Acknowledgment

- Annette Zippelius
- Andrea Fiege
- Timo Aspelmeier
- Matthias Sperl
- Iraj Gholami
Steel balls (∼7 mm ⌀) on a sieve
- Driven by air flow
- Measurement of mean square displacement
\[ \delta r^2(t) = \langle [r(t) - r(0)]^2 \rangle \]
Inelastic, Smooth, Hard Spheres: A Model for Granular Particles

Hard Spheres completely characterized by

- Mass \( m \)
- Radius \( a \)
- Coefficient of restitution \( \epsilon \in [0, 1] \)

Collision law

\[
\mathbf{v}_n' = -\epsilon \mathbf{v}_n, \\
\mathbf{v}_t' = \mathbf{v}_t
\]

Energy Loss on average per collision

\[
\Delta E \propto 1 - \epsilon^2
\]
A Sandstorm for Theoretical Physicists

Random Force $\xi_i(t)$, gaussian distributed

- Average $\langle \xi_i \rangle = 0$
- Driving power $P_D = \langle \xi_i^2 \rangle$

Stationary State as a balance between driving & dissipation

Event Driven Simulations
- 10 000 particles

Bidisperse to avoid crystallization

Coefficient of Restitution $\epsilon = 0.9$

Area Fraction $\eta = 0.1–0.81$

I. Gholami et al. PRE 84 2011

Driven Granular Fluids: Collective Effects
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Driven Granular Fluids: Collective Effects
Outline

1. Static Structure & Momentum Conservation
2. Long-Time Tails
3. The Granular Glass Transition

See also the Lecture on Friday
Static Structure: A Surprise

- Strong increase for $k \rightarrow 0$
- Highly Correlated on large Length Scales
- Implies so called *Giant Number Fluctuations*

> Volume Fraction $\varphi = 0.2$
> $N = 50 \times 400,000$

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2 Kranz, Fiege, Zippelius, in preparation
A Toy Model

\[ \partial_t h(\mathbf{r}, t) = \eta \nabla^2 h(\mathbf{r}, t) + \xi(\mathbf{r}, t) \]

Correlation Function

\[ C(k) = \left\langle \hat{h}(-k)\hat{h}(k) \right\rangle \simeq \frac{\left\langle \hat{\xi}(-k)\hat{\xi}(k) \right\rangle}{\eta k^2} \]  
Grinstein et al., PRL 64 1990

Random Force

\[ \left\langle \xi(\mathbf{r})\xi(\mathbf{r}') \right\rangle \propto \delta(\mathbf{r} - \mathbf{r}') \Rightarrow \left\langle \hat{\xi}(-k)\hat{\xi}(k) \right\rangle = 1. \]

Equilibrium

\[ \left\langle \hat{\xi}(-k)\hat{\xi}(k) \right\rangle \propto \eta k^2 \text{ due to FDT} \]

Local Pairs

\[ \left\langle \xi(\mathbf{r})\xi(\mathbf{r}') \right\rangle \propto \Theta(\mathbf{r} - \mathbf{r}' - \ell) \Rightarrow \left\langle \hat{\xi}(-k)\hat{\xi}(k) \right\rangle \propto \ell^2 k^2. \]
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Divergence under Control

- Volume Fraction $\varphi = 0.2$

- Volume Fraction $\varphi = 0.4$
Insight can be used for Measurements

- Shear viscosity $\eta$
- Volume fraction $\varphi$
- $\varepsilon = 0.9$
- $\varepsilon = 0.8$
- $\varepsilon = 0.7$

Driven Granular Fluids: Collective Effects
Long-Time Tails
The Velocity Autocorrelation Function

\[ \psi(t) = \langle v_s(0) | v_s(t) \rangle \]

**Long-Time Tails**\( \psi(t) \propto t^{-\alpha} \)

(instead of exponential decay)

