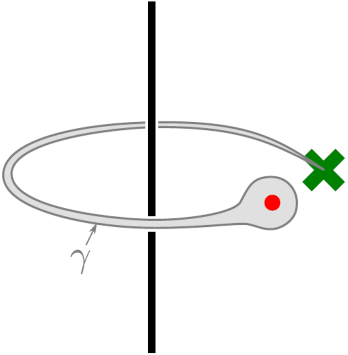
3. (Exotic) Branes in 3D

4. Summary

Introduction: Exotic Branes (1)

String theory contains strings, D-branes; also 7-branes such as D7 in IIB:

- $\tau = C_0 + ie^{-\phi}$
- D7: $C_0 = Q_{D7}\,\theta$
- $Q_{D7} = \int_\infty F_1$
- go around brane: $\tau \to \tau + 1$ "monodromy"



Introduction: Exotic Branes (2)

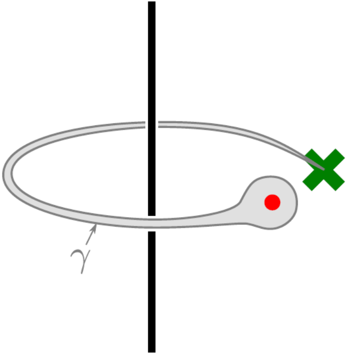
More "exotic" example: 5_2^2:

- $ds_{10}^2 = H(dr^2 + r^2 d\theta^2) + H\ K^{-1} dx_{89}^2 + dx_{034567}^2$

- $K = H(r)^2 + \sigma^2\theta^2$

- $\theta = 0$: $g_{88} = g_{99} = H^{-1}$

- $\theta = 2\pi$: $g_{88} = g_{99} = \frac{H}{H^2 + (2\pi\sigma)^2}$



Introduction: Exotic Branes (3)

In general, for 7-brane ("exotic brane"):

- τ: some fields
- $\tau \to M\tau$ as go around brane
- $M \in$ U-duality group ($E_{8(8)}$)

Introduction: Exotic Branes (4)

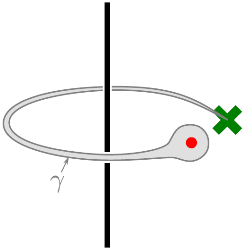
U-duality groups in different dimensions:

d	$G(\mathbb{R})$	K	$\dim(G)$	$rank(G)$	$\dim(K)$
10A	$\mathbb{R}_+$	$\mathbf{1}$	1	1	0
10B	$SL(2)$	$SO(2)$	3	1	1
9	$SL(2) \times \mathbb{R}_+$	$SO(2)$	4	2	1
8	$SL(3) \times SL(2)$	$SO(3) \times SO(2)$	$8+3$	$2+1$	$3+1$
7	$SL(5)$	$SO(5)$	24	4	10
6	$SO(5,5)$	$SO(5) \times SO(5)$	45	5	20
5	$E_{6(6)}$	$USp(8)$	78	6	36
4	$E_{7(7)}$	$SU(8)$	133	7	63
3	$E_{8(8)}$	$SO(16)$	248	8	120

Introduction: Exotic Branes (5)

Properties of exotic branes:

- codim-2 (non-contractible circle around brane)
- non-geometric: even metric can jump! (U-fold)
- tension $\sim g_s^{-n}$, $n \geq 2$



Introduction: Exotic Branes (6)

Non-Abelian charge (=monodromy) lattice: de Boer, Shigemori 1209.6056

- Normal branes: Abelian charge lattice $\mathbb{Z}^n$
  e.g. D1 + D5 = D5 + D1
- Codim-2 branes: monodromy is charge

$$\tau \rightarrow M\tau \quad (1)$$

- Charge $M \in E_{8(8)}(\mathbb{Z})$ - non-Abelian charge lattice
- Not trivial to "add" branes!

