Gradient and Vorticity Banding Phenomena in a Sheared Granular Fluid: Order Parameter Description

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Outline of Talk

• Shear-banding phenomena in granular and complex fluids

• Gradient Banding and Patterns in 2D granular PCF (Landau-Stuart Eqn.)

• Vorticity Banding in 3D gPCF

• Theory for Mode Interactions

• Spatially Modulated Patterns (Ginzburg-Landau Eqn.)

• Summary and Outlook
Sheared granular material (or any complex fluid) does not flow homogeneously like a simple fluid, but forms banded regions having inhomogeneous gradients in hydrodynamic fields.
Origin of Shear-banding?

Multiple Branches in Constitutive Curve

Non-monotonic Steady state Shear Stress vs. Shear Rate Curve

Gradient Banding

Shear Rate > 'Critical' shear rate
Flow breaks into bands of high and low shear rates with same shear stress along the gradient direction.

Vorticity Banding

Shear Stress > Critical Shear Stress
Flow breaks into bands of high and low shear stresses with same shear rates along the vorticity direction.
Gradient Banding in 2D-gPCF

\[ \phi^0 = 0.05 \]

\[ \phi^0 = 0.3 \]

\[ \frac{\partial}{\partial x} = 0, \quad \frac{\partial}{\partial y} \neq 0, \]

Order-parameter description of shear-banding?

Shukla & Alam (2009, 2011a,b, 2013a,b)
Granular Hydrodynamic Equations

(Savage, Jenkins, Goldhirsch, ...)

Balance Equations

- **Mass**
  \[ \frac{D\rho}{Dt} = -\rho \nabla \cdot \mathbf{u} \]

- **Momentum**
  \[ \rho \frac{D\mathbf{u}}{Dt} = -\nabla \cdot \mathbf{\Sigma} \]

- **Pseudo Thermal Energy**
  \[ \frac{\dim}{2} \rho \frac{D\mathbf{T}}{Dt} = -\nabla \cdot \mathbf{q} - \mathbf{\Sigma} : \nabla \mathbf{u} - D \]

- \( \phi \) : Volume fraction of particles
- \( T \) : Granular temperature
- \( u \) : Streamwise velocity
- \( v \) : Normal velocity

Navier-Stokes Order Constitutive Model

- **Stress**
  \[ \mathbf{\Sigma} = (p - \zeta (\nabla \cdot \mathbf{u})) \mathbf{I} - 2\mu \mathbf{S} \]
  \[ S = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) - \frac{1}{\dim} (\nabla \cdot \mathbf{u}) \mathbf{I} \]

- **Flux of pseudo-thermal energy**
  \[ \mathbf{q} = -\kappa \nabla T \]

- **dim**: Dimension of system
- **\( \kappa \)**: Thermal Conductivity
- **\( \mu \)**: Shear Viscosity
- **\( D \)**: Sink of granular energy

\[ \rho = \rho_p \phi \]
**Plane Couette Flow (gPCF)**

- **Base Flow**: Steady, Fully developed.
- **Boundary condition**: No Slip, Zero heat flux.

**Uniform Shear Solution**

\[
\frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) = 0
\]

\[
\frac{\partial p}{\partial y} = 0
\]

\[
\frac{\partial}{\partial y} \left( \kappa \frac{\partial T}{\partial y} \right) + \mu \left( \frac{\partial u}{\partial y} \right)^2 - D = 0
\]

\[
\phi^0 = \text{const.} \quad T^0 = \text{const.}
\]

\[
u^0(y) = y
\]

**Control parameters**

- \( H = \bar{h} / d \)  
  Couette Gap
- \( e \)  
  Restitution Coeff.
- \( \phi^0 \)  
  Volume fraction or mean density

\( \bar{h} \): Reference Length
\( \bar{U} \): Reference velocity
\( \bar{h} / U \): Reference Time

\( d \): Particle diameter
Linear Stability

If the disturbances are of **infinitesimal magnitude**, ‘**nonlinear terms’** in disturbance eqns. can be neglected.

\[ \phi^0 = \text{const.} \quad T^0 = \text{const.} \]
\[ u^0(y) = y \]

\[ X'(x,y,z,t) \sim \exp(\omega t)\exp(i k_x x + i k_z z) \]

**Perturbation (X’)**

\[ + \]

\[ X_{\text{total}} \]

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Can ‘Linear Stability Analysis’ able to predict ‘Gradient-banding’ in Granular Couette flow as observed in Particle Simulations?
Can ‘Linear Stability Analysis’ able to predict ‘Gradient-banding’ in Granular Couette flow as observed in Particle Simulations?

