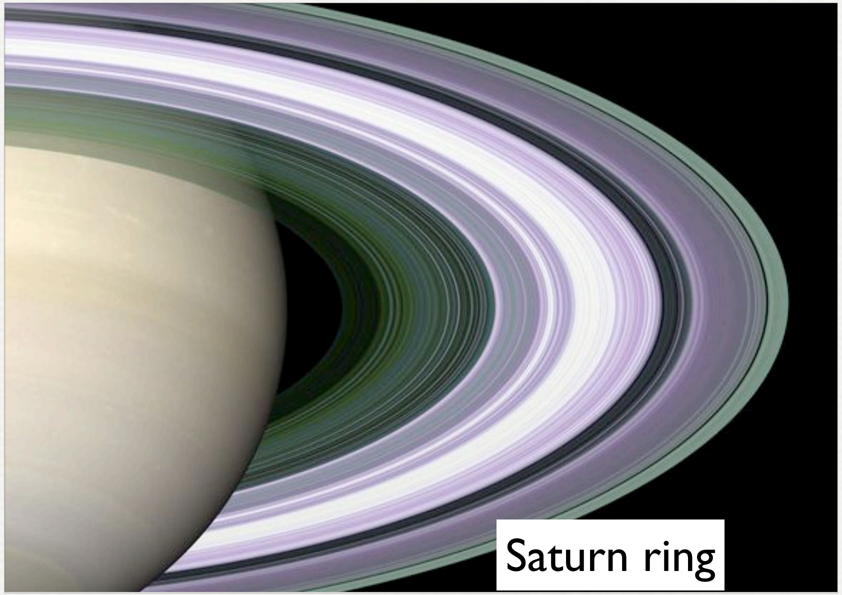
Nonlinear visco-elastic properties of granular materials near jamming transition

Michio Otsuki (Shimane Univ.)
Hisao Hayakawa (Kyoto Univ.)

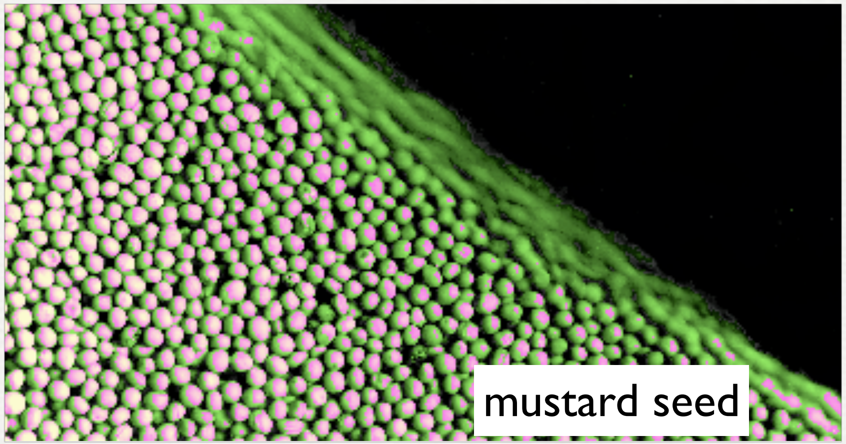
Granular materials

(Assemblies of particles with dissipation)



Saturn ring

Sand

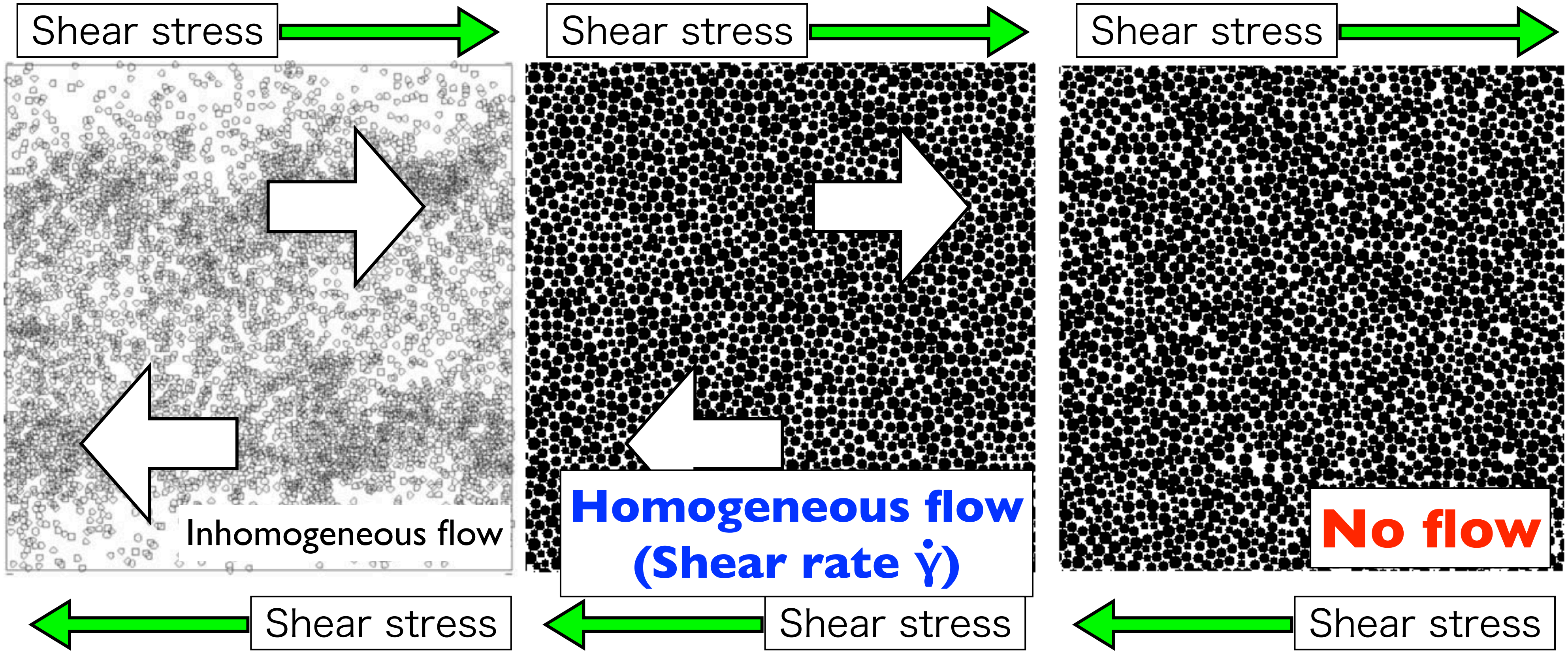


mustard seed

Ginkaku-ji temple

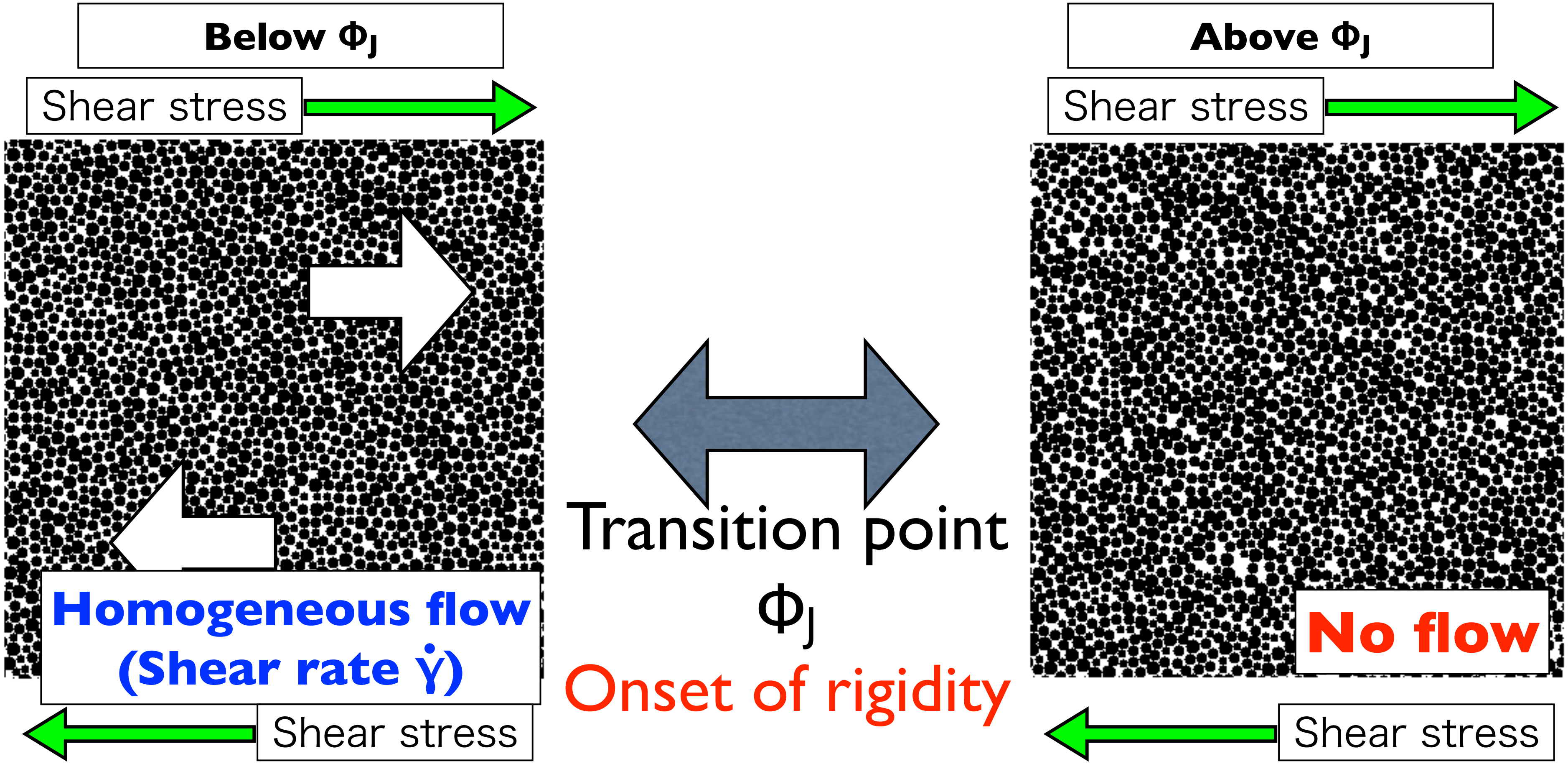
Sheared granular materials

packing fraction : Φ



Gas	Dense liquid	Amorphous solid
(Φ = 0.12)	(Φ = 0.8)	(Φ = 0.85)

Jamming transition

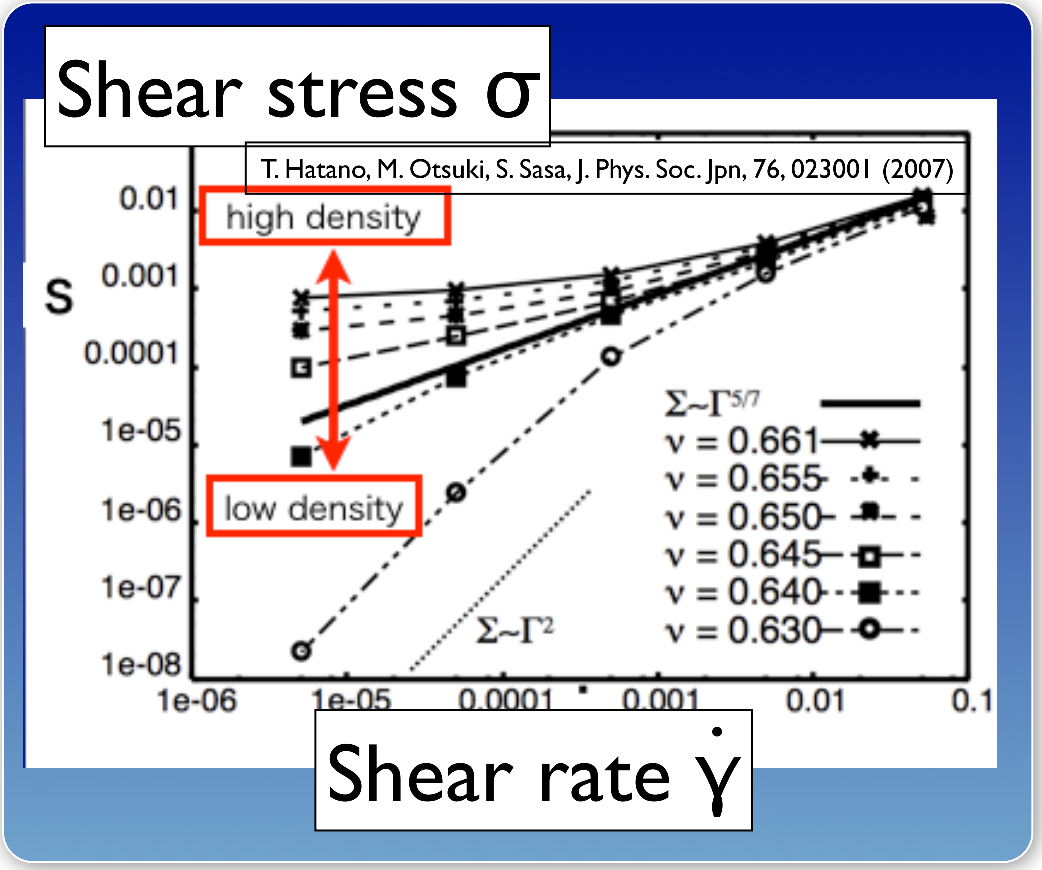


Dense liquid
(Φ = 0.8)

Amorphous solid
(Φ = 0.85)

Rheology under steady shear

frictionless case



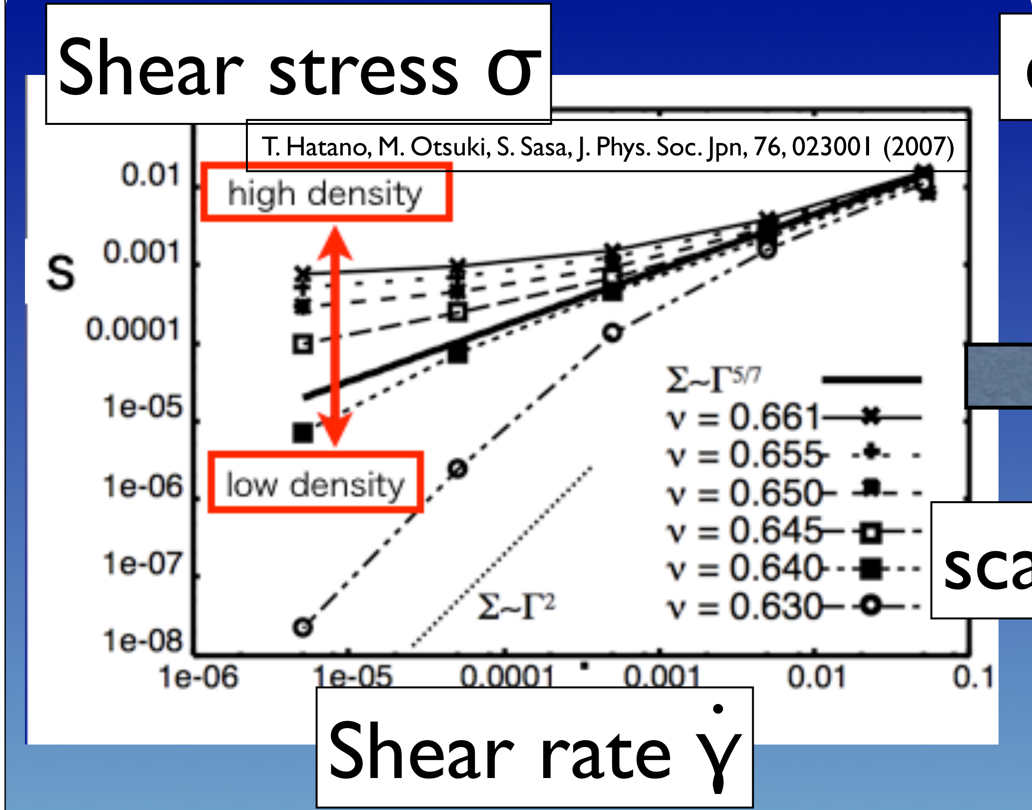
non-linear rheological property

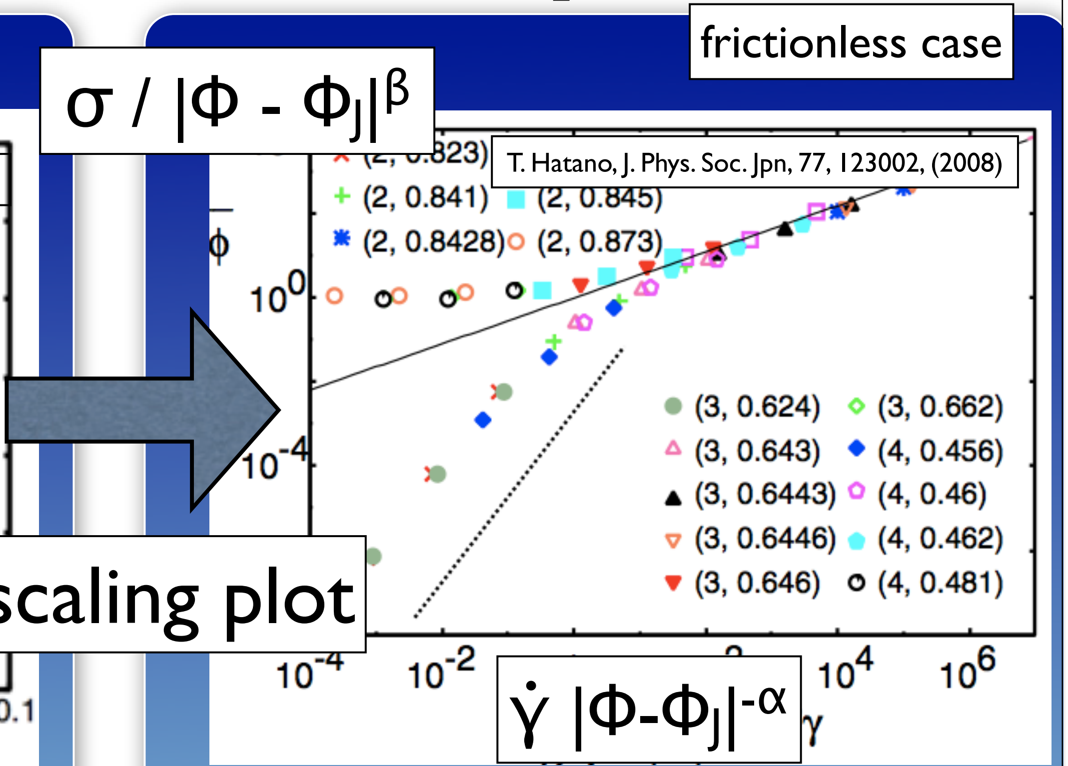
For $\Phi < \Phi_J$, $\sigma \propto \dot{\gamma}^2$ (liquid)