Introduction: Exotic Branes (7)

Results for 1/2 BPS branes in max SUGRA:

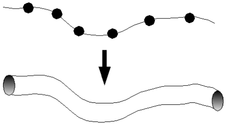
Bergshoeff, Riccioni, {+Romano 1303.0221; +Marrani 1201.5819; +Ortin 1109.4484; 1109.1725; 1009.4657; ...}

- Codim-2 branes $\leftrightarrow$ longest weights in adjoint rep. of U-duality group (3D, $\mathbf{e_8}$: 248-8=240)
- degeneracy of BPS condition $\rightarrow$ can form multi-charge codim-2 brane orbits that are 1/2 BPS
- In principle, all x BPS states should be in some way "bound states" of fundamental 1/2 BPS branes. In practice, difficult to determine all possible states ($\leftrightarrow$ non-Abelian charge lattice!)

Introduction: Exotic Branes (8)

Supertube effect: Mateos & Townsend hep-th/0103030

- 7-branes sometimes considered "pathological"
- Supertube effect: dipole exotic branes de Boer, Shigemori 1004.2521
  - original supertube "puff-up" effect: F1(1)+D0 $\rightarrow$ D2(1ψ)



  - U-dual to e.g.: D4(6789) + D4(4589) $\rightarrow$ $5_2^2(4567\psi,89)$
  - Only ordinary brane charge at infinity (no pathological 7-brane behaviour), but exotic dipole charges!
- Exotic puffed-up microstates may be crucial for constructing (enough) black hole microstates...

Introduction: Going to 3D (1)

7 branes in 10D:

- point particle in 3D
- metric is single-valued
- only scalar monodromies
- would be nice to classify possible monodromies!

Introduction: Going to 3D (2)

Compactify string theory on $T^7 \to$ 3D maximal SUGRA:

- Can ask well-defined question: What are necessary/sufficient conditions for a solution to preserve SUSY?
- Despite complicated scalar structure, we obtain complete necessary/sufficient conditions, complete classification of SUSY solutions
- Afterwards: what is the relation between classification of SUSY solutions and possible monodromies?

Introduction: Going to 3D (3)

Classifying SUSY solutions:

- Previous classifications of SUSY solutions done in e.g. $\mathcal{N} = 1, 2$ SUGRA in 4/5/6D

  Gauntlett, Gutowski, Hull, Pakis, Reall hep-th/0209114 and many others

- Usual strategy:
  - Assume Killing spinor ϵ, construct bilinears, e.g. $V_\mu = \bar{\epsilon}\gamma_\mu\epsilon$
  - Algebraic/differential relations $\rightarrow$ restriction solutions
- Usually can't explicitly construct all SUSY spacetimes
- Still very useful: construct AdS_5 BHs, SUSY microstate geometries, ...

Introduction: Recap

Recap:

- Interested in studying all possible (supersymmetric) exotic branes
- Go to 3D (codim-2: point particles) for richest explicit U-duality structure
- First: classify supersymmetric sols in 3D (pure SUGRA problem)
- Then: relate to classification of point particles

3D Maximal SUGRA (1): $\mathbf{e}_{8(8)}$

Constructing the algebra $\mathbf{e}_8$: GSW, 6.A

- Take the 120 $\mathbf{so}(\mathbf{16})$ generators X^{IJ}:

$$[X^{IJ}, X^{KL}] = \delta^{IL} X^{JK} + \delta^{JK} X^{IL} - \delta^{IK} X^{JL} - \delta^{JL} X^{IK}. \qquad (2)$$

- Append 128 generators Y^A in Majorana-Weyl spinor rep. of $\mathbf{so}(\mathbf{16})$:

$$[X^{IJ}, Y^A] = \Gamma^{IJ}_{AB} Y^B. \qquad (3)$$

- $\mathbf{e}_8 = \{X^{IJ}, Y^A\}$!