Not for all flow regime
One must look beyond Linear Stability

**Linear Theory**

![Graph showing stability regions](image)

```
- Flow remains 'uniform' in dilute limit
- Density segregated solutions are not possible in dilute limit

**Particle Simulation**

```
- Flow is 'non-uniform' in dilute limit
- Density Segregated solutions are possible in dilute limit

\[ \phi^0 = 0.05 \]
```

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Dynamics close to critical situation is dominated by finitely many “critical” modes.

Nonlinear Stability Analysis: Center Manifold Reduction
(Carr 1981; Shukla & Alam, PRL 2009)

Taking the inner product of slow mode equation with adjoint eigenfunction of the linear problem and separating the like-power terms in amplitude, we get Landau-Stuart equation

\[
\frac{dZ}{dt} = c^{(0)}Z + c^{(2)}Z|Z|^2 + c^{(4)}Z|Z|^4 + \ldots
\]

First Landau Coefficient

\[ c^{(2)} = a^{(2)} + ib^{(2)} \]

Second Landau Coefficient

\[ c^{(4)} = a^{(4)} + ib^{(4)} \]

\[ Z(t): \text{complex amplitude of `finite-size' perturbation} \]
Cont...

Other perturbation methods can be used:

\[ c^{(2)} = \frac{\langle Y, N_2(X^{[0;2]}, X^{[1;1]}) + N_2(X^{[2;2]}, \ddot{X}^{[1;1]}) + N_3(X^{[1;1]}, X^{[1;1]}, \dddot{X}^{[1;1]}) \rangle}{\langle Y, X^{[1;1]} \rangle} \]

\[ \left( \frac{\partial}{\partial t} - L \right) \psi = \text{Nonlinear terms} \]

\[ c^{(4)} = \frac{\langle Y, \theta(X^{[1;1]}, X^{[0;2]}, X^{[2;2]}, X^{[1;3]}, X^{[3;3]}, X^{[2;4]}, X^{[0;4]}) \rangle}{\langle Y, X^{[1;1]} \rangle} \]

Enslaved Equation

Represent all non-critical modes

Distortion of mean flow

Second harmonic

\text{Tuesday 2 July 13}

\text{e.g. Amplitude expansion method and multiple scale analysis}
Linear Problem \( L X^{[1;1]} = c^{(0)} X^{[1;1]} \)

Second Harmonic \( L_{22} X^{[2;2]} = G_{22} \)

Distortion to mean flow \( L_{02} X^{[0;2]} = G_{02} \)

Distortion to fundamental \( L_{13} X^{[1;3]} = c^{(2)} X^{[1;1]} + G_{13} \)

Analytical solution exists

Analytically solvable

Shukla & Alam (JFM 2011a)

Analytical expression of first Landau coefficient

\[
c^{(2)} = \frac{\phi^a G_{13}^1 + u^a G_{13}^2 + v^a G_{13}^3 + T^a G_{13}^4}{\phi^a \phi^{[1;1]} + u^a u^{[1;1]} + v^a v^{[1;1]} + T^a T^{[1;1]}}
\]

We have developed a spectral based numerical code to calculate Landau coefficients.
Numerical Method: comparison with analytical solution

(i) Spectral collocation method,
(ii) SVD for inhomogeneous eqns.
(iii) Gauss-Chebyshev quadrature for integrals.

This validates spectral-based numerical code

Real part of first Landau coefficient

Shukla & Alam (JFM, 2011a)

Distorted density eigenfunction

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Equilibrium Amplitude and Bifurcation

Cubic Landau Eqn: \( Z = Ae^{i\theta} \)

\[
\begin{align*}
\frac{dZ}{dt} &= c^{(0)}Z + c^{(2)}Z | Z |^2 \\
\frac{dA}{dt} &= a^{(0)}A + a^{(2)}A^3, \\
\frac{d\theta}{dt} &= b^{(0)} + b^{(2)}A^2
\end{align*}
\]

Real amplitude eqn.

Phase eqn.

Cubic Solution

\[
A = 0, \quad A = \pm \sqrt{-\frac{a^{(0)}}{a^{(2)}}}
\]

Supercritical Bifurcation \( a^{(0)} > 0, a^{(2)} < 0 \)

Subcritical Bifurcation \( a^{(0)} < 0, a^{(2)} > 0 \)

\( b^{(0)} = 0 \)

\( b^{(0)} \neq 0 \)

Pitchfork (stationary) bifurcation

Hopf (oscillatory) bifurcation

