Generalised geometry and consistent truncations

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Consistent truncations: some history... [*CGLP 2003*]

Kaluza and Klein: $M_4 \times S^1$ (1919)

$$\mathrm{d}s^2 = \mathrm{e}^{\phi/\sqrt{3}} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{-2\phi/\sqrt{3}} (\mathrm{d}y + A_\mu \mathrm{d}x^\mu)^2$$

where reducing with $\phi(x)$, $g_{\mu\nu}(x)$ and $A_\mu(x)$ gives

$$S_4 = \int_{M_4} \sqrt{-g} \left(R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{4}\mathrm{e}^{-\sqrt{3}\phi} F^2 \right)$$

E&M but "problematic" new scalar – Klein just set $\phi = 0$

Consistency

are all solutions in 4d also solutions to 5d equations?

- For KK, Jordan (1947) and Thiry (1948) showed that consistent only if keep both A_μ and ϕ
- What about more general compactifications?

"DeWitt" group manifold reductions (1963)

$$M_{D+d} = M_D \times G \qquad G \text{ is } d\text{-dim semi-simple Lie group}$$

use left-invariant one-forms e^a

$$\mathrm{d}e^a = -\tfrac{1}{2} f_{bc}{}^a e^b \wedge e^c \qquad \text{Lie algebra } \mathfrak{g}$$

to define $(D+d)$ metric ("Scherk–Schwarz" reduction)

$$\mathrm{d}s^2 = (\det M)^q g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + M_{ab}(e^a + A^a_\mu \mathrm{d}x^\mu)(e^b + A^b_\nu \mathrm{d}x^\nu)$$

gives gauge theory with group G and scalar "moduli" M_{ab}

- Scherk–Schwarz (1979)

  DeWitt reduction is consistent if unimodular ${f_{ab}}^a = 0$

  (true for compact G)
- for Bianchi cosmologies: Hawking (1969), Sneddon (1976)
- generalises to G/Γ with free discrete group Γ
- consistent truncation of any theory if keep all singlet modes

"Pauli" coset reductions

$$M_{D+d} = M_D \times (G/H) \qquad G/H \text{ coset space}$$

consistent with gauge group G?

e.g. $\quad S^2 = SO(3)/SO(2) \quad$ Pauli 1953

showed reduction of pure gravity not consistent

generically doesn't work

Remarkable "sporadic" $S^d = SO(d+1)/SO(d)$ examples

- $d = 11$ sugra on S^7 (de Wit & Nicloai 1987)
- $d = 11$ sugra on S^4 (NVvN 1999)
- type IIB sugra on S^5 (not completely proven consistent)

all require particular field content, complicated ansatzë

$$\mathrm{d}s'^2 = \frac{R^2}{(T^{kl}y_k y_l)^{2/(d-1)}}\, T^{-1}_{ij}\mathrm{d}y^i\mathrm{d}y^j$$

$$A' = -\frac{1}{2(T^{kl}y_k y_l)}\frac{R^{d-1}}{(d-2)!}\,\epsilon_{i_1\dots i_{d+1}}(T^{i_1 j}y_j)y^{i_2}\mathrm{d}y^{i_3}\wedge\cdots\wedge\mathrm{d}y^{i_{d+1}} + A$$

...

where $y^i y^i = 1$ and $F = \mathrm{d}A \simeq \mathrm{vol}(S^d)$ and T^{ij} moduli

Why do sporadic examples work?

> *"systematic understanding of coset reductions is lacking ... no known algorithmic prescription" [CGLP 2001]*

New ingredient: generalised geometry. We will see

> *consistent Pauli reductions are just "generalised" DeWitt reductions*

Consistent truncations

$GL^+(d+1,\mathbb{R})$ generalised geometry

Leibniz generalised parallelisations

- General definition
- Sphere generalised parallelisations

Examples

Conclusions

$GL^+(d+1,\mathbb{R})$ generalised geometry

Simplified theory

Key ingredient is d-form field strength $F = \mathrm{d}A$

$$S = \int \sqrt{-g}\,\left(R - \tfrac{1}{2d!}F^2\right)$$

symmetries parametrised by $v \in TM$ and $\lambda \in \Lambda^{d-2}T^*M$

$\delta g = \mathcal{L}_v g$	diffeomorphisms
$\delta A = \mathcal{L}_v A + \mathrm{d}\lambda$	and gauge transf.