- In 3D elastic & inelastic hard spheres have \( \alpha = 3/2 \)
Equation of Motion

\[ \partial_t \psi(t) + \omega_E \psi(t) + \int_0^t d\tau M(t - \tau) \psi(\tau) = 0 \]

Collision Frequency \( \omega_E \)

Memory Kernel \( M(t) \)

Incoherent Scattering Function \( \phi_s(k, t) \) contains more information about the tagged particle
Mode-Coupling Approximation

Consider coupling of tagged particle to:
- Collective Density Modes $\phi(k, t)$
- Longitudinal Current Modes $\phi_L(k, t)$
- Transverse Current Modes $\phi_T(k, t)$

Transverse Mode yields slowest decay (in 3D)

$$M(t \to \infty) = M_T(t \to \infty) \propto (1 + \varepsilon)^2 \int_0^\infty dk j''(k) \phi_T(k, t) \phi_s(k, t)$$

and indeed $\psi(t \to \infty) \propto t^{-3/2}$

Situation in 2D is very subtle

3Kranz, Zippelius, in preparation
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\(^3\)Kranz, Zippelius, in preparation
The Glass Transition
The Glass Transition

Amorphous Solid from either
1. Supercooled Melt
2. Supersaturated Suspension
3. Dense Granular Fluid?

- No Static Order Parameter

Debenedetti & Stillinger, Nature 2001
Scattering Function $\phi(q, t) = \langle \rho^*_q(\tau) \rho_q(\tau + t) \rangle$ independent of $\tau$

Density $\rho(r, t) = \sum_i \delta(r_i(t) - r)$

Order Parameter $f_q = \phi(q, t \to \infty)$

- Fluid: $f_q = 0$
- Glass: $f_q > 0$
Equation of Motion\(^4\)

\[
(\partial_t^2 + q^2 v_q^2) \phi(q, t) + \int_0^t d\tau M(q, t - \tau) \partial_\tau \phi(q, \tau) = 0
\]

Speed of Sound \(v_q\)

Memory Kernel \(M(q, t)\)

\(^4\)Kranz, Sperl, Zippelius, PRL 104, 225701 (2010); PRE 87, 022207 (2013)
Mode-Coupling Approximation

\[(\partial_t^2 + q^2 v_q^2)\phi(q, t) + \int_0^t d\tau M(q, t - \tau) \partial_\tau \phi(q, \tau) = 0\]

- Interpretation as interacting undamped sound waves

\[M(q, t) \approx \sum_{q=k+p} \nu_{qkp} \nu_{qkp} \phi(k, t) \phi(p, t)\]

**Loss of Detailed Balance** implies

- Rate of creation \(\nu_{qkp} \neq \text{rate of annihilation} \ \nu_{qkp}\)
- Can still be calculated explicitly
The Granular MCT Glass Transition

- Percus-Yevick static structure factor
- Iterative Numerical Solution
- Standard Discretization Parameters
Nonuniversal Dynamics

$\phi(2q_\alpha=4.2,t)$

$\varepsilon = 0.5$

Critical exponents $a, b$ depend on $\varepsilon$. 
Nonuniversal Dynamics

- Critical exponents $a, b$ depend on $\varepsilon$. 
The Mean Square Displacement $5$

$\text{MSD} < \delta r^2 > (t)$

$\varepsilon = 0.0$
$\varepsilon = 0.5$
$\varepsilon = 1.0$

$5$ Sperl, Kranz, Zippelius, EPL 98, 28001 (2012)
Work in Progress

- Understand & Use Integration Through Transients (with Matthias Fuchs/Sperl)
- Simulation Results for the Granular Glass Transition (with Stefan Luding, Vitaliy Ogarko)
- Active Particles/Mobile Cells
Summary

- Momentum Conservation matters
- Violations of FDT may be useful
- Long-Time Tails are the same as in Equilibrium
- There is a Granular Glass Transition
- It has nontrivial Properties
Thank you for your attention

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