For $\Phi > \Phi_J$, $\sigma \simeq$ const (solid)

For $\Phi \simeq \Phi_J$, $\sigma \propto \dot{\gamma}^{y_\gamma}$

Rheology under steady shear





non-linear rheological property

For $\Phi < \Phi_J$, $\sigma \propto \dot{\gamma}^2$ (liquid)

For $\Phi > \Phi_J$, $\sigma \simeq$ const (solid)

For $\Phi \simeq \Phi_J$, $\sigma \propto \dot{\gamma}^{y_\gamma}$

Critical scaling law

$$\sigma(\dot{\gamma}, \Phi) = |\Phi - \Phi_J|^{y_\Phi} S_\pm(\dot{\gamma} |\Phi - \Phi_J|^{-\alpha})$$

α, y_Φ : Critical exponents

Theory for exponents

M. Otsuki and H. Hayakawa, PRE, 80, 011308, (2009)

Three Critical scaling laws

$$T(\dot{\gamma}, \Phi) = |\Phi - \Phi_J|^{x_\Phi} \mathcal{T}_\pm(\dot{\gamma} |\Phi-\Phi_J|^{-\alpha})$$

Kinetic energy

$$\sigma(\dot{\gamma}, \Phi) = |\Phi - \Phi_J|^{y_\Phi} S_\pm(\dot{\gamma} |\Phi-\Phi_J|^{-\alpha})$$

Shear stress

$$P(\dot{\gamma}, \Phi) = |\Phi - \Phi_J|^{y_\Phi'} p_\pm(\dot{\gamma} |\Phi-\Phi_J|^{-\alpha})$$

Pressure

Four Assumptions

- S / P is constant.

  Coulomb's friction : Hatano (2007)

- P in high density region : $P \sim \Phi$

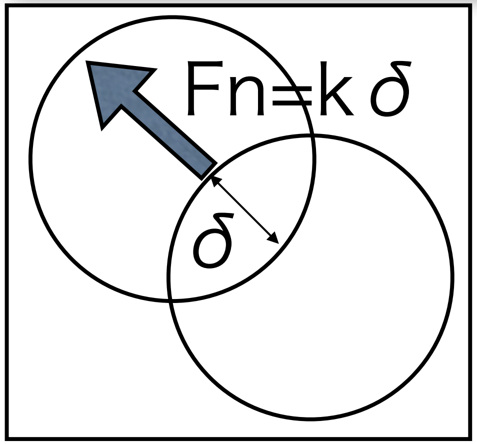
  O'Hern, et al., (2003)

- Characteristic time : $P^{-1/2}$

  Wyart, et al. (2005)

- Low density region : collision frequency $\propto T^{1/2}$

  Kinetic theory



Theoretical prediction for critical exponents

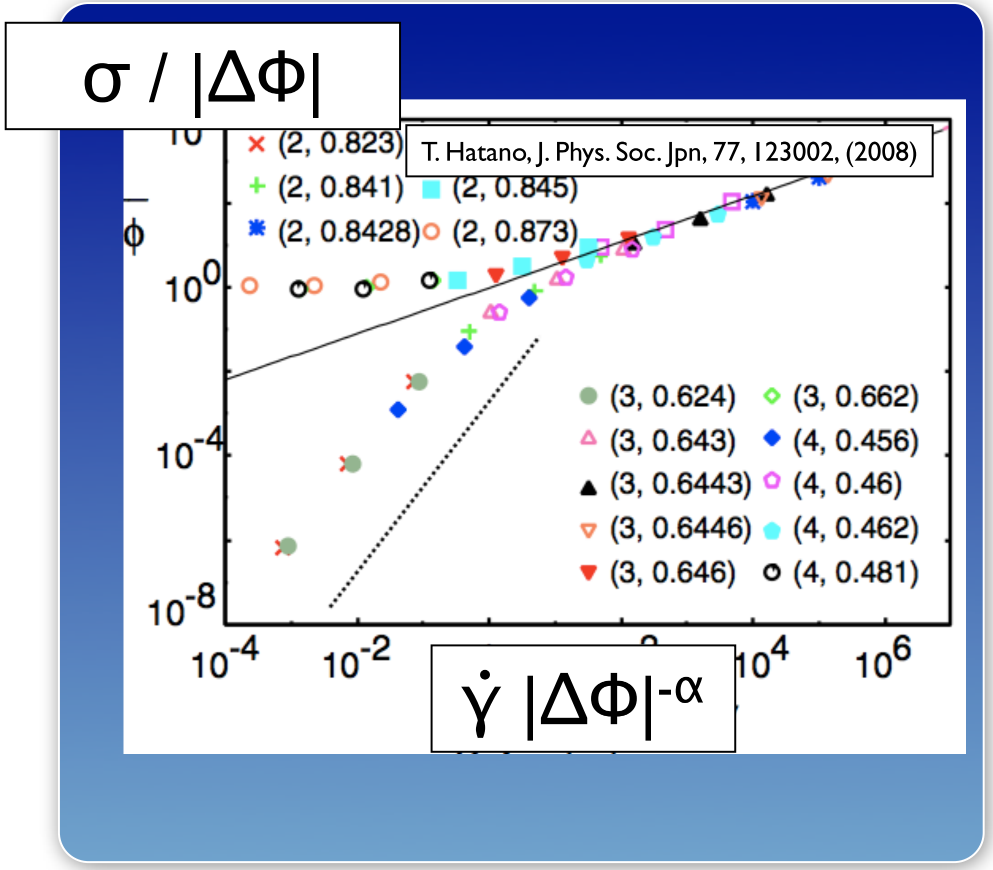
$$x_\Phi = 3,\ y_\Phi = 1,\ y_\Phi' = 1,\ \alpha = 5/2 \text{ (for disks)}$$

Linear repulsive force

Rheology under steady shear

frictionless case

M. Otsuki and H. Hayakawa, PRE, 80, 011308, (2009)



$$\sigma(\dot{\gamma}, \Phi) = |\Phi - \Phi_J|^{y_\Phi} S_{\pm}(\dot{\gamma}\, |\Phi - \Phi_J|^{-\alpha})$$

Theoretical prediction :

$\alpha = 1, y_\Phi = 2/5$ (for disk)

linear repulsive force

$$\sigma(\dot{\gamma}, \Phi) = \Delta\Phi\, S_{\pm}(\dot{\gamma} / \Delta\Phi^{5/2})$$

$$\Delta\Phi = \Phi - \Phi_J$$

Problem

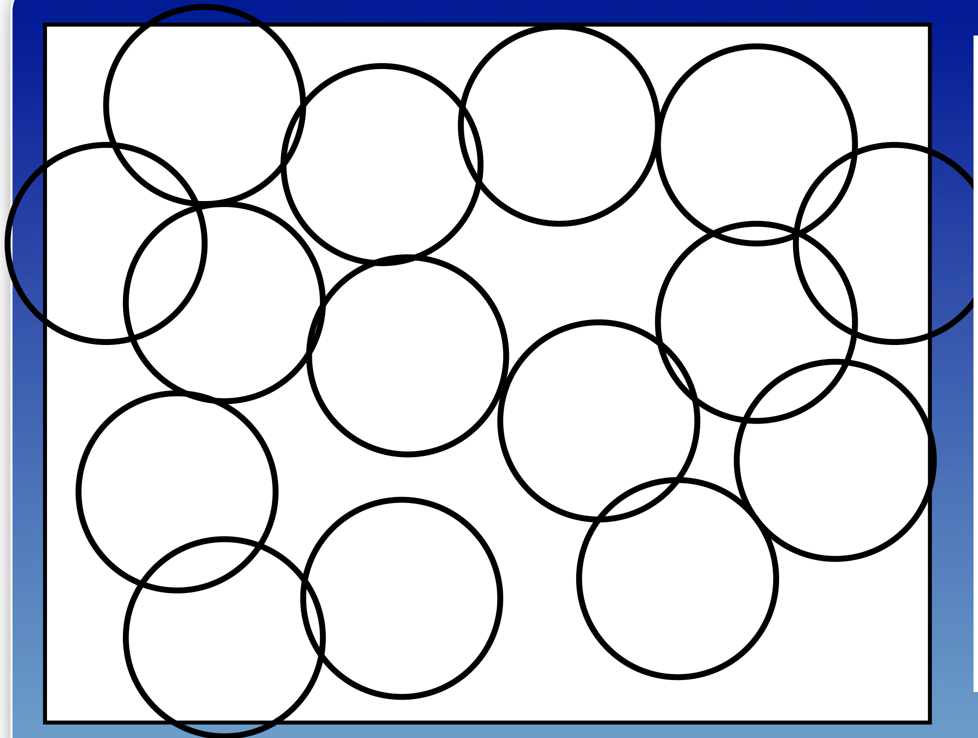
- The system under steady shear is not suitable to study the rigidity near the jamming transition.
- In experiments, the steady shear is hard to realize.

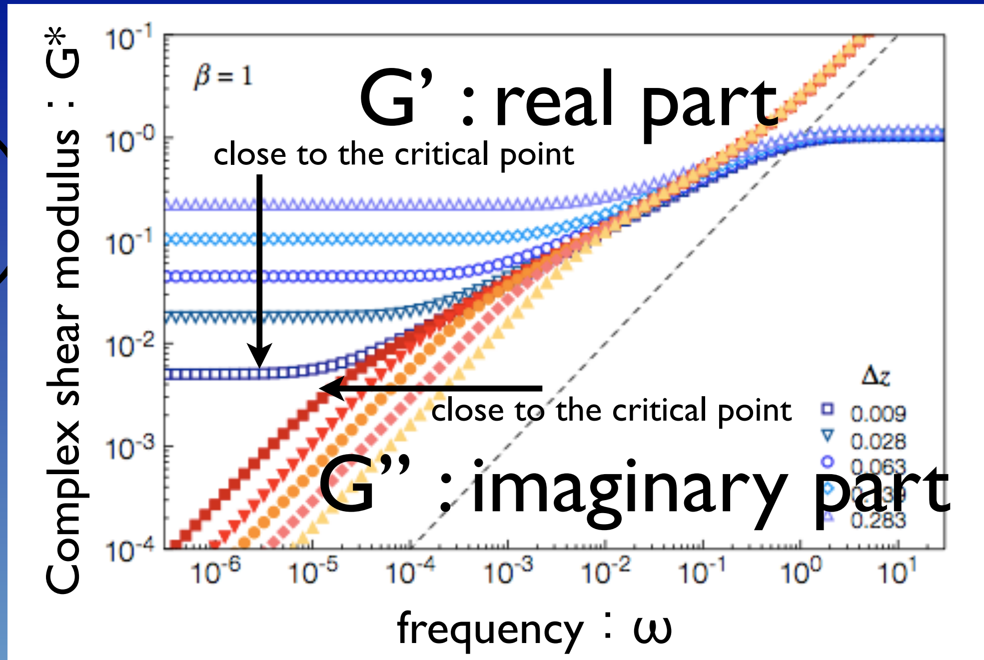
↓

We numerically investigate the rheological properties **under oscillatory shear (OS)**

Previous study on the system under OS

B. Tighe, PRL 107,158303 (2011)



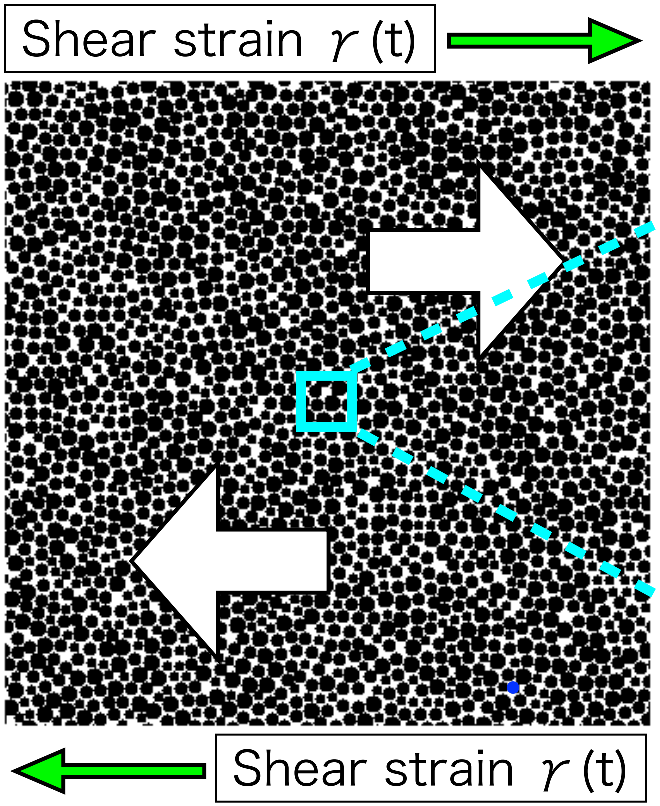


- System : no mass, fixed contact networks, tangential friction
- Complex shear modulus exhibits critical scalings.

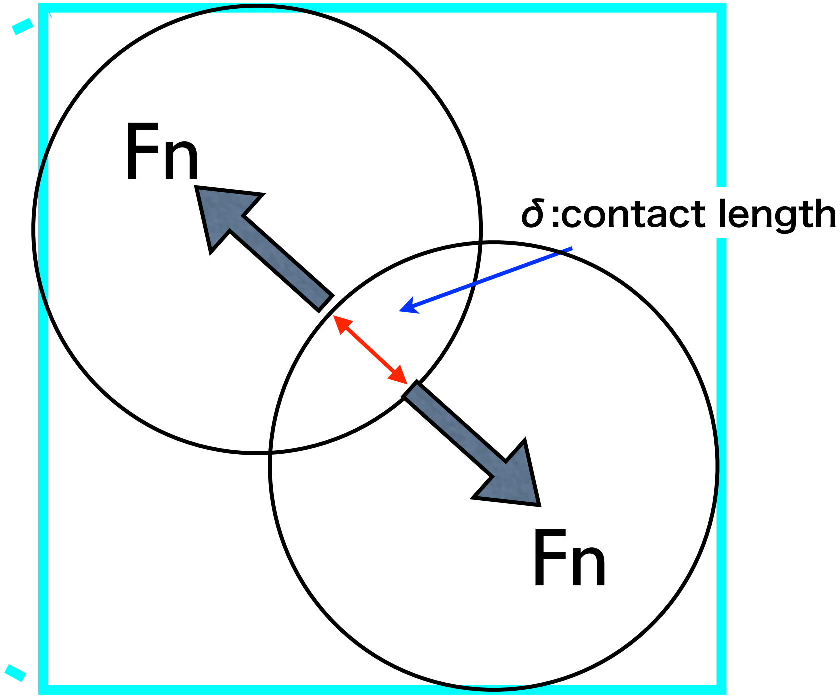
Purpose of this work

- In the previous work, the attention is restricted to the small shear limit and the change of the contact network is not considered.
- However, the change of the network dominates the rheological property near the jamming transition point.