Generalised geometry (... à la Hitchin/Gualtieri)

extension of differential geometry that unifies diffeos and gauge transfs.

Unified reformulation of supergravity theories

- type II with $O(d,d)\times\mathbb{R}^+$
  [*Siegel; Hohm, Hull, Kwak, Zweibach; Jeon, Lee, Park; CSW*]
- $d=11$ (or type II) on M_d with $E_{d(d)}\times\mathbb{R}^+$
  [*Hull; Pacheco & DW; Berman, G^2, Perry; CSW*]

existence of torsion-free, compatible generalised connection D so

$$S=\int |\operatorname{vol}_G|\mathrm{R} \qquad \text{generalised Ricci flat}$$

Not just $O(d,d)$ and $E_{d(d)}$ [*Baraglia; Strickland-Constable*]

families of different generalised geometries $G \times \mathbb{R}^+$

here focus on $GL^+(d+1,\mathbb{R})$

NB: since $E_{4(4)} \simeq SL(5,\mathbb{R})$ the $d=4$ case already in [Hull; Berman, G^2 & Perry; CSW] and gen. case also recently in [Park & Suh]

Generalised tangent space

$$E \simeq TM \oplus \Lambda^{d-2}T^*M \qquad \tfrac{1}{2}d(d+1) \text{ dimensions}$$

$$V^M = V^{\underline{mn}} = \begin{cases} V^{m,d+1} = v^m \\ V^{mn} = \frac{1}{(d-2)!}\epsilon^{mnp_1\dots p_{d-2}}\lambda_{p_1\dots p_{d-2}} \end{cases}$$

since $\Lambda^2 TM \otimes \det T^*M \simeq \Lambda^{d-2}T^*M$

$$V'^{\underline{mn}} = R^{\underline{m}}{}_{\underline{p}} R^{\underline{n}}{}_{\underline{p}} V^{\underline{mp}} \qquad GL^+(d+1,\mathbb{R}) \text{ action}$$

Partial derivative

$$\partial_M = \partial_{\underline{mn}} = \begin{cases} \partial_{m,d+1} = \partial_m \\ \partial_{mn} = 0 \end{cases}$$

with section condition $\partial_{[\underline{mn}} f \, \partial_{\underline{pq}]} g = 0$

Dorfman derivative

$$L_V V' = [v, v'] + \mathcal{L}_v \lambda' - i_{v'} \mathrm{d}\lambda$$
$$= (V \cdot \partial) V' - (\partial \times_{\text{ad}} V) V'$$

takes standard form [*CSW*]

Generalised metric: $SO(d+1)$ structure

$$G_{\underline{mn}} = \frac{\mathrm{e}^{-\Delta}}{\sqrt{g}} \begin{pmatrix} g_{mn} & g_{mn}A^n \\ g_{np}A^p & \det g + g_{pq}A^pA^q \end{pmatrix}$$

where

$$A^m = \tfrac{1}{(d-1)!}\epsilon^{mp_1\dots p_{d-1}}A_{p_1\dots p_{d-1}} \qquad \in TM\otimes \det T^*M$$

so that

$$G(V,V) = \tfrac{1}{2}G_{\underline{mp}}G_{\underline{nq}}V^{\underline{mn}}V^{\underline{pq}}$$

encodes metric g, gauge potential A and warp factor Δ

Leibniz generalised parallelisations

deWitt reductions

1. global frame $\{\hat{e}_a\} \in TM$ $\Leftrightarrow$ M is parallelisable
2. structure const. $[\hat{e}_a, \hat{e}_b] = f_{ab}{}^c \hat{e}_c$ $\Leftrightarrow$ local group manifold

gauging of reduced theory fixed by Lie algebra, scalar ansatz

$$g_{mn} = M_{ab} e^a{}_m e^b{}_n$$

moduli M_{ab} parametrise $GL(d,\mathbb{R})/SO(d)$.