- For split real form $\mathbf{e}_{8(8)}$, can take Y^A as non-compact generators

3D Maximal SUGRA (2): Action

Fields in the theory:

- 128 non-compact scalars $\phi := \phi^A Y^A \in \mathbf{e_{8(8)}} \ominus \mathbf{so(16)}$
- $V = e^{\phi} \in E_{8(8)}/SO(16)$
- Global symmetry (U-duality): $e^{\phi} \to g e^{\phi} h(\phi)$, $g \in E_{8(8)}$ (h compensating gauge transformation to keep gauge)

3D Maximal SUGRA (3): Action

Constructing a bosonic action: Marcus, Schwarz '83

- $V^{-1}\partial_\mu V = P_\mu + Q_\mu$
- $Q_\mu \in \mathbf{so(16)}$: compact "gauge field"
- $P_\mu \in \mathbf{e_8} \ominus \mathbf{so(16)}$: non-compact part
- E_8 invariant action: $\mathcal{L} = R - g^{\mu\nu} tr(P_\mu P_\nu)$

3D Maximal SUGRA (4): Action

$SL(2)$ toy example:

- $V \in SL(2)/SO(2)$
- Can choose (Borel) gauge:
$$V = \begin{pmatrix} 1 & \tau_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{\tau_2} & 0 \\ 0 & \sqrt{\tau_2}^{-1} \end{pmatrix}$$
- $V^{-1}dV = P + Q$:
  - $P = \frac{1}{2\tau_2} \begin{pmatrix} d\tau_2 & d\tau_1 \\ d\tau_1 & -d\tau_2 \end{pmatrix}$
  - $Q = \frac{1}{2\tau_2} \begin{pmatrix} 0 & d\tau_1 \\ -d\tau_1 & 0 \end{pmatrix}$
- Kinetic term: $tr(P^2) = \frac{1}{2\tau_2^2} \left(d\tau_1^2 + d\tau_2^2 \right)$

3D Maximal SUGRA (5): Variations

SUSY transformations:

- SUSY parameters ϵ^I_α
- $I = 1, \cdots 16$ - $\mathbf{so(16)}$ vector; $\alpha = 1, 2$ - 3D Majorana spinor
- $\delta\psi^I_\mu = \nabla_\mu \epsilon^I + Q^{IJ}_\mu \epsilon^J$
- $\delta\chi^{\dot{A}} = P^A_\mu \gamma^\mu \Gamma^I_{\dot{A}A} \epsilon^I$

Classification - Null and Timelike Solutions (1)

- Find ϵ^I such that $\delta\psi^I_\mu = \delta\chi^A = 0$
- Construct $V_\mu = (\epsilon^I)^T \gamma_{\hat{0}} \gamma_\mu \epsilon^I$
- $\nabla_\mu V_\nu = 0$ - V is Killing, but also stronger
- $V^2 \leq 0$

Classification - Null and Timelike Solutions (2)

Null class: $V = \partial_u$ null

- $ds^2 = -dudv - 2\omega(v,x)dvdx + h(v,x)dx^2$
- pp-waves, all 1/2-SUSY
- scalars $\phi(v)$

Classification - Null and Timelike Solutions (3)

Timelike class: $V = \partial_t$ timelike

- $ds^2 = -dt^2 + e^{U(z,\bar{z})} dz d\bar{z} \qquad (\partial_t \phi = 0)$
- SUSY condition ($\xi = \epsilon_1 + i\epsilon_2$):

$$P_z^A \Gamma^I_{\dot{A}A} \xi^I = 0 \tag{4}$$

Classification (Timelike) - Nilpotent Orbits (1)

Necessary and sufficient SUSY condition timelike solutions:

$$\boxed{P_z^A \Gamma^I_{\dot{A}A} \xi^I = 0} \quad (5)$$

- Invariant under (compact) conjugation $K \in SO(16)$:
  $P \to K^{-1} P K$
- SUSY determined by conj. class = orbit of P_z in $\mathbf{p} \equiv \mathbf{e}_8 \ominus \mathbf{so}(16)$
- Lots of math results about these objects! Rich (Zariski) topology of orbits... Kostant & Rallis '71, Collingwood & McGovern '93, Djokovic '00, '03 '05

Classification (Timelike) - Nilpotent Orbits (2)

First results:

- a) ("smaller orbit = more SUSY")
  If $\overline{\mathcal{O}}_1 \subseteq \overline{\mathcal{O}}_2$, then (#SUSY $\mathcal{O}_1$) $\geq$ (#SUSY $\mathcal{O}_2$)
  Example:
  - 1/4 BPS orbit $\mathcal{O}$ of (N_1 D1-branes) + (N_5 D5-branes)
  - 1/2 BPS orbit $\mathcal{O}' \subset \overline{\mathcal{O}}$ of (N_1 D1-branes)
- b) For SUSY, P needs to be nilpotent ($P^n = 0$)
  - e.g.: trivial orbit $\mathcal{O}_0 = \{0\}$ is max. SUSY; $\mathcal{O}_0 \subseteq \overline{\mathcal{O}}_X$ for all nilpotent X

Classification (Timelike) - Nilpotent Orbits (3)

Proof of (a) ("smaller orbit = more SUSY")
If $\overline{\mathcal{O}}_1 \subseteq \overline{\mathcal{O}}_2$, then $(\#\text{SUSY } \mathcal{O}_1) \geq (\#\text{SUSY } \mathcal{O}_2)$

- $P_z^A \Gamma^I_{\dot{A}A} \xi^I = 0$ is essentially matrix equation $M\xi = 0$ with $M = P_z^A \Gamma^I_{\dot{A}A}$
- $\exists$ non-trivial solutions for ξ^I $\rightarrow$ all 16×16 det. of M vanish
- More SUSY preserved $\rightarrow$ smaller and smaller sub-determinants must vanish
- det $= 0$ $\rightarrow$ (homogeneous) equations involving P; solutions form closed set in Zariski topology

Classification (Timelike) - Nilpotent Orbits (4)

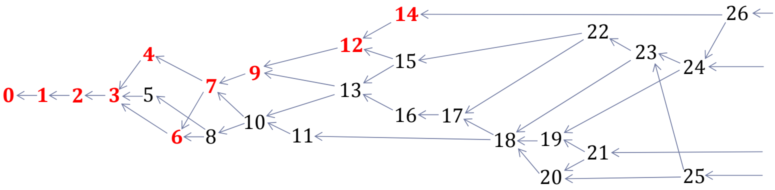
Proof of (b) For SUSY, P needs to be nilpotent ($P^n = 0$)

- Unique Jordan decomposition algebra element: $X = X_N + X_S$; X_N nilpotent, X_S semisimple; $[X_N, X_S] = 0$
- (math result) $\mathcal{O}_{X_S} \subseteq \overline{\mathcal{O}}_X$
- (math result) some element in CSA $c_i H_i \in \mathcal{O}_{X_S}$
- Solve for $c_i \rightarrow$ all $c_i = 0$ for SUSY $\rightarrow X_S = 0$

Classification (Timelike) - Nilpotent Orbits (5)

Narrowed problem down to nilpotent orbits of P_z

- Nilpotent orbits very special
- Only finitely many - $\mathbf{0}, \cdots, \mathbf{115}$
- Partial ordering (Hasse diagram) $\mathcal{O}_i \leq \mathcal{O}_j$ if $\overline{\mathcal{O}}_i \subseteq \overline{\mathcal{O}}_j$ Collingwood & McGovern '93; Djokovic '00, '03, '05



- No SUSY preserved by **5** $\rightarrow$ no other orbits preserve SUSY!
  - Recall: (a) "smaller orbit = more SUSY"

Classification - Summary

Complete classification of SUSY solutions in 3D maximal SUGRA:

- Null class: pp-waves, 1/2-SUSY
- Timelike class:
  - $ds^2 = -dt^2 + e^{U(z,\bar{z})} dz\bar{z} \; (\partial_t \phi = 0)$
  - Orbit of P_z completely determines SUSY preserved
  - P_z nilpotent, "smaller orbit = more SUSY"
    - max. SUSY: **0**
    - 1/2 SUSY: **1**
    - 1/4 SUSY: **2**
    - 1/8 SUSY: $\mathbf{3}, \mathbf{4}$
    - 1/16 SUSY: $\mathbf{6}, \mathbf{7}, \mathbf{9}, \mathbf{12}, \mathbf{14}$