⋆We investigate the rheological properties under OS in a wide range of shear amplitude.



Contact force



$$F_n = k\,\delta - \eta\,\dot{\delta}$$

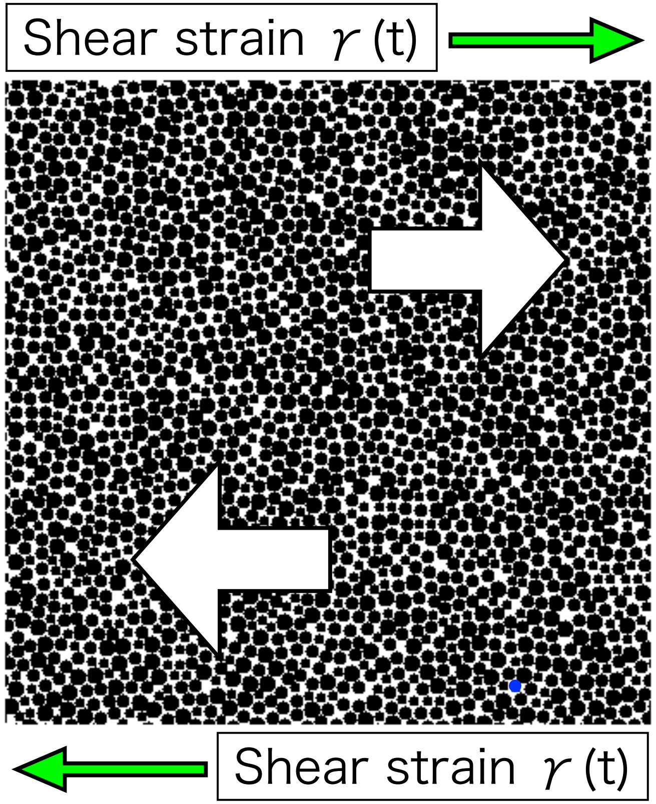
Elastic part

Dissipative part

Model of granular materials

(frictionless)

Oscillatory shear



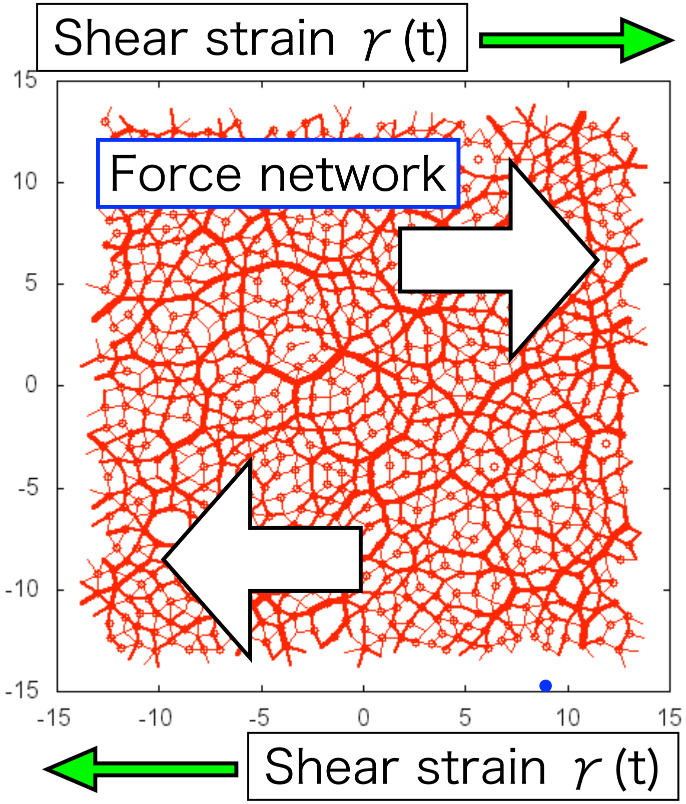
- Shear strain : $\gamma(t) = \gamma_0 \cos(\omega t)$
- Amplitude : γ_0,   Frequency : ω
- Shear stress : $\sigma(t)$
- Volume fraction : Φ
- Shear modulus : $G^* = G' + i\, G''$
- $G' \propto \int dt\, \sigma(t) \cos(\omega t) / \gamma_0$

  Real part : Storage modulus
- $G'' \propto -\int dt\, \sigma(t) \sin(\omega t) / \gamma_0$

  Imaginary part : Loss modulus

We numerically investigate $G^*(\gamma_0, \omega, \Phi)$.

Oscillatory shear



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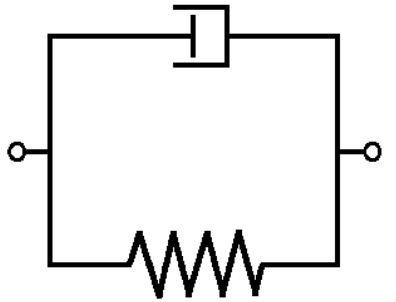
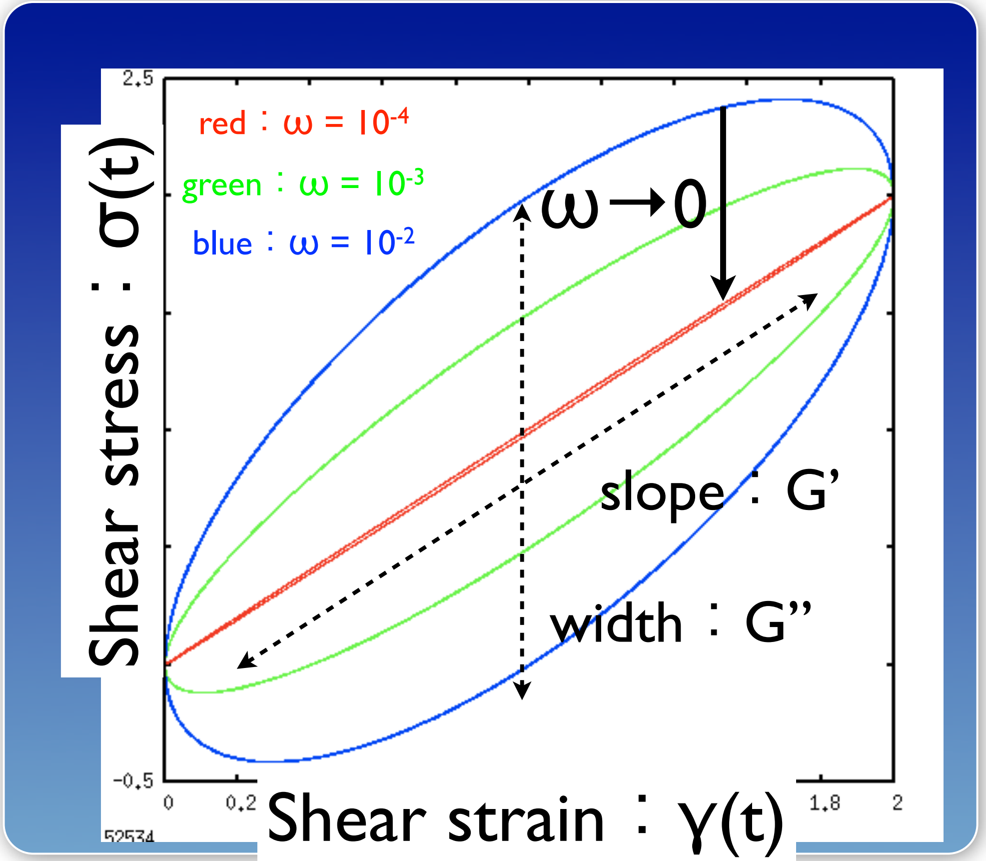
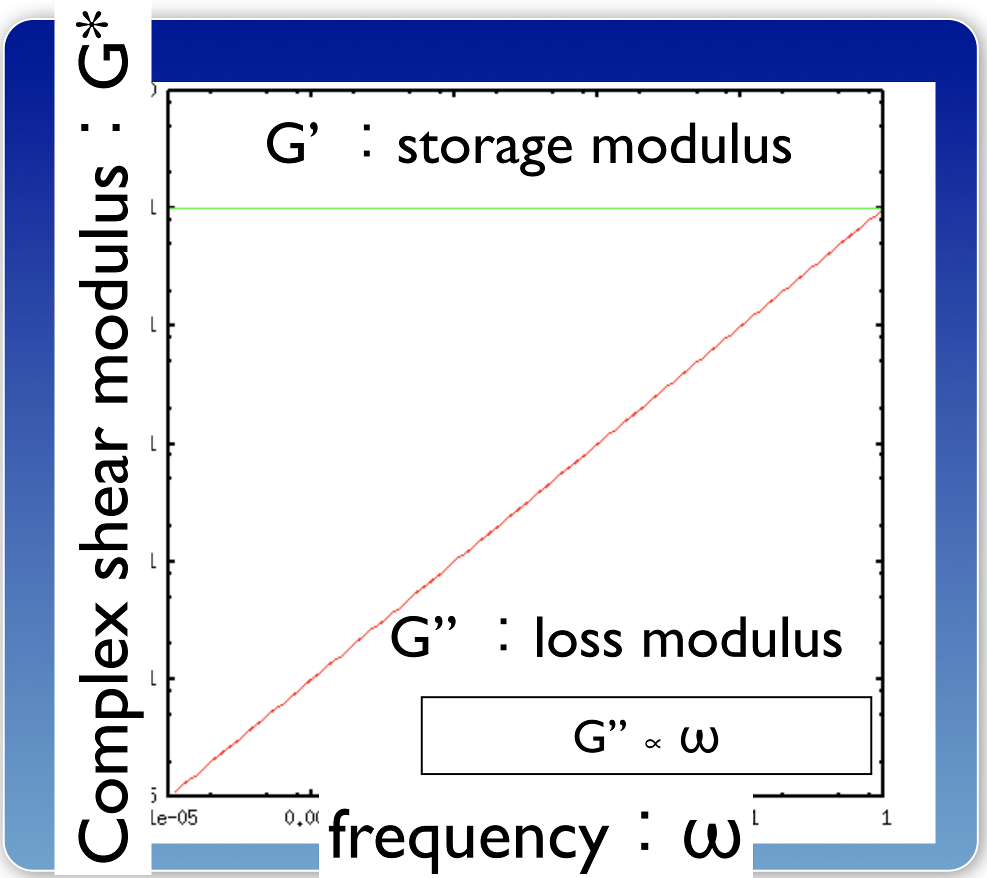


$$\sigma = \sigma_E + \sigma_K,\ \sigma_E = E\,\gamma,\ \sigma_K = \eta\,\dot{\gamma}$$

$$G' = E,\quad G'' = \eta\,\omega$$





G* for the Voigt model

Model of typical visco-elastic materials

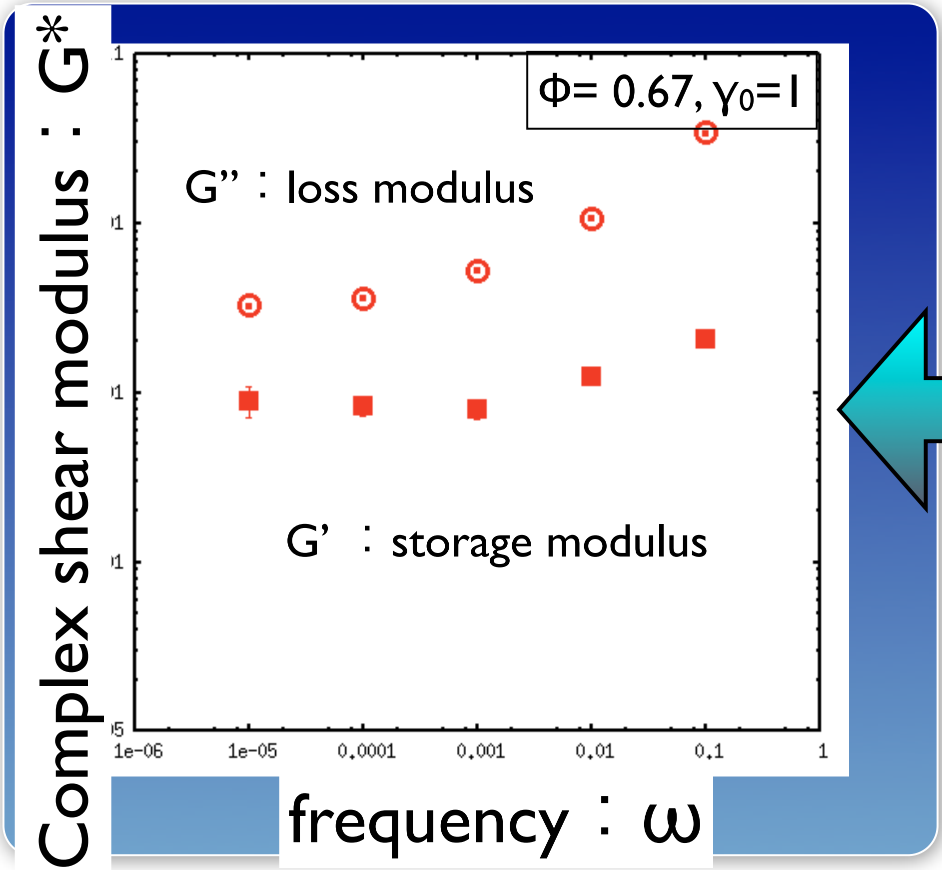
Critical scalings of G^*

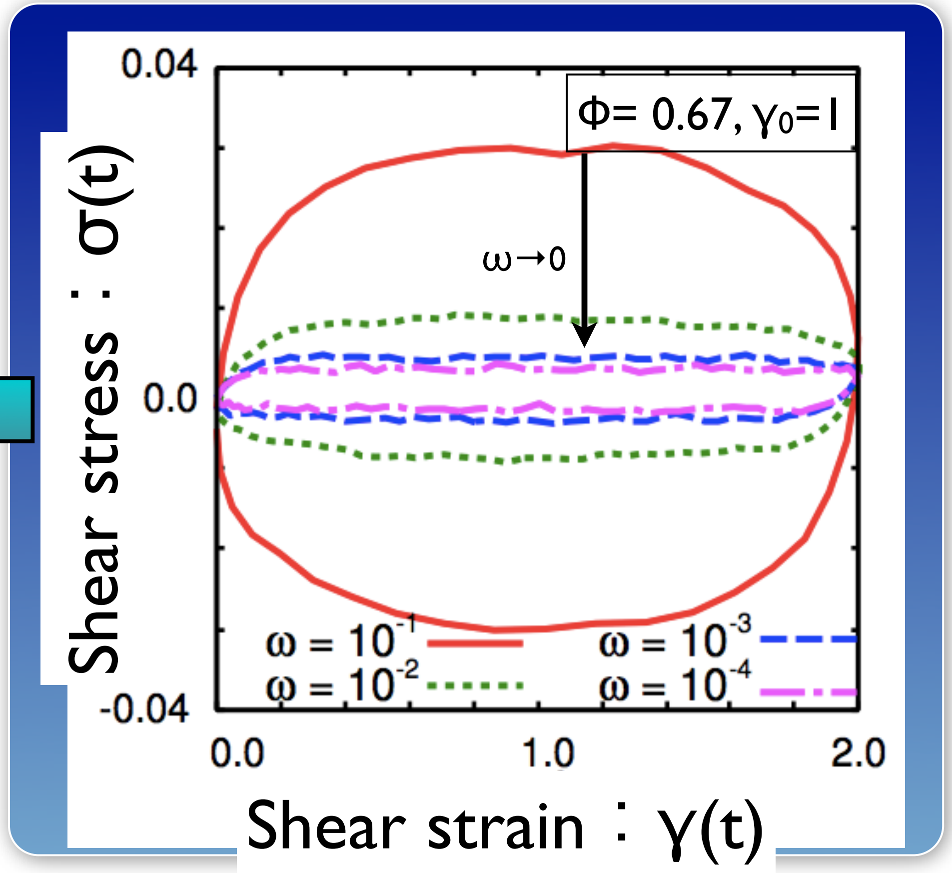
- We find three critical behaviors.

1. $G^*(\gamma_0, \omega, \Phi)$ for $\gamma_0 \geqq 1$. (Large amplitude region)
2. $G^*(\gamma_0, \omega, \Phi)$ for $\gamma_0 < 1$. (Small amplitude region)
3. $G^*(\gamma_0, \omega, \Phi)$ for $\omega \to 0$. (Quasi static limit)

$G^*(\gamma_0, \omega, \Phi)$ for $\gamma_0 \geqq 1$

ω-dependence





- G" remains for $\omega \to 0$.