Generalised analogue

$$\{\hat{E}_A\} \in E \qquad \Leftrightarrow \text{“generalised parallelisable”}$$

$$L_{\hat{E}_A}\hat{E}_B = X_{AB}{}^C\hat{E}_C \qquad \Leftrightarrow \text{“Leibniz parallelisation”}$$

only guaranteed Leibniz algebra

much weaker requirement

for example M is only required to be a coset G/H

Conjecture

Given a Leibniz generalised parallelisation $\{\hat{E}_A\}$ there is a consistent truncation on M preserving the same number of supersymmetries as the original theory with embedding tensor given by $X_{AB}{}^C$ and gen. Scherk–Schwarz

$$G_{MN} = M_{AB} E^A{}_M E^B{}_N$$

gives scalar moduli

[c.f. Kaloper & Myers; Hull & Reid–Edwards; GMPW; . . . ; ABMN; CSW; Giessbuhler; Berman, Musaev, Thompson; . . .]

deWitt reductions

local group manifold ⇒ Leibniz gen. parallelisation

so deWitt reductions are of this type and [*CSW*]

using Weitzenböck connection prove embed. tensor is X

but . . .

New result

All round spheres S^d are Leibniz generalised parallelisable in $GL^+(d+1,\mathbb{R})$ with a Lie algebra $so(d+1)$

Stark contrast to conventional geometry

- only S^1, S^3 and S^7 are parallelisable (Bott, Milner; Kervaire)
- only $S^1 = U(1)$ and $S^3 = SU(2)$ are group manifolds

Sphere generalised parallelisations

Round sphere conventions

$$g = R^2\delta_{ij}\mathrm{d}y^i\mathrm{d}y^j = R^2\mathrm{d}s^2(S^d)$$
$$F = \frac{d-1}{R}\,\mathrm{vol}_g,$$

where $y^i y^i = 1$

$$[v_{ij}, v_{kl}] = R^{-1}\left(\delta_{ik}v_{lj} - \delta_{il}v_{kj} - \delta_{jk}v_{li} + \delta_{jl}v_{ki}\right)$$

$SO(d+1)$ Killing vectors

Global frame

$$\hat{E}_{ij} = v_{ij} + \sigma_{ij} + i_{v_{ij}} A$$

where $\sigma_{ij} = * \left(R^2 \mathrm{d}y_i \wedge \mathrm{d}y_j \right)$ and index $A = [ij]$

$$v_{ij} = 0 \qquad \text{when } y_i = y_j = 0$$
$$\mathrm{d}y_i \wedge \mathrm{d}y_j = 0 \qquad \text{when } y_i^2 + y_j^2 = 1$$

so global and is generalised orthonormal

$$G(\hat{E}_{ij}, \hat{E}_{kl}) = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

Algebra is $so(d+1)$

$$L_{\hat{E}_{ij}} \hat{E}_{kl} = [\![\hat{E}_{ij}, \hat{E}_{kl}]\!] = R^{-1} \left(\delta_{ik} \hat{E}_{lj} - \delta_{il} \hat{E}_{kj} - \delta_{jk} \hat{E}_{li} + \delta_{jl} \hat{E}_{ki} \right)$$