Branes in 3D - Single Center Ansatz

Take simple single-center spherically symmetric ansatz (timelike class):

- metric $ds^2 = -dt^2 + e^{U(r)}(dr^2 + r^2 d\theta^2)$
- scalars $M = e^{\theta X} m(r) e^{\theta X^T}$
  - $M = e^{\phi} e^{\phi^T}$ $(\in E_{8(8)}/SO(16))$
  - $X \in \mathfrak{e}_{8(8)}$ gives monodromy (through $g = e^{2\pi X}$)
    (Remember $e^{\phi} \to g e^{\phi} h$)
- with any solution m, X; can conjugate by $U \in E_{8(8)}$ to get solution UXU^{-1}, UmU^T

Branes in 3D - Easy Brane Representatives (1)

Identify $X = E_\alpha$ with 240 "fundamental" 1/2 BPS branes (D7-brane U-duality multiplet):

- Root 8-vector α: $(\pm1, \pm1, 0, 0, 0, 0, 0, 0)$, $\frac{1}{2}(\pm1, \pm1, \pm1, \pm1, \pm1, \pm1, \pm1, \pm1)$ (and perm.)
- Brane tension $2\alpha_8 - 2$
- General mass formula:

$$M \sim g_s^{2\alpha_8 - 2} \prod_{i=1}^{7} R_i^{\alpha_i - \alpha_8 + 1} \tag{6}$$

Branes in 3D - Easy Brane Representatives (2)

$$M \sim g_s^{2\alpha_8 - 2} \prod_{i=1}^{7} R_i^{\alpha_i - \alpha_8 + 1} \tag{7}$$

Examples:

- $D3(123) \leftrightarrow \frac{1}{2}(+1, +1, +1, -1, -1, -1, -1, | +1)$
  $\leftrightarrow M \sim g_s^{-1} R_1 R_2 R_3$
- $F1(1) \leftrightarrow (+1, 0, 0, 0, 0, 0, 0, | +1)$
  $\leftrightarrow M \sim g_s^{0} R_1$
- $KK(12345; 6) \leftrightarrow (0, 0, 0, 0, 0, +1, -1, |0)$
  $\leftrightarrow M \sim g_s^{-2} R_1 R_2 R_3 R_4 R_5 (R_6)^2$

Branes in 3D - Easy Brane Representatives (3)

$$M \sim g_s^{2\alpha_8 - 2} \prod_{i=1}^{7} R_i^{\alpha_i - \alpha_8 + 1} \tag{8}$$

Allows us to construct brane reps for each (SUSY) nilpotent orbit: e.g. orbit **14** (1/16 BPS):

M5	X			X		X	X	X
M5		X	X			X	X	X
M5		X		X	X		X	X
M5	X		X		X		X	X
M5			X	X	X	X		X
M5	X	X			X	X		X
M5	X	X	X	X				X
P								X

Branes in 3D - All Nilpotent & Semi-Simple Charges (1)

$sl(2)$ solutions: we can find (SUSY) solutions for: Bergshoeff, Hartong, Ortin, Roest hep-th/0612072

- $X_N = \begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix}$ "N-brane"
- $X_K = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$ "K-brane"
- $X_A = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ "A-brane" (not globally well-defined)
- Always $P \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Branes in 3D - All Nilpotent & Semi-Simple Charges (2)

Scalars: $M = e^{\theta X} m(r) e^{\theta X^T}$

N-brane:

$$X = \begin{pmatrix} 0 & \pm 1 \\ 0 & 0 \end{pmatrix} \qquad m = \begin{pmatrix} \log r & 0 \\ 0 & (\log r)^{-1} \end{pmatrix}$$

K-brane:

$$X = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix} \qquad m = \begin{pmatrix} \frac{1+r^{2|\lambda|}}{1-r^{2|\lambda|}} & 0 \\ 0 & \frac{1-r^{2|\lambda|}}{1+r^{2|\lambda|}} \end{pmatrix}$$