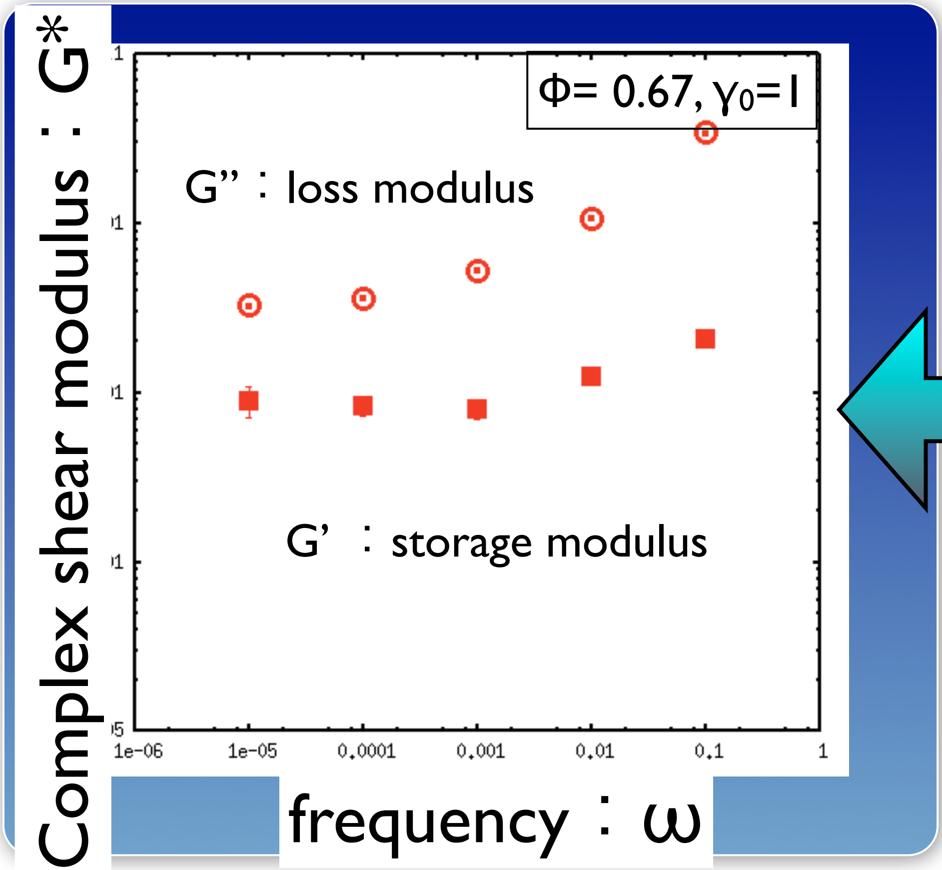
⇒Energy dissipation in the quasi-static limit.

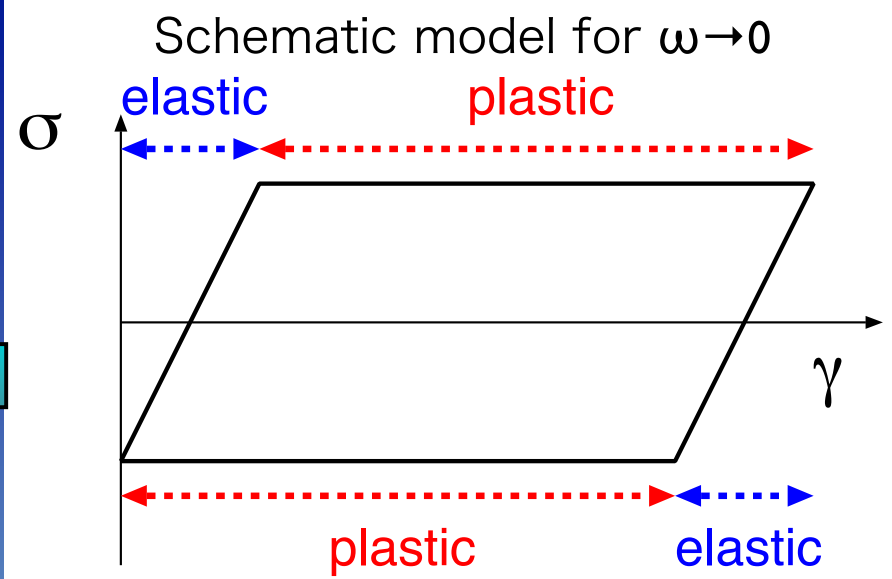
c.f. the Voigt model : G" ∝ ω

- The width in the plot of the σ-γ relation remains in $\omega \to 0$.

$G^*(\gamma_0, \omega, \Phi)$ for $\gamma_0 \geqq 1$

ω-dependence





$$\sigma = \begin{cases} E\gamma, & (\gamma < \gamma_c) \\ \sigma_Y, & (\gamma_c < \gamma) \end{cases}$$

- G" remains for $\omega \to 0$.

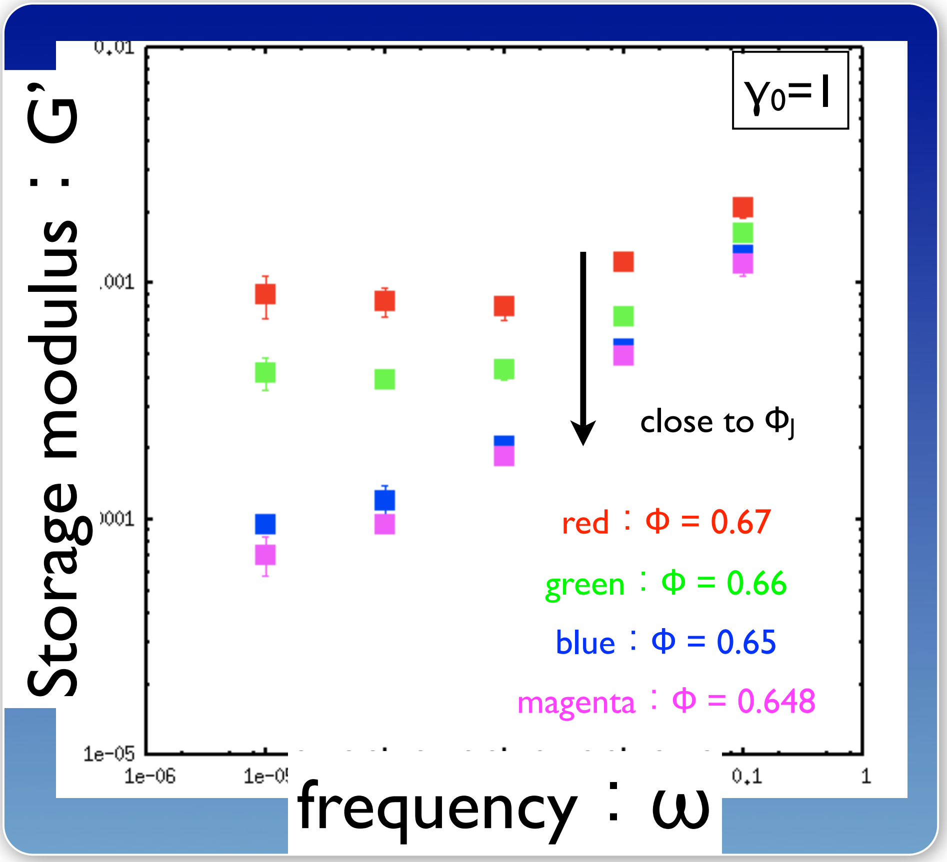
⇒Energy dissipation in the quasi-static limit.

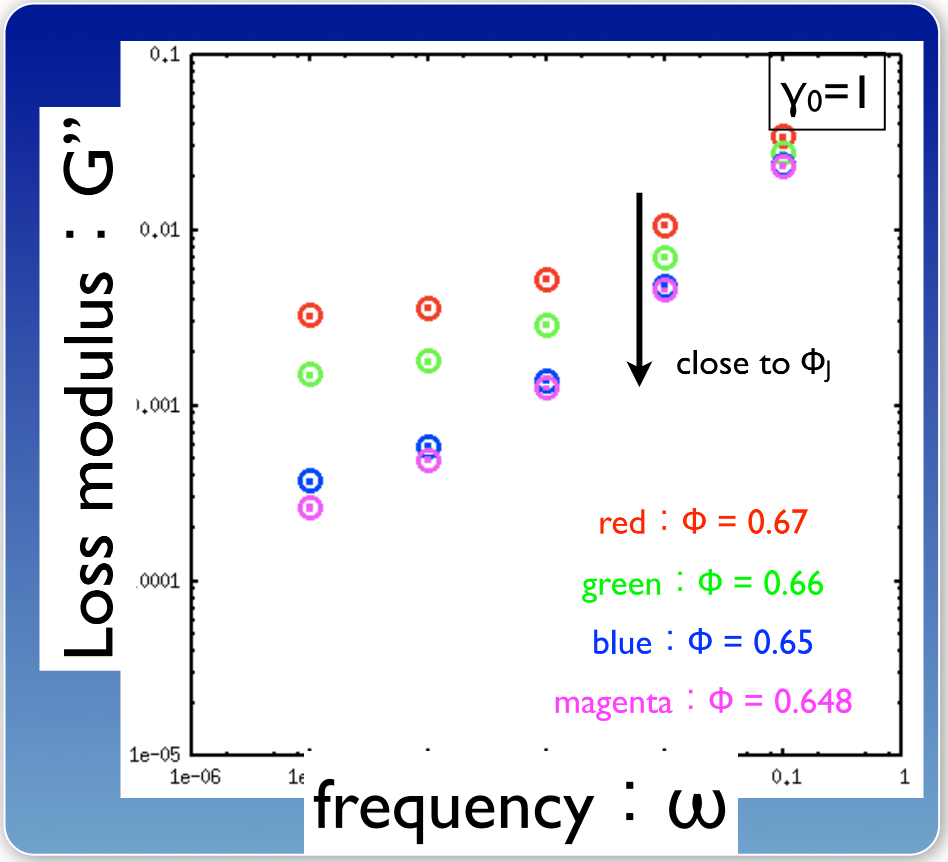
c.f. the Voigt model : G" ∝ ω

- The width in the plot of the σ-γ relation remains in $\omega \to 0$.

$G^*(\gamma_0, \omega, \Phi)$ for $\gamma_0 \geqq 1$

Φ-dependence





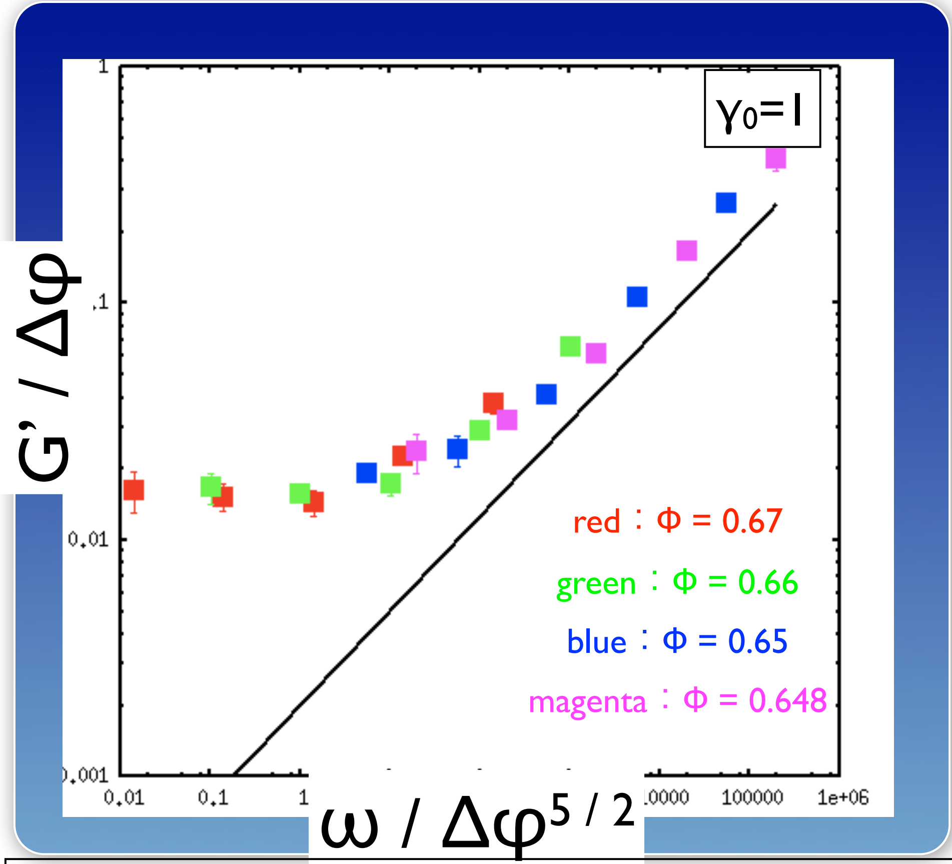
As Φ approaches Φ_J, G^* shows a power-law dependence on ω with a non-trivial exponent.

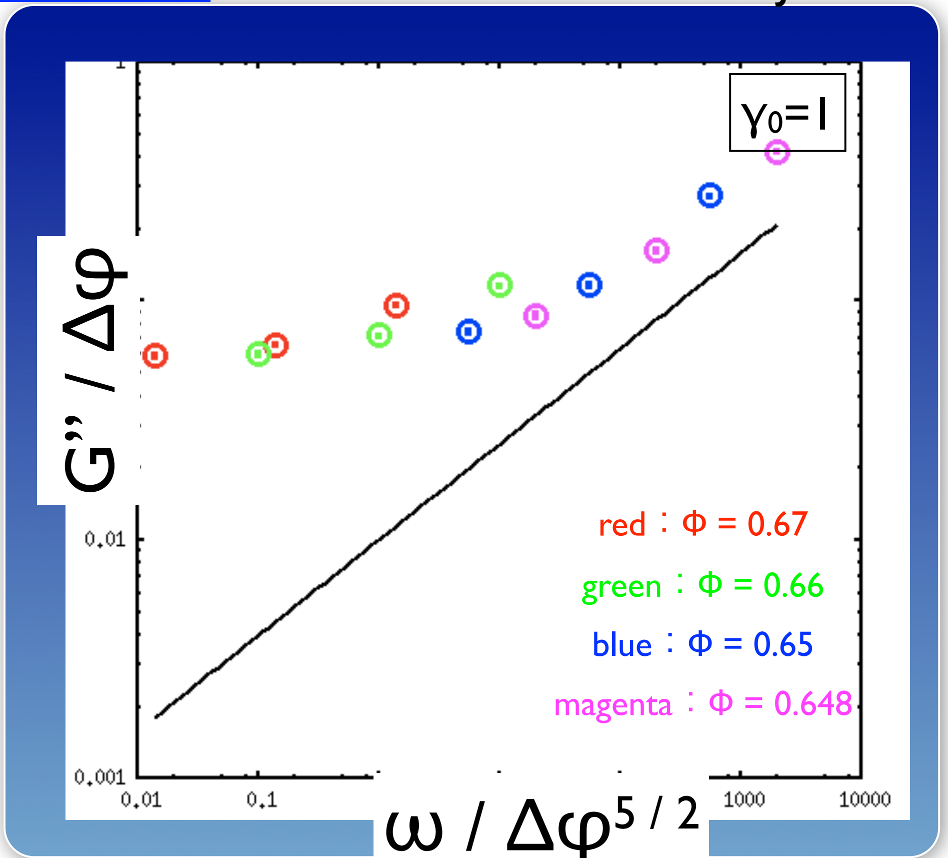
$G^*(\gamma_0, \omega, \Phi)$ for $\gamma_0 \geqq 1$

$G^*(\omega, \Phi) = \Delta\Phi \, g(\omega / \Delta\Phi^{5/2})$

Critical scaling

$\Delta\Phi = \Phi - \Phi_J$





We assume the rheological property under OS with a large γ_0 is dominated by that under steady shear.

$\sigma(\dot{\gamma}, \Phi) = \Delta\Phi \, F_{\pm}(\dot{\gamma} / \Delta\Phi^{5/2})$

Critical scalings of G^*

- We find three critical behaviors.