Generalised Scherk–Schwarz

$$G'^{MN} = \tfrac{1}{2} T^{ik} T^{jl} \hat{E}^M_{ij} \hat{E}^N_{kl}$$

where $\det T = 1$

$$T \in SL(d+1,\mathbb{R}) / SO(d+1)$$

gives

$$g' = \frac{R^2}{(T^{kl} y_k y_l)^{2/(d-1)}} \; T^{-1}_{ij} \mathrm{d}y^i \mathrm{d}y^j$$

$$A' = -\frac{1}{2(T^{kl} y_k y_l)} \frac{R^{d-1}}{(d-2)!} \; \epsilon_{i_1 \dots i_{d+1}} (T^{i_1 j} y_j) y^{i_2} \mathrm{d}y^{i_3} \wedge \dots \wedge \mathrm{d}y^{i_{d+1}} + A$$

$$\mathrm{e}^{2\Delta'} = (T^{kl} y_k y_l)^{(d-3)/(d-1)}$$

agrees with all known sphere reductions including warp factor

Example: S^3 and $SO(3,3)$

Near-horizon NS fivebranes

$$
\begin{aligned}
\mathrm{d}s_{10}^2 &= \mathrm{d}s^2(\mathbb{R}^{5,1}) + \mathrm{d}t^2 + R^2\mathrm{d}s^2(S^3), \\
H &= 2R^{-1}\,\mathrm{vol}_g, \\
\phi &= -t/R,
\end{aligned}
$$

with generalised tangent space exactly as Hitchin

$$
\begin{aligned}
E &\simeq TM \oplus T^*M \\
SL(4,\mathbb{R}) &\simeq SO(3,3) \\
SO(4)/\mathbb{Z}_2 &\simeq SO(3)\times SO(3)
\end{aligned}
$$

$SU(2) \times SU(2)$ gauging

(Anti-)self-dual combinations $\hat{E}^L_{\bar{a}}$ and $\hat{E}^R_a$ basis for $E = C^+ \oplus C^-$

$$
\begin{aligned}
L_{\hat{E}^L_{\bar{a}}} \hat{E}^L_{\bar{b}} &= R^{-1} \epsilon_{\bar{a}\bar{b}\bar{c}} \hat{E}^L_{\bar{c}} \\
L_{\hat{E}^R_a} \hat{E}^R_b &= R^{-1} \epsilon_{abc} \hat{E}^R_c \\
L_{\hat{E}^L_{\bar{a}}} \hat{E}^R_a &= 0
\end{aligned}
$$

matches embedding tensor and ansatz of [*Cvetic, Lu, Pope*]

Conventional parallelisation

Since $S^3 = SU(2)$ can also do conventional deWitt

$$\hat{E}^L_{\bar{a}} = l_{\bar{a}} - \lambda_{\bar{a}} - i_{l_{\bar{a}}} B$$
$$\hat{E}^R_{a} = l_{a} + \lambda_{a} - i_{l_{a}} B$$

with algebra giving $SU(2)$ gauging

$$L_{\hat{E}^L_{\bar{a}}} \hat{E}^L_{\bar{b}} = R^{-1} \epsilon_{\bar{a}\bar{b}\bar{c}} \hat{E}^L_{\bar{c}}$$
$$L_{\hat{E}^R_{a}} \hat{E}^R_{b} = R^{-1} \epsilon_{ab\bar{c}} \hat{E}^L_{\bar{c}}$$
$$L_{\hat{E}^R_{a}} \hat{E}^L_{\bar{a}} = R^{-1} \epsilon_{a\bar{b}\bar{c}} \hat{E}^L_{\bar{c}}$$

Example: S^4 and $E_{4(4)}$

Near-horizon M5-branes: $\mathrm{AdS}_7 \times S^4$

$$E \simeq TM \oplus \Lambda^2 T^* M$$
$$E_{4(4)} \times \mathbb{R}^+ \simeq GL^+(5,\mathbb{R})$$
$$H_4 \simeq SO(5)$$

everything goes straight through to match [*NVvN*]