A-brane:

$$X = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \qquad m = \begin{pmatrix} \sec v & \tan v \\ \tan v & \sec v \end{pmatrix},$$

$v \equiv c_1 + 2\lambda \log r$

Branes in 3D - All Nilpotent & Semi-Simple Charges (3)

Any nilpotent monodromy $X_N \in \mathbf{e}_{8(8)}$:

- X_N is part of an $\mathbf{sl}(2)$-subalgebra of $\mathbf{e}_{8(8)}$ (Jacobson-Morozov theorem)
- embed $\mathbf{sl}(2)$ N-brane solution into $\mathbf{e}_{8(8)}$ so that $X_{sl(2)} \rightarrow X_N$

  Djokovic '00
- SUSY of resulting solution depends on nilpotent orbit of X_N

Branes in 3D - All Nilpotent & Semi-Simple Charges (4)

Any semisimple monodromy $X_S \in \mathbf{e_{8(8)}}$:

- X_S is conjugate to the sum of at most 8 X_K's and X_A's, where X_K, X_A are the corresponding $\mathbf{sl(2)}$ elements; moreover all these (8) $\mathbf{sl(2)}$ algebras commute Sugiura '59
- can paste (up to 8) K- and A-branes together (and conjugate) to construct X_S as monodromy
- relative orientation of K/A-branes determines SUSY preservation

Branes in 3D - General Monodromies

General monodromy X:

- Unique Jordan decomposition $X = X_S + X_N$, $[X_S, X_N] = 0$
- Can't just "paste" brane sols for X_S, X_N separately together (except special cases)
- $\rightarrow$ case by case analysis?

Branes in 3D - Other Monodromies

Questions:

- How do we find solutions for general monodromies?
- Are the solutions we found for nilpotent/semisimple monodromies the only ones possible for these monodromies?
- Can we construct a SUSY solution for any monodromy?

No definitive answers... but:

- It is possible to find multiple solutions for the same monodromy
- Different solutions for same monodromy can preserve different SUSY

Branes in 3D - Plethora of Examples

Study plethora of examples:

subalgebra	X	semisimple	nilpotent	compact	SUSY	global
$\mathbf{sl}(2)$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$		◯		◯	◯
$\mathbf{sl}(2)$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	◯		◯	◯	◯
$\mathbf{sl}(2)$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	◯			◯	
$(\mathbf{sl}(2) \subset)\ \mathbf{e}_8$	any nilpotent		◯		◯	◯
$(\mathbf{sl}(2)^n \subset)\ \mathbf{e}_8$	any semisimple	◯		(◯)	◯	(◯)
$\mathbf{sl}(2)^n$	any (in $\mathbf{sl}(2)^n$)	(◯)	(◯)	(◯)	◯	◯
$\mathbf{sl}(3)$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$		◯			
$(\mathbf{sl}(2) \subset \mathbf{sl}(3))$			◯		◯	◯
$\mathbf{sl}(3)$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$					◯
$\mathbf{sl}(3)$	$\begin{pmatrix} 1 & 0 & \pm 2 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$				◯	
$\mathbf{sl}(3)$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$		◯		◯	
$(\mathbf{sl}(2) \subset \mathbf{sl}(3))$			◯			◯

Summary

- Exotic branes: objects in string theory with weird properties
- e.g. spacetime/other fields can change when we loop around; monodromy
- Classify these $\rightarrow$ compactify to 3D; only scalar monodromies $\phi \rightarrow \phi + X$
- $P \sim \partial\phi$ completely determines SUSY of 3D; particular nilpotent orbits allowed
- Relation $P \leftrightarrow X$? No easy relation $\rightarrow$ no easy way to classify "allowed" SUSY exotic branes; lots of examples

Outlook

- Multi-center solutions? In general multi-punctured Riemann surface, difficult mathematical problem
- Codim-1 branes like D8-brane? Need 3D massive/gauged SUGRA
- Repeat analysis for other non-maximal 3D SUGRA (e.g. $SO(8,22)$ theory from heterotic on T^7)?
- Place in double/extended field theory of such objects, natural place to discuss them
- Relevance for BH microstates