1. $G^*(\gamma_0, \omega, \Phi)$ for $\gamma_0 > 1$. (Large amplitude region)
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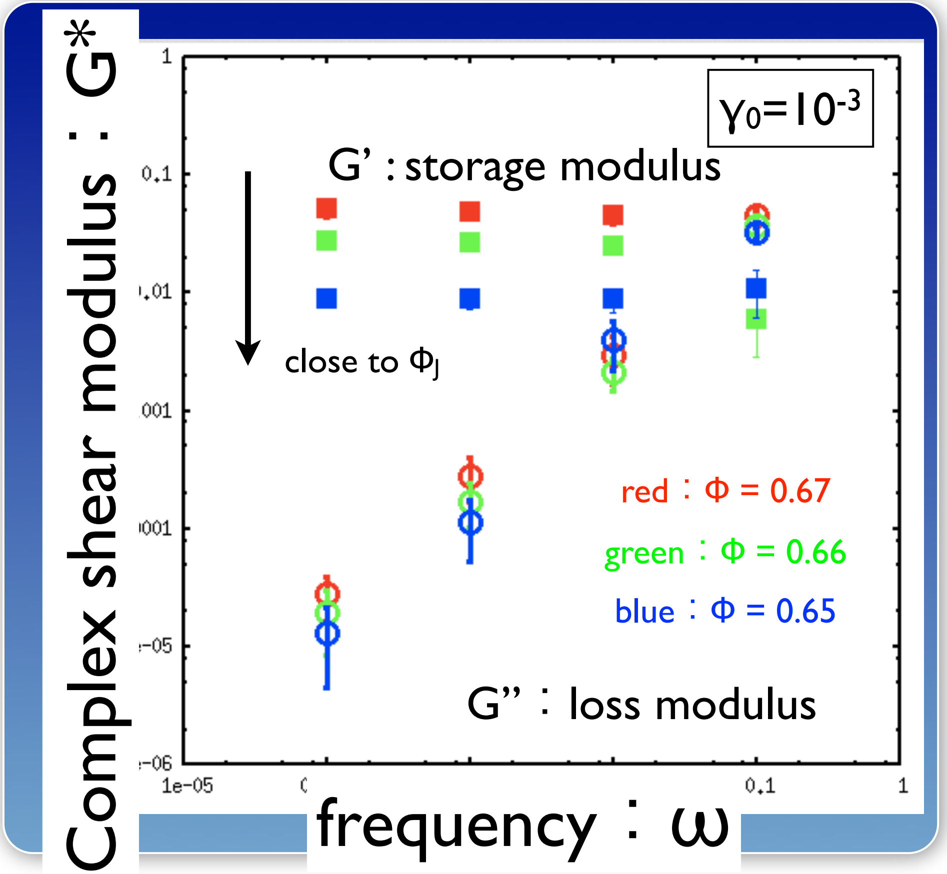
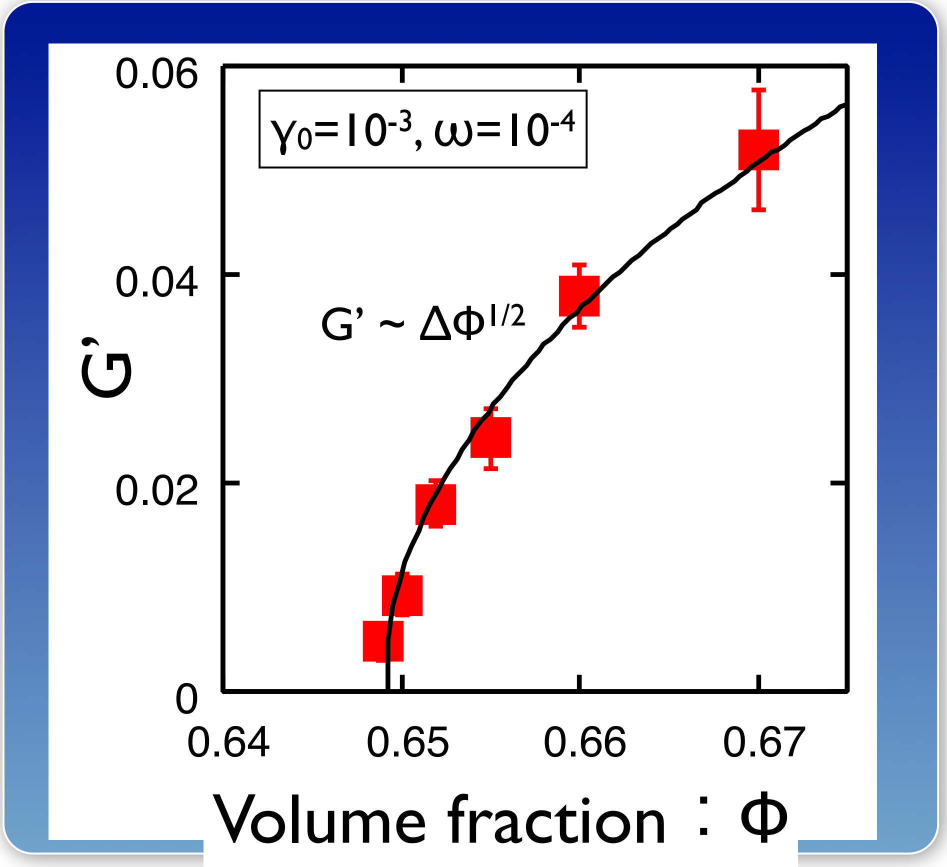
3. $G^*(\gamma_0, \omega, \Phi)$ for $\omega \to 0$. (Quasi static limit)

$G^*(\gamma_0, \omega, \Phi)$ for $\gamma_0 \ll 1$





The behavior of G* is consistent with the Voigt model.

Storage modulus : $G' \propto (\Phi - \Phi_J)^{1/2}$ (small ω-dependence)

Loss modulus : $G'' \propto \omega$

C. O'Hern, et al., Phys. Rev. Lett. 88, 075507 (2002)

Critical scalings of G^*

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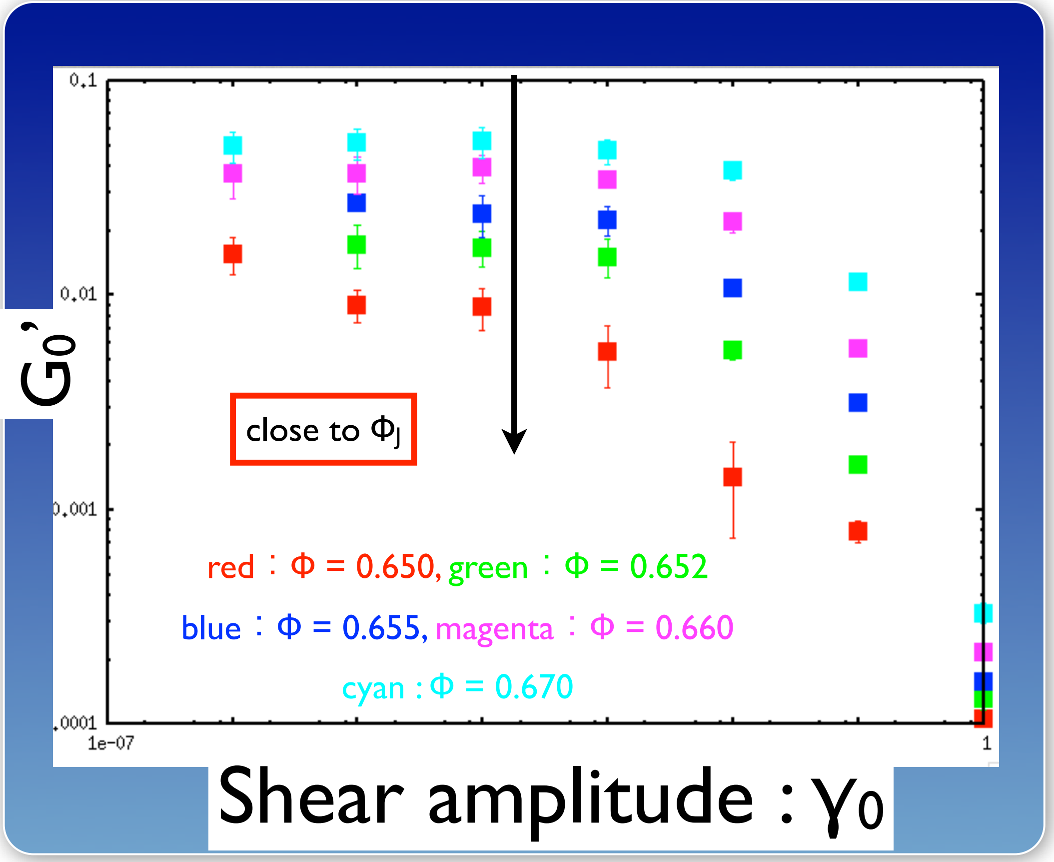
$G^*(\gamma_0, \omega, \Phi)$ for $\omega \to 0$

Quasi-static limit

$$G_0'(\gamma_0, \Phi) \equiv \lim_{\omega \to 0} G'(\gamma_0, \omega, \Phi)$$



$\gamma_c(\Phi)$: yield strain

- G_0' = const. for $\gamma_0 < \gamma_c(\Phi)$.
- G_0' decreases as γ_0 increases for $\gamma_0 > \gamma_c(\Phi)$.
- G_0' decreases as Φ approaches Φ_J.

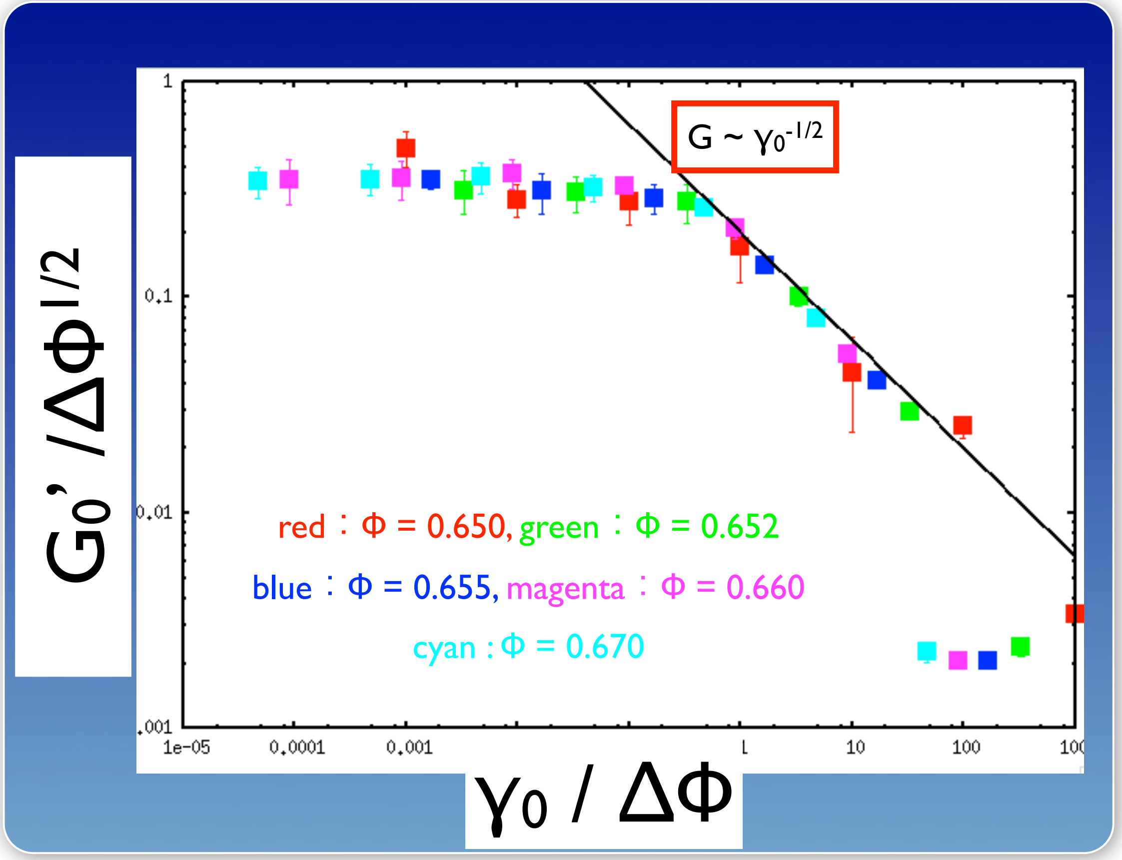
$G^*(\gamma_0, \omega, \Phi)$ for $\omega \to 0$

Theoretical prediction

$$G_0'(\gamma_0, \Phi) = \Delta\Phi^{1/2}\, h(\gamma_0 / \Delta\Phi)$$

$$\lim_{x \to \infty} h(x) \propto x^{-1/2}$$

Critical scaling



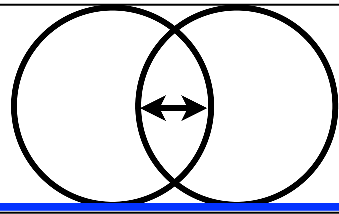
Three Assumptions

$G' \sim \Delta\Phi^{1/2}$, for $\gamma_0 \to 0$

C. O'Hern, et al., Phys. Rev. Lett. 88, 075507 (2002)

The yield strain γ_c is proportional to the contact length.

$\gamma_c(\Phi) \sim \Delta\Phi$



B. Tighe, et al., Phys. Rev. Lett. 105, 088303 (2010)

G_0' is independent of Φ for $\Delta\Phi \to 0$.

$G^*(\gamma_0, \omega, \Phi)$ for $\omega \to 0$

Implication

$$G_0'(\gamma_0, \Phi) = \Delta\Phi^{1/2}\, h(\gamma_0 / \Delta\Phi), \quad \lim_{x\to\infty} h(x) \propto x^{-1/2}$$

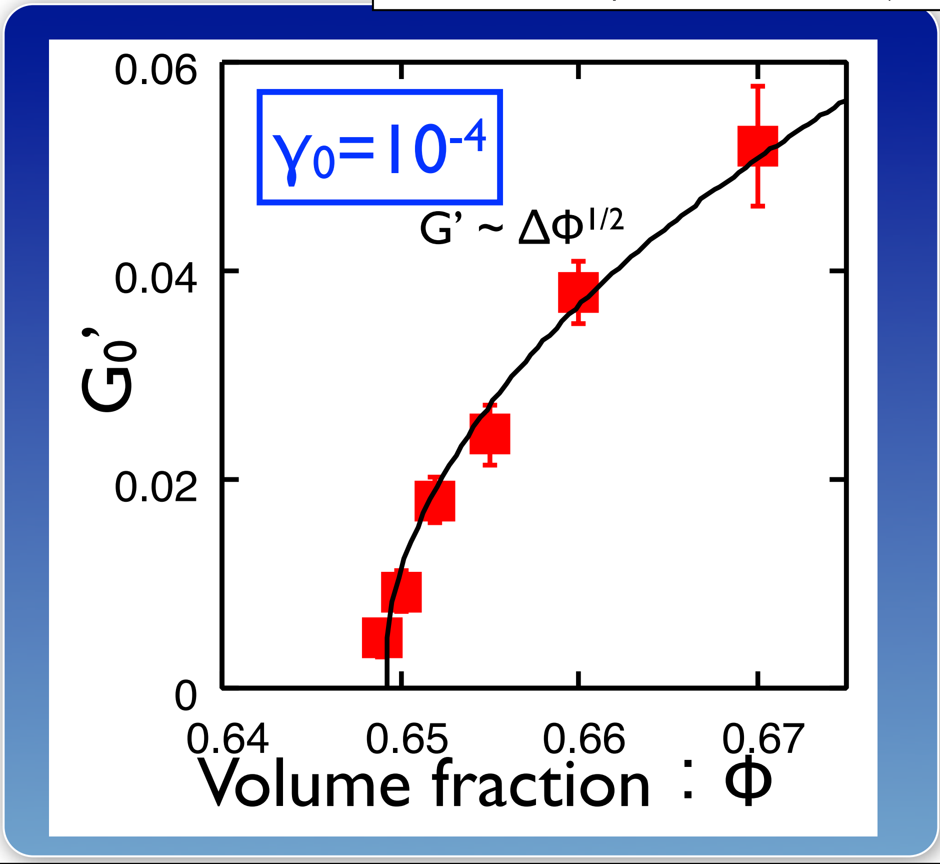
Scaling changes on the order of limits.