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Phase Diagram

Constitutive equations are function of radial distribution function (RDF)

\[ \phi^0 : \text{Mean Density} \]
\[ \phi_m : \text{Maximum Mean Density} \]
\[ H : \text{Couette Gap} \]

\[ \chi(\phi) = \frac{1}{1 - (\phi / \phi_m)^{1/3}} \]

\[ H^* = H\beta^{-1}\sqrt{(1 - e^2)} \]

**Gradient-banding in dilute flows**

This agrees with MD simulations of Tan & Goldhirsch 1997

\[ \Delta \text{Nonlinear Stability theory and MD simulations both support gradient banding in 2D-GPCF (PRL 2009)} \]
Cont...

Carnahan-Starling RDF

\[ \chi(\phi) = \frac{1 - \phi / 2}{(1 - \phi / \phi_m)^3} \]

Change of constitutive relations leads to three degenerate points

![Diagram](image)

0.196
Subcritical -> supercritical

0.467
Supercritical -> subcritical

0.559
Subcritical -> supercritical

(JFM 2011a)
Paradigm of Pitchfork Bifurcations

Supercritical
\[ \phi^0 > \phi^{s2} \]

Subcritical
\[ \phi^{s1} < \phi^0 < \phi^{s2} \approx 0.559 \]

Supercritical
\[ \phi^s < \phi^0 < \phi^{s1} \approx 0.467 \]

Subcritical
\[ \phi^l < \phi^0 < \phi^s \approx 0.196 \]

Bifurcation from infinity
\[ \phi^0 < \phi^l \approx 0.174 \]
Incompressible Newtonian Fluids

Plane Couette Flow
- Bifurcation From Infinity

Plane Poiseuille Flow
- Subcritical Bifurcation

Rayleigh-Bénard Convection
- Supercritical Bifurcation

Taylor Couette Flow
- Supercritical Bifurcation

All in one!
Granular Plane Couette flow
admits all types of Pitchfork bifurcations
Conclusions

- Problem is analytically solvable.
- Landau-Stuart equation describes gradient-banding transition in a sheared granular fluid.
- Landau coefficients suggest that there is a “sub-critical” (bifurcation from infinity) finite amplitude instability for “dilute” flows even though the dilute flow is stable according to linear theory.
- This result agrees with previous MD-simulation of gPCF.
- gPCF serves as a paradigm of pitchfork bifurcations.
- Analytical solutions have been obtained.
- An spectral based numerical code has been validated.

References: Shukla & Alam (2011a), J. Fluid Mech., vol 666, 204-253
``Gradient-banding” and Saturn’s Ring?

Self gravity?... other effects needed...

Salo, Schmidt & Spahn (2001) Icarus,
Patterns in 2D-gPCF

Flow is linearly unstable due to stationary and traveling waves, leading to particle clustering along the flow and gradient directions.

Shukla & Alam, JFM (2011b) vol. 672, 147-195
Particle Simulations of Granular PCF

(Conway and Glasser 2006)
Stability of 2D-gPCF when subject to “finite amplitude perturbation”

Seeking an order parameter theory for stationary and traveling wave instabilities...
Linear Theory

Standing wave instability

1st peak $k_x \approx O(1)$

Long-wave instability

$k_x \approx 0$

2nd peak $k_x \approx O(1)$

Traveling wave instability

Phase velocity

$\phi = 0.2, H = 100, e = 0.8$

$a^{(0)}$ Growth rate

$c_{ph}$ Phase velocity

$c_{ph} = -\frac{b^{(0)}}{k_x}$

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Long-Wave Instabilities

Supercritical pitchfork/Hopf bifurcation

\[ k_x \approx 0 \]

\[ \phi = 0.2, H = 100, e = 0.8 \]

\[ k_x = 2.1 \times 10^{-5} \]

Growth Rate

\[ k_x = 10^{-5} \]

SW Density Patterns

\[ k_x = 4 \times 10^{-5} \]

TW Density Patterns

Real and Imag. Part of first LC

Amplitude

Non-linear

Linear