Example: S^7 and $E_{7(7)}$

Near-horizon M2-branes: $\mathrm{AdS}_4 \times S^7$

$$E \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^7 T^*M)$$
$$E_{7(7)} \times \mathbb{R}^+ \supset GL^+(8,\mathbb{R})$$
$$H_7 \simeq SU(8)/\mathbb{Z}_2 \supset SO(8)/\mathbb{Z}_2$$

more general frame (cf. [*de Wit & Nicolai; G^2 & Nicolai*])

$$\hat{E}_A = \begin{cases} \hat{E}_{ij} = v_{ij} + \sigma_{ij} + i_{v_{ij}}\tilde{A} & \text{for } E^{(0)} \\ \hat{E}'^{\,ij} = \omega_{ij} + \tau_{ij} - j\tilde{A}\wedge\omega_{ij} & \text{for } E^{(1)} \end{cases}$$

$$\omega_{ij} = R^2 \mathrm{d}y_i \wedge \mathrm{d}y_j \text{ and } \tau_{ij} = R(y_i \mathrm{d}y_j - y_j \mathrm{d}y_i) \otimes \mathrm{vol}_g$$

Leibniz algebra

$$
\begin{aligned}
L_{\hat{E}_{ij}}\hat{E}_{kl} &= R^{-1}\big(\delta_{ik}\hat{E}_{lj} - \delta_{il}\hat{E}_{kj} - \delta_{jk}\hat{E}_{li} + \delta_{jl}\hat{E}_{ki}\big) \\
L_{\hat{E}_{ij}}\hat{E}'^{\,kl} &= R^{-1}\big(\delta_i^k\delta_{jp}\hat{E}'^{\,lp} - \delta_i^l\delta_{jp}\hat{E}'^{\,kp} - \delta_j^k\delta_{ip}\hat{E}'^{\,lp} + \delta_j^l\delta_{ip}\hat{E}'^{\,kp}\big) \\
L_{\hat{E}'^{\,ij}}\hat{E}_{kl} &= 0 \\
L_{\hat{E}'^{\,ij}}\hat{E}'^{\,kl} &= 0
\end{aligned}
$$

embedding tensor and moduli agree with [*de Wit & Nicolai*]

Example: S^5 and $E_{6(6)}$

Near-horizon D3-branes: $\mathrm{AdS}_5 \times S^5$

$$E \simeq TM \oplus (T^*M \oplus T^*M) \oplus \Lambda^3 T^*M \oplus (\Lambda^5 T^*M \oplus \Lambda^5 T^*M)$$
$$E_{6(6)} \times \mathbb{R}^+ \supset GL^+(6,\mathbb{R}) \times SL(2,\mathbb{R})$$
$$H_6 = USp(8)/\mathbb{Z}_2 \supset SU(4)/\mathbb{Z}_2 \times SO(2) \simeq SO(6) \times SO(2),$$

again more general frame

$$\hat{E}_A = \begin{cases} \hat{E}_{ij} = v_{ij} + \sigma_{ij} - i_{v_{ij}} A & \text{for } E^{(0)} \\ \hat{E}^i_{\hat{\alpha}} = \hat{f}_{\hat{\alpha}}{}^{\alpha} \left(R\mathrm{d}y^i + y^i \,\mathrm{vol}_g + R\mathrm{d}y^i \wedge A \right) & \text{for } E^{(\alpha)} \end{cases}$$

Leibniz algebra

$$
\begin{aligned}
L_{\hat{E}_{ij}} \hat{E}_{kl} &= R^{-1}\big(\delta_{ik}\hat{E}_{jl} - \delta_{il}\hat{E}_{jk} - \delta_{jk}\hat{E}_{il} + \delta_{jl}\hat{E}_{ik}\big) \\
L_{\hat{E}_{ij}} \hat{E}^{k}_{\hat{\alpha}} &= R^{-1}\big(\delta_{il}\delta^{k}_{j}\hat{E}^{l}_{\hat{\alpha}} - \delta_{jl}\delta^{k}_{i}\hat{E}^{l}_{\hat{\alpha}}\big) \\
L_{\hat{E}^{i}_{\hat{\alpha}}} \hat{E}_{jk} &= 0 \\
L_{\hat{E}^{i}_{\hat{\alpha}}} \hat{E}^{j}_{\hat{\beta}} &= 0
\end{aligned}
$$