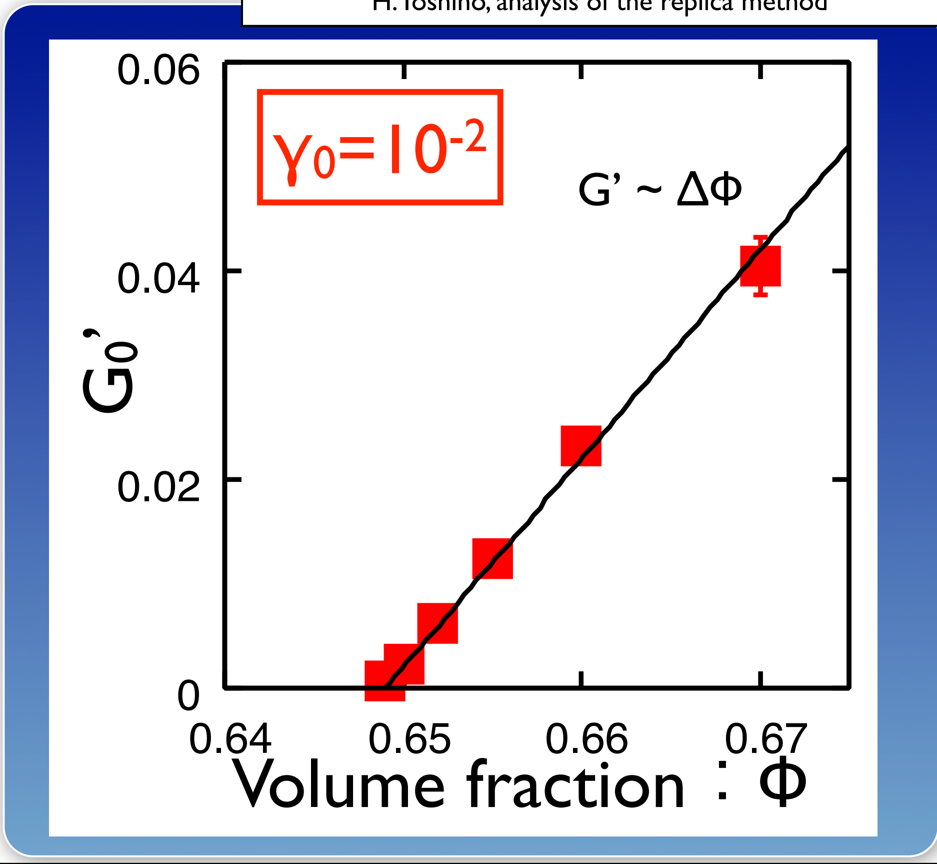
$$\lim_{\Delta\Phi \to 0} \lim_{\gamma_0 \to 0} G_0'(\gamma_0, \Phi) \propto \Delta\Phi^{1/2}$$

$$\lim_{\Delta\Phi \to 0} G_0'(\gamma_0, \Phi) \propto \Delta\Phi$$

C. O'Hern, et al., Phys. Rev. Lett. 88, 075507 (2002)

c.f. T.G. Mason et al. PRE (1997), experiments of emulsions
H.Yoshino, analysis of the replica method

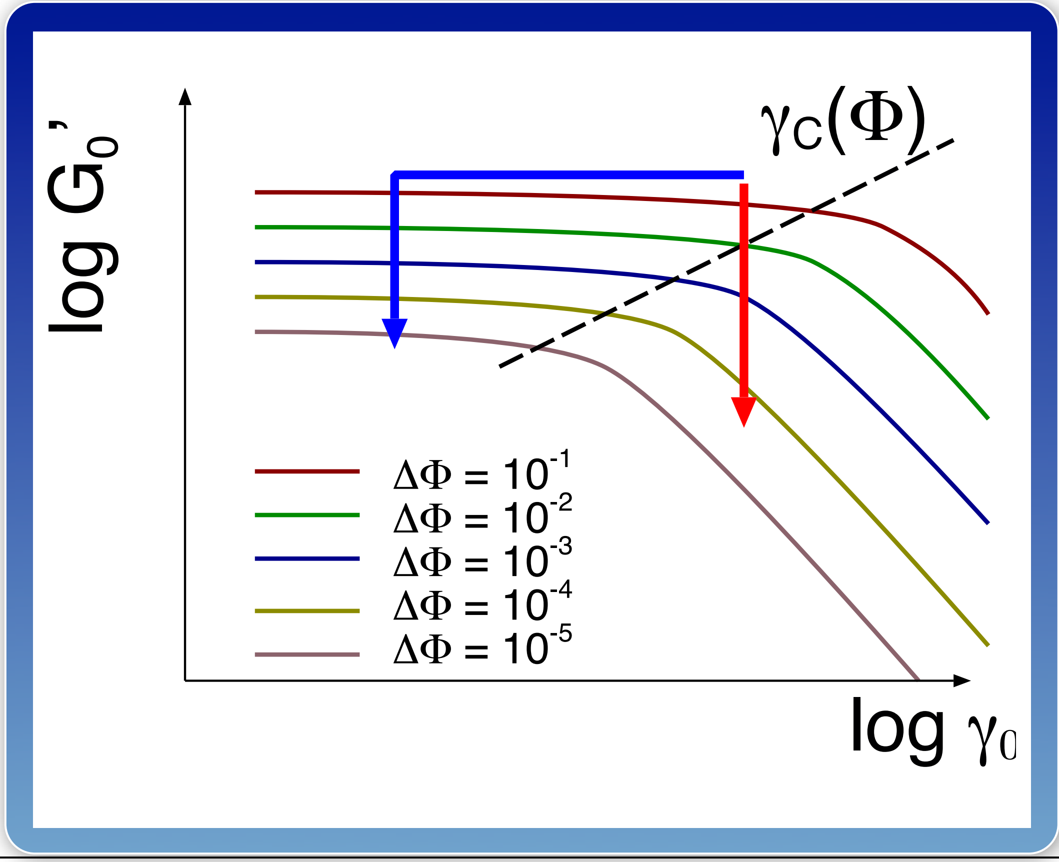




$G^*(\gamma_0, \omega, \Phi)$ for $\omega \to 0$

Implication

$$G_0'(\gamma_0, \Phi) = \Delta\Phi^{1/2}\, h(\gamma_0 / \Delta\Phi), \quad \lim_{x\to\infty} h(x) \propto x^{-1/2}$$



Scaling changes on the order of limits.

$$\lim_{\Delta\Phi\to 0} \lim_{\gamma_0\to 0} G_0'(\gamma_0, \Phi) \propto \Delta\Phi^{1/2} \qquad \lim_{\Delta\Phi\to 0} G_0'(\gamma_0, \Phi) \propto \Delta\Phi$$

$G^*(\gamma_0, \omega, \Phi)$ for $\omega \to 0$

Implication

$$G_0'(\gamma_0, \Phi) = \Delta\Phi^{1/2}\, h(\gamma_0 / \Delta\Phi), \quad \lim_{x\to\infty} h(x) \propto x^{-1/2}$$

Scaling changes on the order of limits.

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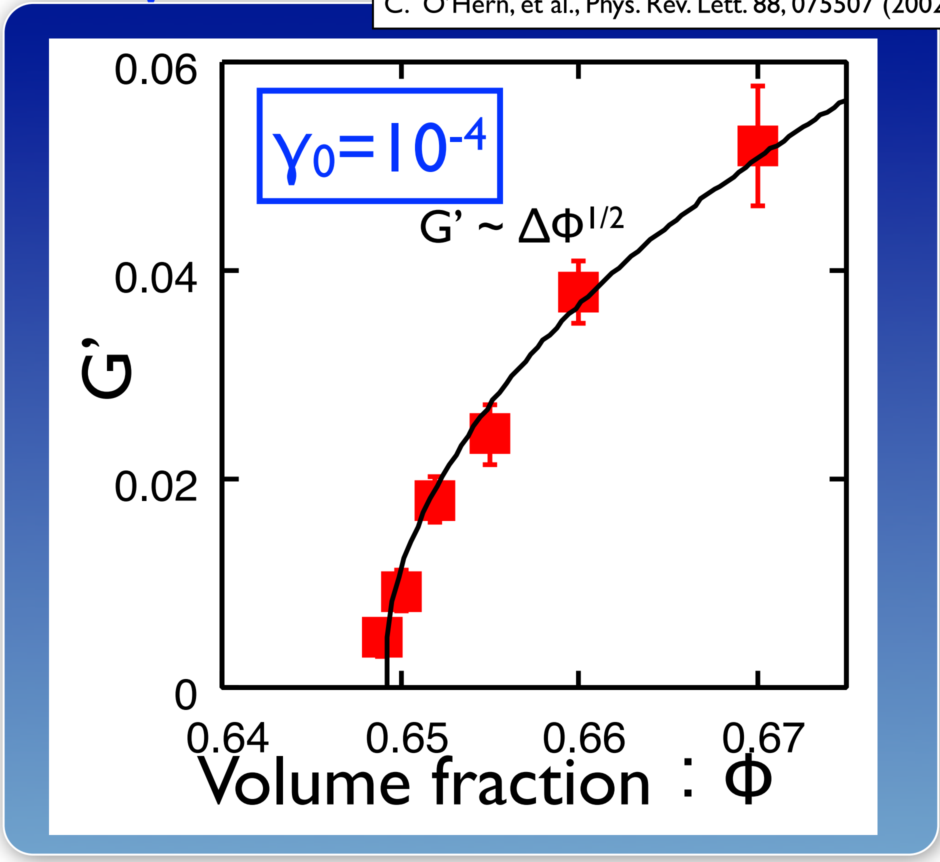
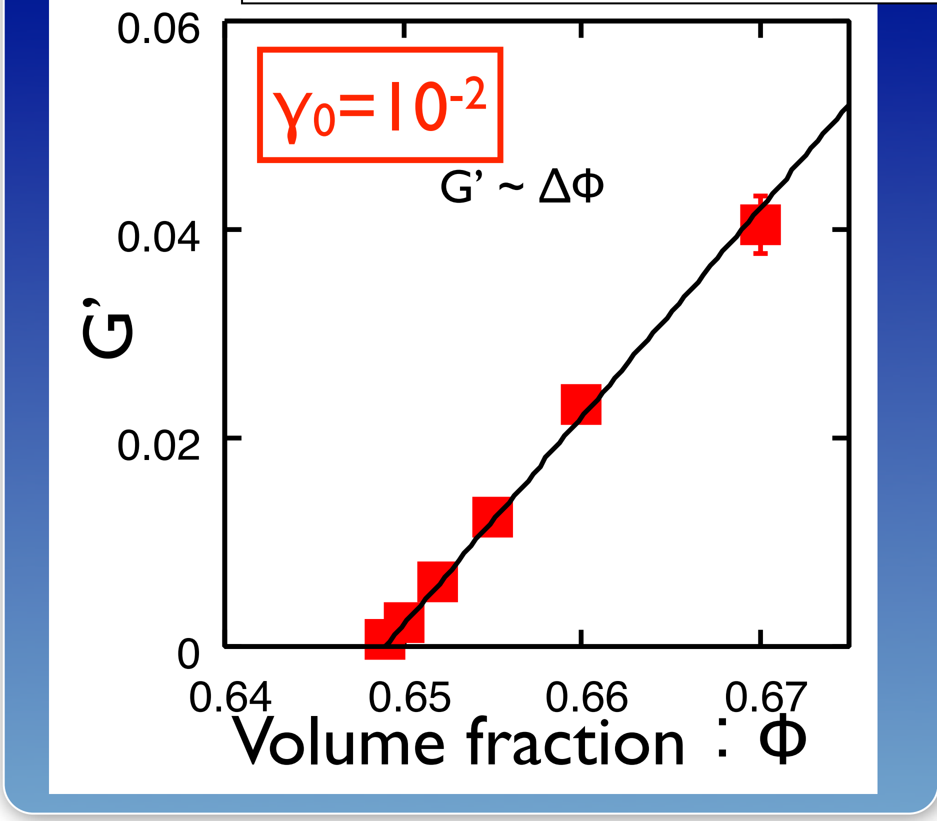


$$\lim_{\Delta\Phi \to 0} G_0'(\gamma_0, \Phi) \propto \Delta\Phi$$

c.f. T.G. Mason et al. PRE (1997), experiments of emulsions
H. Yoshino, analysis of the replica method



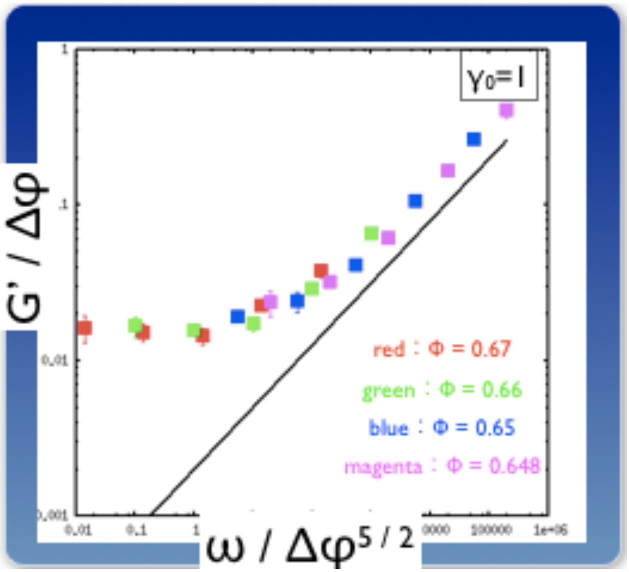
Summary

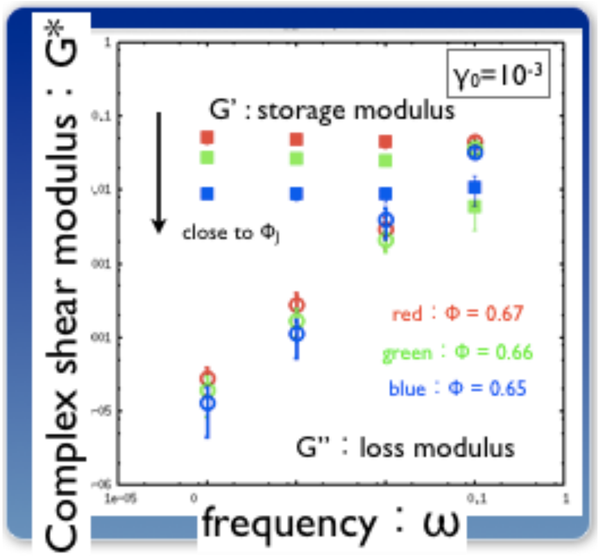
- We numerically investigate complex shear modulus of oscillatory sheared system.
- We find three critical scalings.

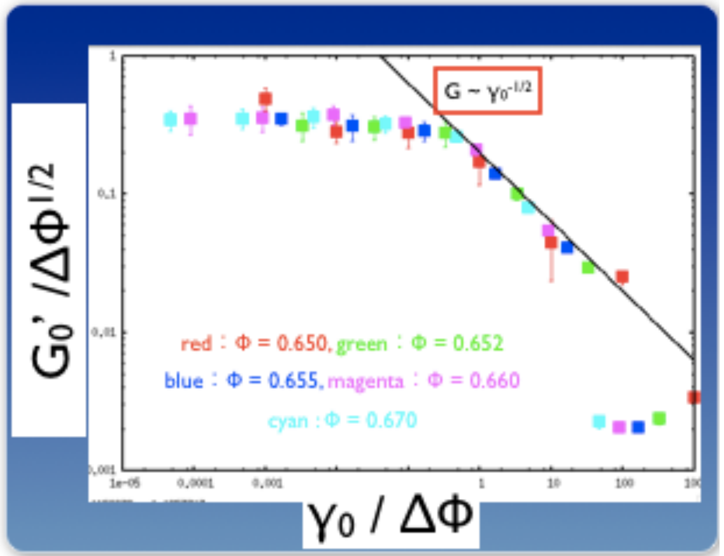
$G^*(\omega, \Phi) = \Delta\Phi \, g(\omega / \Delta\Phi^{5/2})$

$G' \propto (\Phi - \Phi_J)^{1/2}, \quad G'' \propto \omega$

$G_0'(\gamma_0, \Phi) = \Delta\Phi^{1/2} \, h(\gamma_0 / \Delta\Phi)$

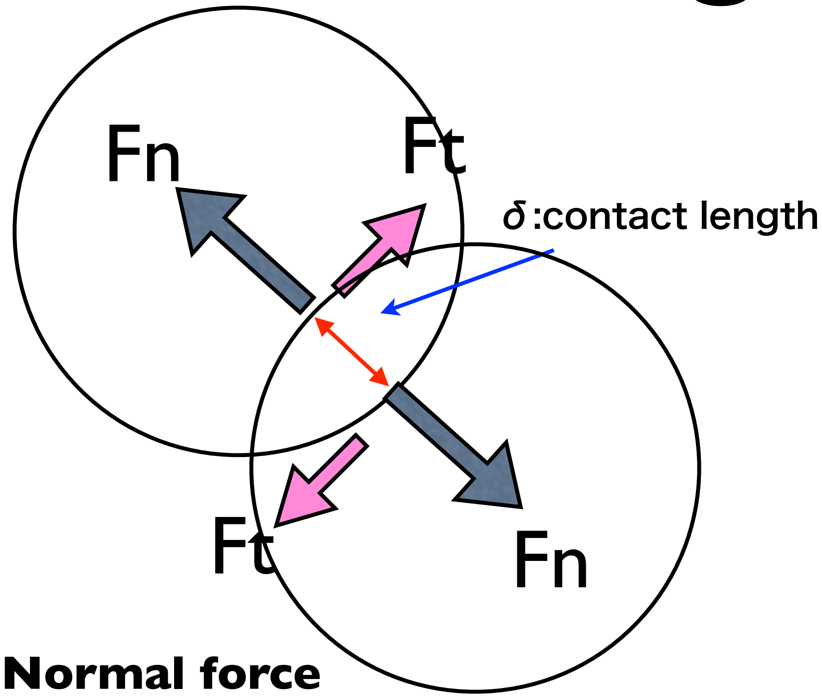


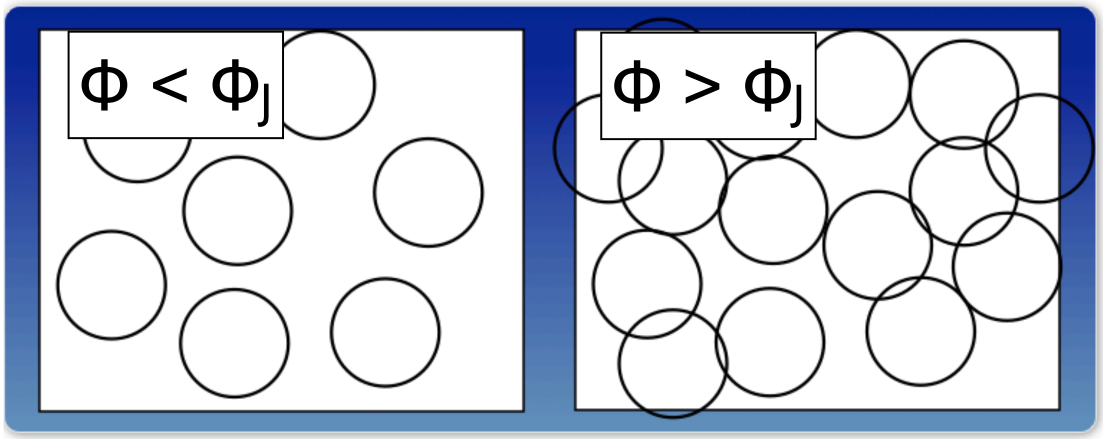




Thank you for your attention.