Non-linear

Linear

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Stationary Instability

$\phi = 0.2, \ H = 100, \ e = 0.8$

Supercritical pitchfork bifurcation

$k_x \approx O(1)$

SW density patterns

Amplitude

Non-linear

Linear

Structural features are different from long-wave stationary instability
Travelling Instabilities

Supercritical Hopf bifurcation

\[ c_{ph}^e = - \frac{\omega}{k_x} = c_{ph} - \frac{b^{(2)} A^2}{k_x} \]

Non-linear

Linear

\[ k_x = 0.935 \]

\[ \phi = 0.2, \ H = 100, \ e = 0.8 \]

\[ k_x \approx O(1) \]
Supercritical Hopf Bifurcation/ Limit Cycle Solutions

\[ \frac{dZ}{dt} = c^{(0)}Z + c^{(2)}Z \mid Z \mid^2 \]

\[ \frac{dA}{dt} = a^{(0)}A + a^{(2)}A^3, \]

\[ \frac{d\theta}{dt} = b^{(0)} + b^{(2)}A^2 \]

\[ A^2(t) = \frac{a^{(0)}A_0^2}{[a^{(0)} + a^{(2)}A_0^2] \exp(-2a^{(0)}t) - a^{(2)}A_0^2}, \]

\[ \theta(t) = \theta_0 + b^{(0)}t - \frac{b^{(2)}a^{(0)}}{a^{(2)}} \ln \left[ \frac{a^{(0)} + a^{(2)}A_0^2(1 - \exp(-2a^{(0)}t))}{a^{(0)}} \right] \]

\( A_0 = A(t = 0), \theta_0 = \theta(t = 0) \)

\( \phi = 0.2, H = 100 \)

\( k_x = 0.935 \)

\( A_0 > A_e \) Stable limit cycle

\( A_0 < A_e \) Stable limit cycle
Subcritical Hopf Bifurcation/ Limit Cycle Solutions

Both orbits spiral away from the unstable limit cycle

moderate values of $k_x$

$k_x = 0.215$
Dominant Stationary Instabilities

Density patterns are structurally similar at all densities.

Non-linear $\phi^0 = 0.1, k_x = 0.52$

Non-linear $\phi^0 = 0.3, k_x = 0.77$

Non-linear $\phi^0 = 0.45, k_x = 1.39$
Dominant Traveling Instabilities

Supercritical Hopf Bifurcation

Subcritical Hopf Bifurcation

Resonance

Non-linear

\( \phi^0 = 0.15, k_x = 0.85 \) Stable

Non-linear

\( \phi^0 = 0.33, k_x = 0.13 \) Unstable

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Conclusions

- The origin of nonlinear states at long-wave lengths is tied to the corresponding subcritical / supercritical nonlinear gradient-banding solutions (discussed in 1st Part of talk).

- For the dominant stationary instability nonlinear solutions appear via supercritical bifurcation.

- Structure of patterns of supercritical stationary solutions look similar at any value of density and Couette gap.

- For the dominant traveling instability, there are supercritical and subcritical Hopf bifurcations at small and large densities.

- Uncovered mean flow resonance at quadratic order.

Vorticity Banding in 3D-gPCF

Pure Spanwise Perturbations

\[ \frac{\partial}{\partial x} = 0, \quad \frac{\partial}{\partial y} = 0, \quad \frac{\partial}{\partial z} \neq 0 \]

Shukla & Alam (2013b, JFM)
Linear Vorticity Banding

Gradient-banding modes stationary modes at all density.

Vorticity-banding modes stationary at dilute limit & traveling in moderate-to-dense limit.

Dispersion relation

$$\left( \omega + \frac{\mu^0 k_z^2}{\phi^0 H^2} \right)^2 \left( \omega^3 + a_2 \omega^2 + a_1 \omega + a_0 \right) = 0$$

Pure spanwise GPCF

Stable

Unstable

Pitchfork bifurcation

Supercritical Hopf bifurcation

Analytically solvable

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Nonlinear Stability

Linear Problem
\[ L X^{[1;1]} = c^{(0)} X^{[1;1]} \]

Second Harmonic
\[ L_{22} X^{[2;2]} = G_{22} \]

Distortion to mean flow
\[ L_{02} X^{[0;2]} = G_{02} \]

Distortion to fundamental