taking $E_{6(6)} \times \mathbb{R}^+ / USp(8)$ generalised metric

$$
G^{MN} = \delta^{AB}({U_A}^{C}\hat{E}_C{}^{M})({U_B}^{D}\hat{E}_D{}^{N}) = T^{AB}\hat{E}_A{}^{M}\hat{E}_B{}^{N}
$$

gives

$$e^{2\Delta'} g'^{mn} = \delta^{AB} U_A{}^{jj'} U_B{}^{kk'} v^m_{jj'} v^n_{kk'}$$

$$e^{2\Delta'} B'^{\alpha}_{mn} = \delta^{AB} U_A{}^{jj'} U_B{}_k{}^{\hat{\gamma}} \hat{f}_{\hat{\gamma}}{}^{\alpha} R \, v_{jj'\,[m} \partial_{n]} y^k$$

$$e^{2\Delta'} \left(A'_{mnpq} - \tfrac{3}{2} B'_{\alpha m[n} B'^{\alpha}_{pq]} - A_{mnpq}\right) = -\delta^{AB} U_A{}^{jj'} U_B{}^{kk'} v_{jj'\,m} \lambda_{kk'\,npq}$$

$$e^{2\Delta'} \left(e^{-\phi'} h'^{\alpha\beta} g'_{mn} - B'^{\alpha}_{mp} g'^{pq} B'^{\beta}_{qn}\right) = \delta^{AB} U_A{}_i{}^{\hat{\alpha}} U_B{}_j{}^{\hat{\beta}} \hat{f}_{\hat{\alpha}}{}^{\alpha} \hat{f}_{\hat{\beta}}{}^{\beta} R^2 \partial_m y^i \partial_n y^j$$

new complete ansatz for the scalar fields

DFT/extended geometry description?

Sphere examples highlights key question [*cf Park, BCP, Cederwall*]

what is extended geometry of generic sugra background?

what are winding modes? if locally a manifold then for example

$$\mathrm{d}s^2_{O(d,d)} = \mathrm{d}x^m \left(\mathrm{d}y_m + \Gamma^p{}_{mn} y_p \mathrm{d}x^n\right)$$

model doubled space $\mathcal{M} = T^*M$, extra data of connection Γ

- for flat metric Γ must be pure gauge so M is parallelisable
- for gen. geom. take section $y_p = 0$ and metric always exists

Conclusions

- natural conjecture

all consistent truncations are gen. DeWitt reductions

- analogue of unimodular condition ${f_{ab}}^b = 0$?

$$X \in N \oplus E \qquad \text{gen. torsion rep.}$$

for $N = \Lambda^3 E$ for $O(d,d)$, **351** for $E_{6(6)}$, **912** for $E_{7(7)}$, etc

$${X_{AB}}^B = 0 \qquad \text{component in } E$$

satisfied for all examples

- full ansatz should follow from expansion in $\{\hat{E}_A\}$ (c.f. [*Hohm, Samtleben*])

$$A_\mu = A_\mu{}^A \hat{E}_A$$
$$\delta A_\mu = \partial_\mu \Lambda - L_{A_\mu}\Lambda = \left(\partial_\mu \Lambda^C - X_{AB}{}^C A_\mu^A \Lambda^B\right)\hat{E}_C$$

full tensor hierarchy

- if can reformulate the *full* theory using generalised geometry

group theoretic proof of consistency

- classification of Leibniz generalised parallelisations
  - reduces to problem in Lie algebras
  - M is necessarily coset G/H
  - quite constrained
- understand new $SO(8)$ gaugings of Dell'Agata et al.?