Model of granular materials





Normal force

- $F_n = k\,\delta^{\Delta} - \eta\, v_n$
  Elastic part / Dissipative part
  - $\Delta = 1$ (Disk)
  - $\Delta = 3/2$ (Sphere)

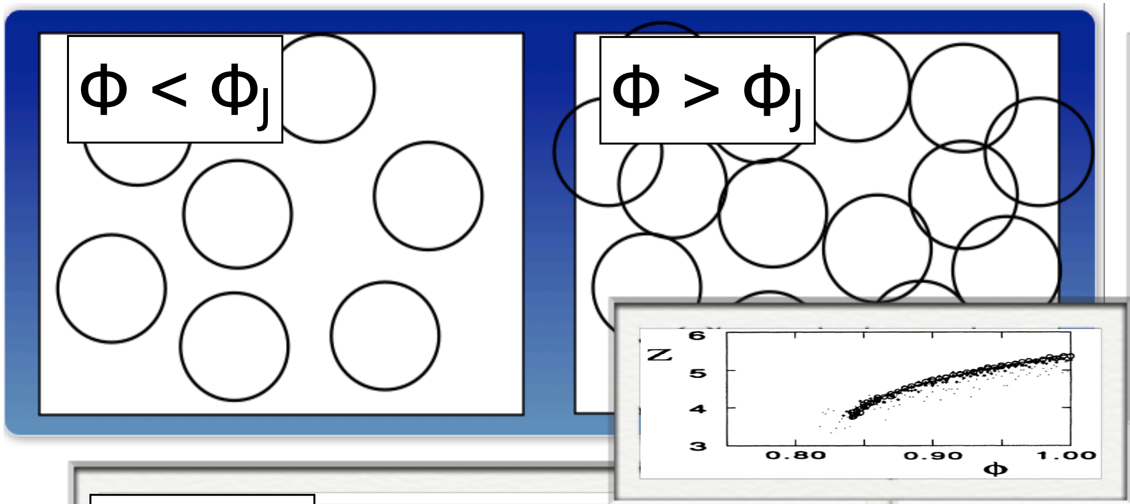
Tangential force

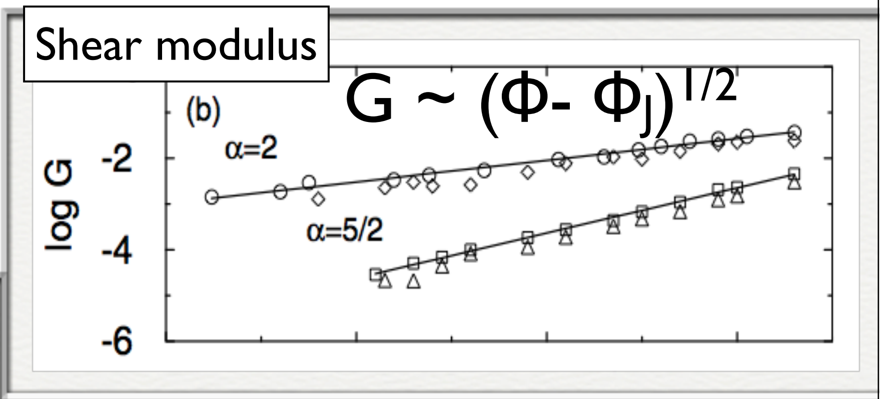
- Friction coefficient : μ
- $F_t < \mu F_n$ (Coulomb's friction)
- Frictionless : $\mu = 0$
- Frictional : $\mu > 0$

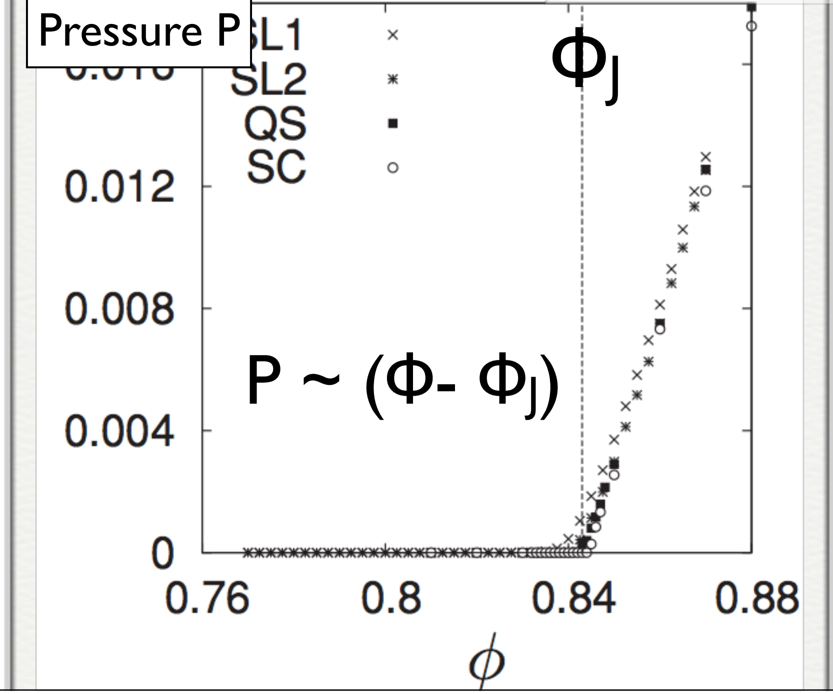
Important parameters : Δ, μ

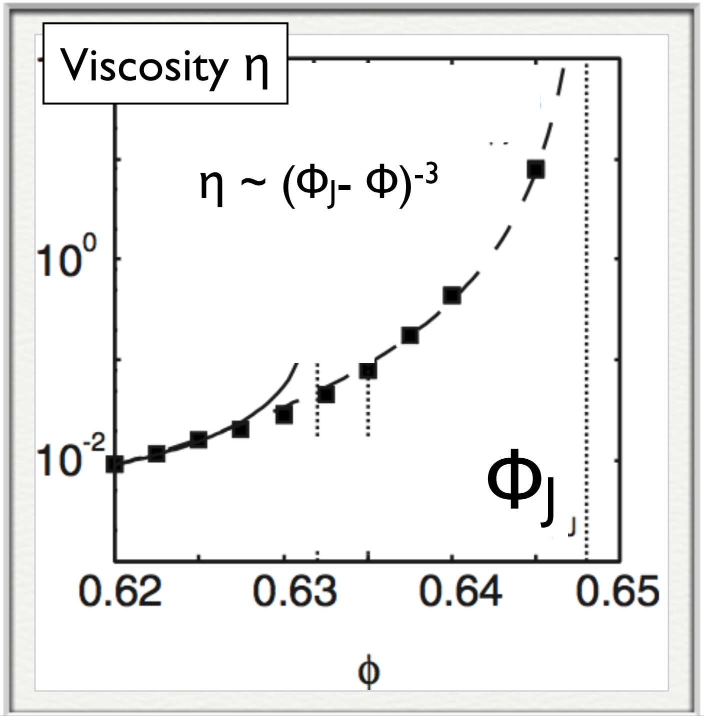
Critical property (without shear)

Frictionless case, Δ = 1



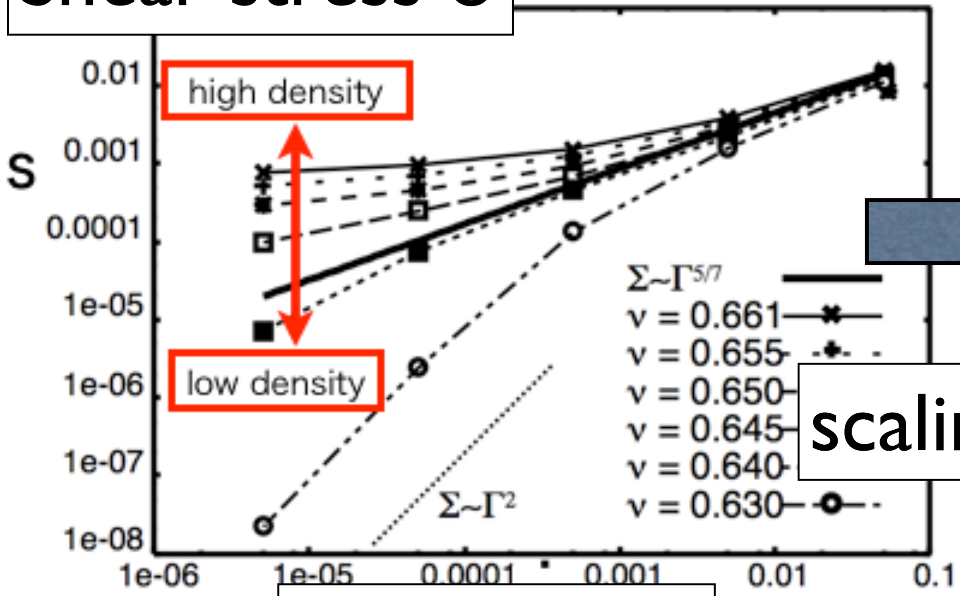






Critical scalings

Shear stress σ

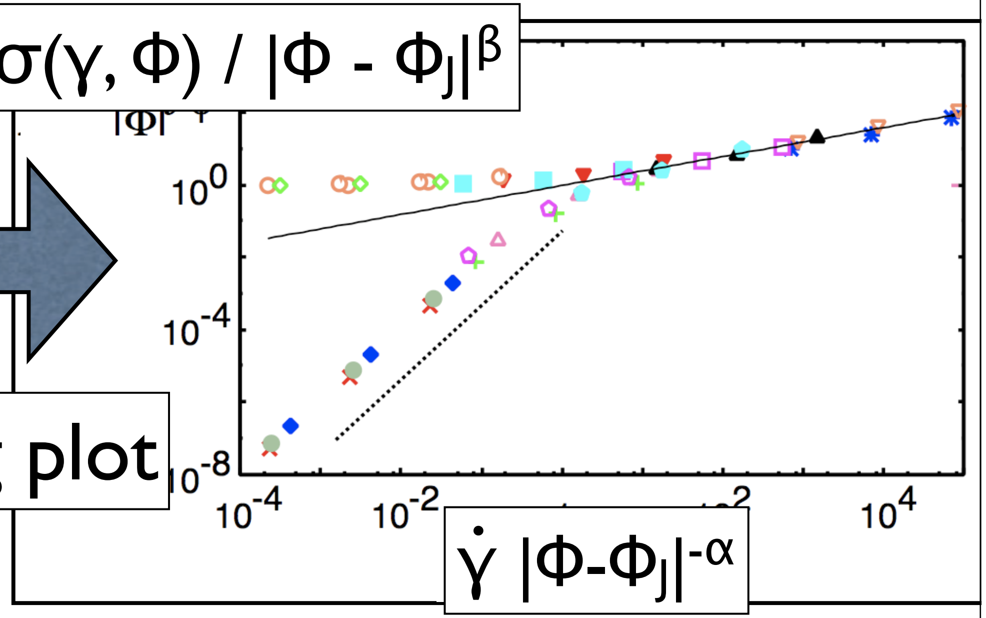


Shear rate $\dot{\gamma}$

scaling plot

Frictionless case, $\Delta = 1$

$\sigma(\gamma, \Phi) \,/\, |\Phi - \Phi_J|^{\beta}$



$\dot{\gamma}\, |\Phi-\Phi_J|^{-\alpha}$

Hatano, 2008

non-linear transport property

For $\Phi < \Phi_J$, $\sigma \propto \dot{\gamma}^2$ (liquid)

For $\Phi > \Phi_J$, $\sigma \simeq$ const (solid)

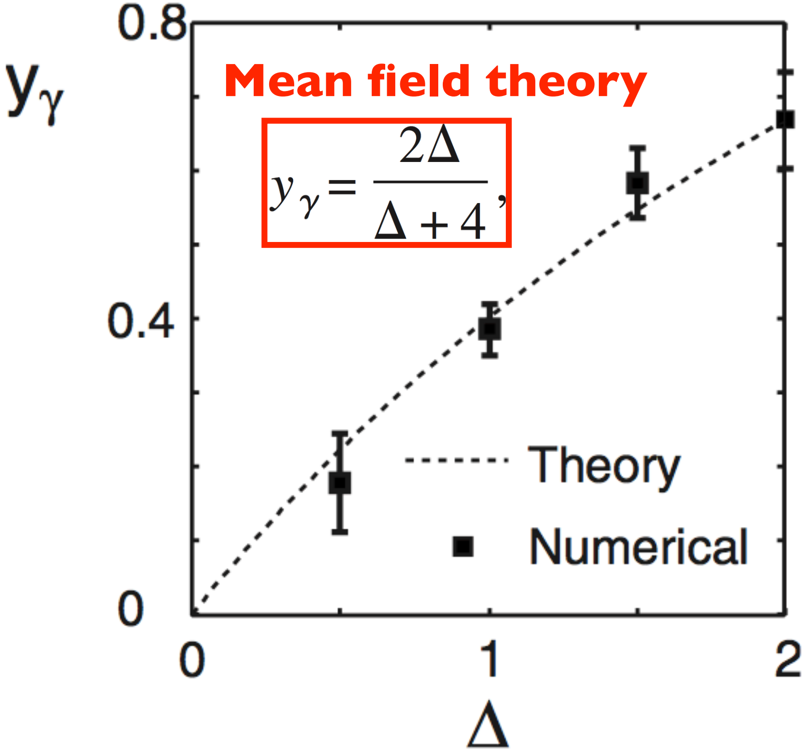
For $\Phi \simeq \Phi_J$, $\sigma \propto \dot{\gamma}^{y_\gamma}$

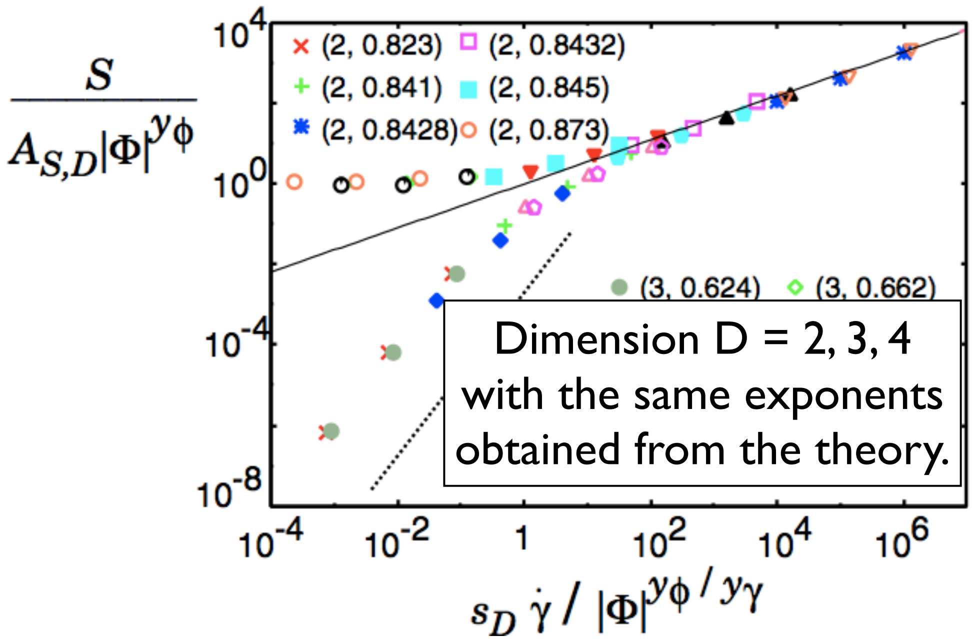
$$\sigma(\gamma, \Phi) = |\Phi - \Phi_J|^{\beta}\, S_{\pm}(\dot{\gamma}\, |\Phi-\Phi_J|^{-\alpha})$$