\[ L_{13} X^{[1;3]} = c^{(2)} X^{[1;1]} + G_{13} \]

Analytical expression for first Landau coefficient
\[ c^{(2)} = \frac{\phi^a G_{13}^1 + w^a G_{13}^4 + T^a G_{13}^5}{\phi^a \phi^{[1;1]} + w^a w^{[1;1]} + T^a T^{[1;1]}} \]

Analytically solvable

Analytical solution exists at any order in amplitude

Shukla & Alam (2013b, JFM)
Nonlinear Vorticity Banding

\[ A_e = \pm \sqrt{-\frac{a(0)}{a^{(2)}}} \]

\[ A_e = \pm \sqrt{-a^{(2)} \pm \sqrt{a^{(2)} ^2 - 4a(0)a^{(4)}}} \]

\[ \phi^d = 0.08 \]

\[ 0.08 < \phi^0 \leq 0.1 \]

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Significance of higher order nonlinear correction

Nonlinear disturbance

\[
\left( \frac{\partial}{\partial t} - L \right) X' = \sum_{i=2}^{\infty} N_i
\]

Cubic correction

\[ c^{(4)} = c^{(4)}(N_2, N_3, N_4, N_5) \]

Quintic correction

\[ c^{(4)} = c^{(4)}(N_2, N_3, N_4, N_5) \]

**n-th order Landau coeff. need (n+1)-th order nonlinear term for correct results.**
Particle density iso-surfaces for $\phi = 0.05, e = 0.6$

\[
\frac{L}{W} = \frac{D}{W} = 3
\]
Vorticity Banding in Dense 3D Granular Flow
(Grebenkov, Ciamarra, Nicodemi, Coniglio, PRL 2008, vol 100)

Ordered state           lower viscosity
Disordered state

Disordered
Ordered

Metastable
Ordered
Disordered

Viscosity

φ = 0.608

Time

V_{shear}

Ordered state  lower viscosity
Disordered state  higher viscosity

Width (z)
Length (x)
Depth (y)

φ^{-1}

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Conclusions

Higher order nonlinear terms are important to get correct bifurcation scenario.

Supercritical Region
Subcritical Region

Pitchfork Bifurcation
Hopf Bifurcation

Subcritical and supercritical
Subcritical

Density

$\phi^0 = 0.1$

Gradient Banding

$H^* = H\beta^{-1}\sqrt{1-e^2}$

Analytical solution exists at any order.

Higher order nonlinear terms are important to get correct bifurcation scenario.
Patterns in three-dimensional gPCF

Shukla & Alam (2013c, preprint)
Nonlinear Stability

Linear Problem \( L X^{[1;1]} = c^{(0)} X^{[1;1]} \)

Second Harmonic \( L_{22} X^{[2;2]} = G_{22} \)

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Distortion to fundamental \( L_{13} X^{[1;3]} = c^{(2)} X^{[1;1]} + G_{13} \)

Analytical Expression of first Landau Coefficient

\[
c^{(2)} = \frac{\phi^{a} G_{13}^1 + u^{a} G_{13}^2 + v^{a} G_{13}^3 + w^{a} G_{13}^4 + T^{a} G_{13}^5}{\phi^{a} \phi^{[1;1]} + u^{a} u^{[1;1]} + v^{a} v^{[1;1]} + w^{a} w^{[1;1]} + T^{a} T^{[1;1]}}
\]

Adjoint Eigenfunction \((\phi^{a}, u^{a}, v^{a}, w^{a}, T^{a})\)

Analytical solution exists
Dilute Flows

Supercritical bifurcation

Subcritical bifurcation

Linear Stability Curve

$\phi^0 = 0.02$

$k_z = 40$

$A_e$ vs $k_z$

$k_z = 2$

$A_e$ vs $H - H_c$

 Stable

 Unstable

Large wavenumbers

Small wavenumbers

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Moderate-to-Dense Flows

Linear Stability Curve

Supercritical bifurcation

\[ k_z = 2, \quad e = 0.8 \]

\[ \phi^0 = 0.2 \]

Dense Flows

Subcritical bifurcation

\[ \phi^0 = 0.5 \]

\[ k_z = 2, \quad e = 0.8 \]
Linear and Nonlinear Density Patterns

Stable, supercritical patterns

\[ H = 135, \phi^0 = 0.2, \]
\[ e = 0.8, k_z = 2 \]

Unstable, subcritical patterns

\[ H = 140, \phi^0 = 0.5, \]
\[ e = 0.8, k_z = 2 \]

- Patterns exists in streamwise and gradient direction.
- Nonlinear pattern looks very different from linear patterns.
Conclusions

- In dilute limit finite amplitude solutions occur via supercritical bifurcation for large wavenumbers and via subcritical bifurcation for small wavenumbers.

- Transition from supercritical to subcritical in moderate-to-dense limit.

- The finite amplitude nonlinear patterns look very different from its linear analogue.

*Shukla & Alam (2013c, preprint for PoF)*
Theory for Spatially Modulated Patterns

Complex Ginzburg Landau Equation (CGLE)

Landau Equation

\[
\frac{dA}{dt} = c^{(0)} A + c^{(2)} A | A |^2
\]

Holds for \textit{spatially periodic} patterns

Complex Ginzburg Landau Equation

\[
\frac{\partial A}{\partial t} = \varepsilon^2 A + a_2 \frac{\partial^2 A}{\partial X^2} + c^{(2)} A | A |^2
\]

Partial differential equation

Holds for \textit{spatially modulated} patterns
Under which condition CGLE arises?

For $H < H_c$, all modes are decaying: Homogeneous state is stable, at $k_x = k_{xc}$ a critical wave number gains neutral stability, there is a narrow band of wavenumbers around the critical value where the growth rate is slightly positive.

width of the unstable wavenumbers: $\propto (H - H_c)^{1/2}$
Theory (Multiple scale analysis)

\[
\left( \frac{\partial}{\partial t} - L \right) X' (x, y, t) = \sum_{i=2}^{\infty} N_i
\]

Growth rate is of order \( H - H_c \)  \, \text{Stewartson & Stuart (1971)}

The timescale at which nonlinear interaction affects the evolution of fundamental mode is of order \(1/(\text{growth rate})\)

\[\varepsilon^2 = d_{1r} \mid H - H_c \mid\]

\[\tau = \varepsilon^2 t \quad \rightarrow \text{Slow time scale}\]

\[\xi = \varepsilon (x - c_g t) \quad \rightarrow \text{Slow length scale}\]

\[\downarrow \quad \text{Group velocity}\]

\[\frac{\partial A}{\partial \tau} = \frac{d_1}{d_{1r}} A + a_2 \frac{\partial^2 A}{\partial \xi^2} + c^{(2)} |A|^2 A\]

\[X = x - c_g t\]
Patterns in Vibrated Bed

Conclusions

- Complex Ginzburg Landau equation has been derived that describes spatio-temporal patterns in a "bounded" sheared granular fluid.

- Numerical results awaited...

Recent work of Saitoh and Hayakawa (Granular Matter 2011) on TDGL in "unbounded" shear flow.
Summary

- Landau-type order parameter theory for the gradient banding in gPCF has been developed using center manifold reduction.  

- Analytical solution for the shearbanding instability, comparison with numerics & bifurcation scenario have been obtained. 
  Ref: JFM, vol. 666, 204-253, (2011a)

- The order parameter theory for the 2D and 3D gPCF has been developed. Nonlinear patterns and bifurcations have been studied. 
  Ref: JFM, vol. 672, 147-195 (2011b)

- Nonlinear states and bistability for vorticity banding have been analysed. 
  Ref: JFM, vol. 718 (2013b)

- Coupled Landau equations for resonating and non-resonating cases have been derived. 
  Preprint

- Complex Ginzburg Landau equation has been derived for bounded shear flow. 
  Preprint
Outlook

Present order-parameter theory can be modified for other pattern forming problems, e.g. granular convection, granular Taylor-Couette flow, inclined Chute flow, etc.
Revisit nonlinear theory of Saturn’s Ring

- Non-isothermal model with spin, stress
- Anisotropy & self-gravity ...
- Spatially modulated waves ...
- Wave interactions ...
- Secondary instability, ....