α, β : Critical exponents

second order transition

Characteristic features





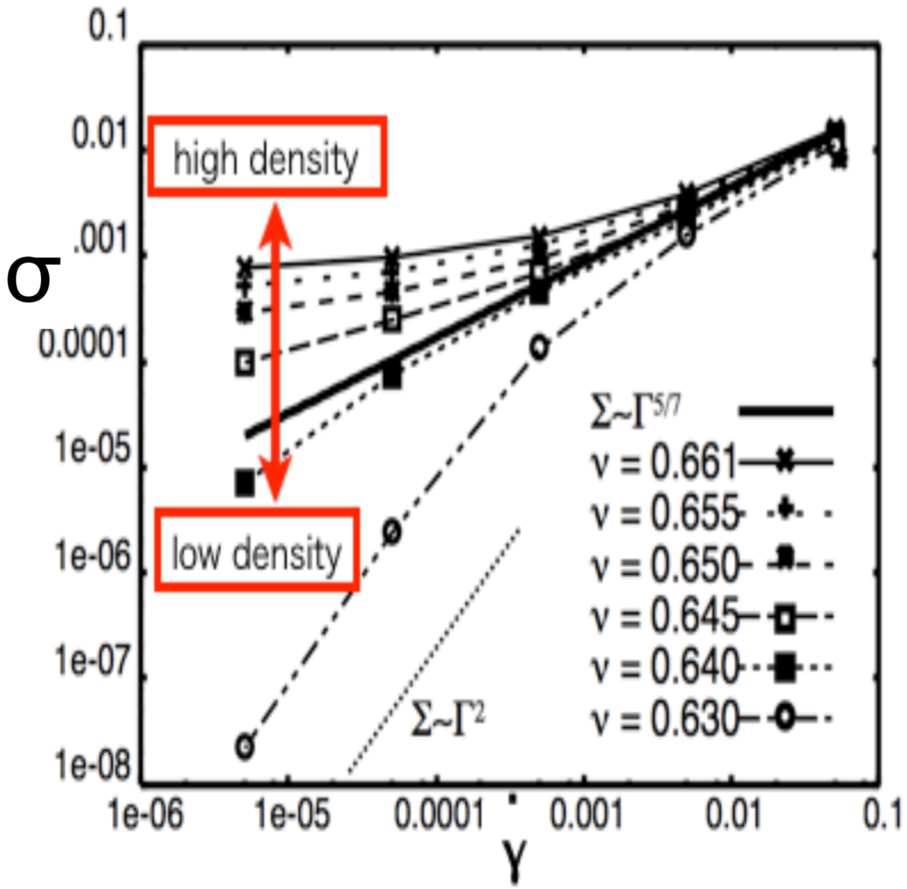
The critical exponents depend on the type of the contact force.

The critical exponents are independent of the dimension.

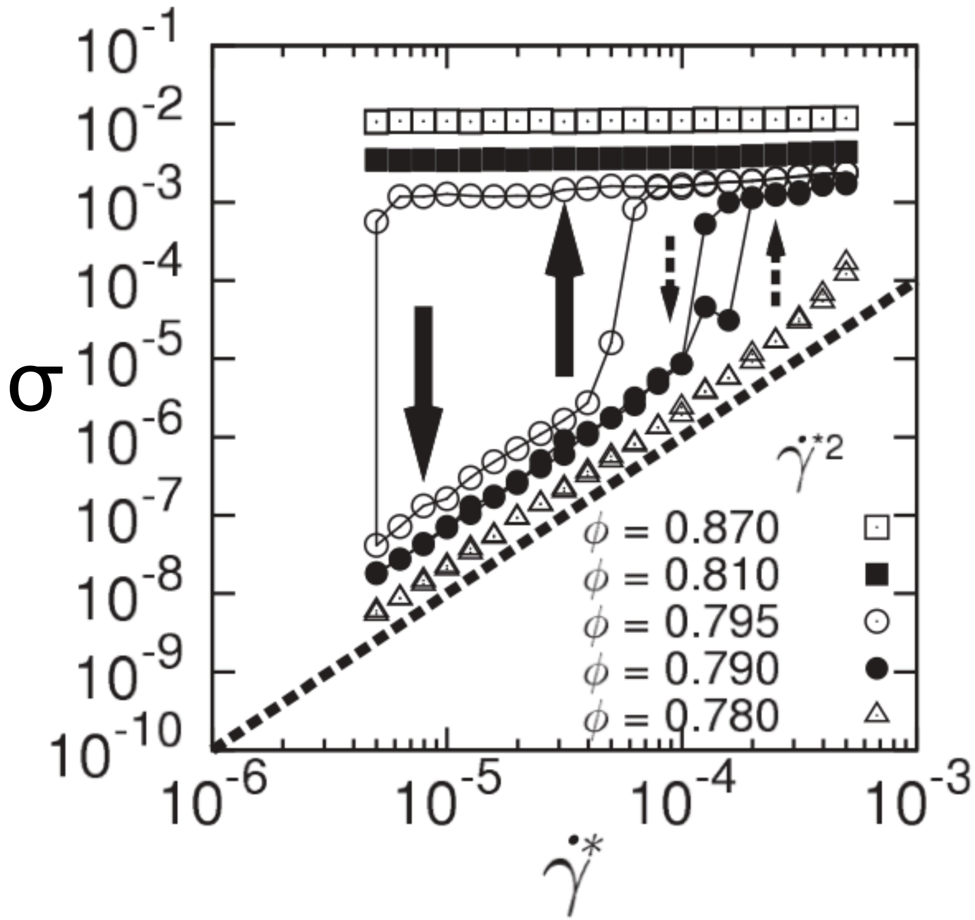
$F_n = k\,\delta^{\Delta}$

M. Otsuki and H. Hayakawa, Phys. Rev. E, 80, 011308, (2009)

Effect of Friction



Frictionless (μ = 0.0)



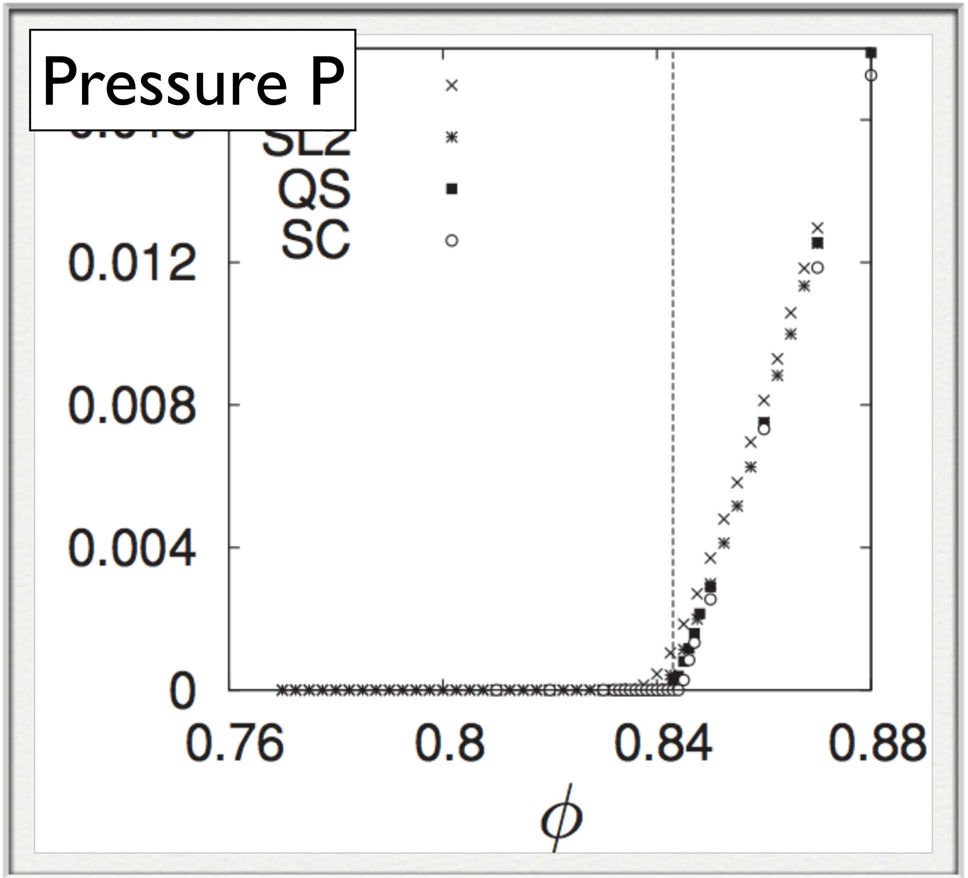
Frictional (μ = 2.0)

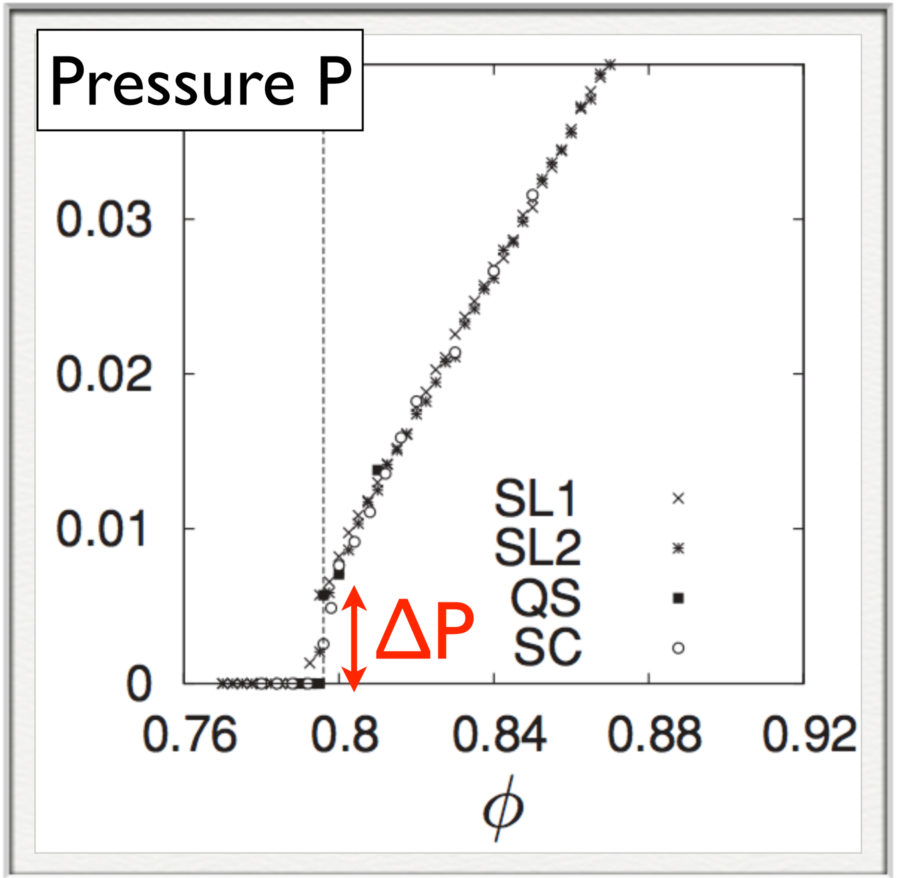
Hysteresis loop for frictional case

M. Otsuki and H. Hayakawa, Phys. Rev. E 83, 051301 (2011)

Effect of friction

(pressure in the zero shear limit)

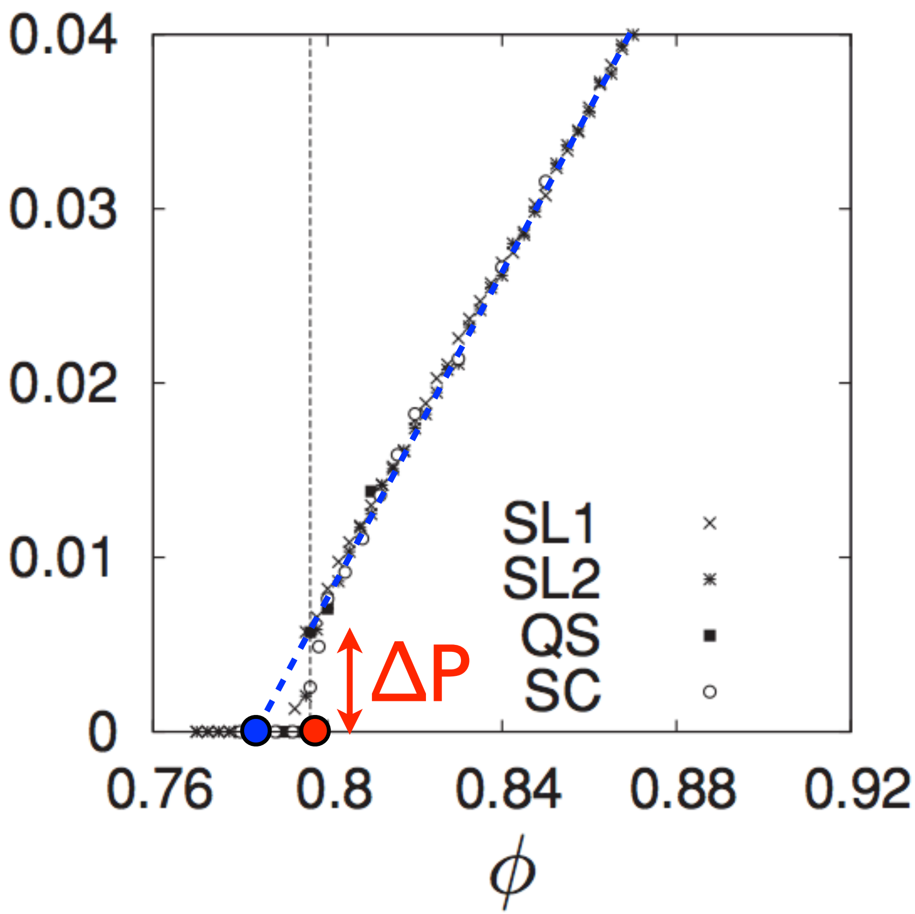


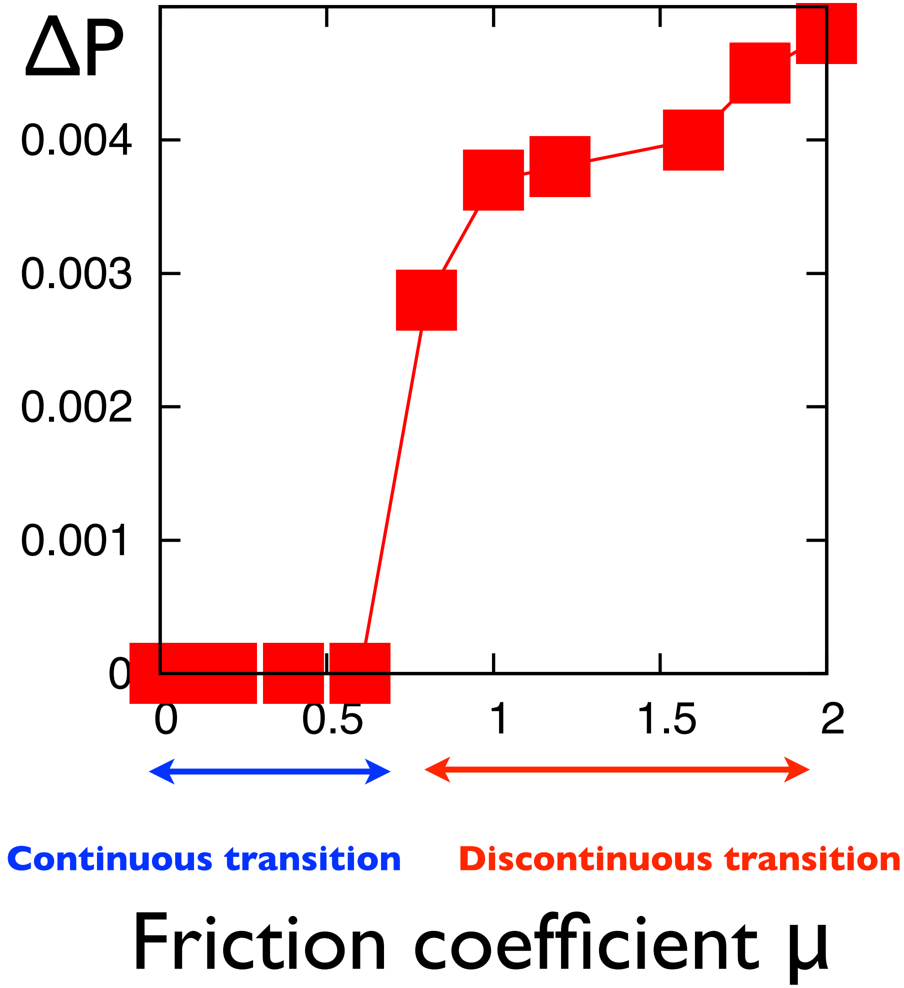


Frictionless (μ = 0.0)
Continuous transition

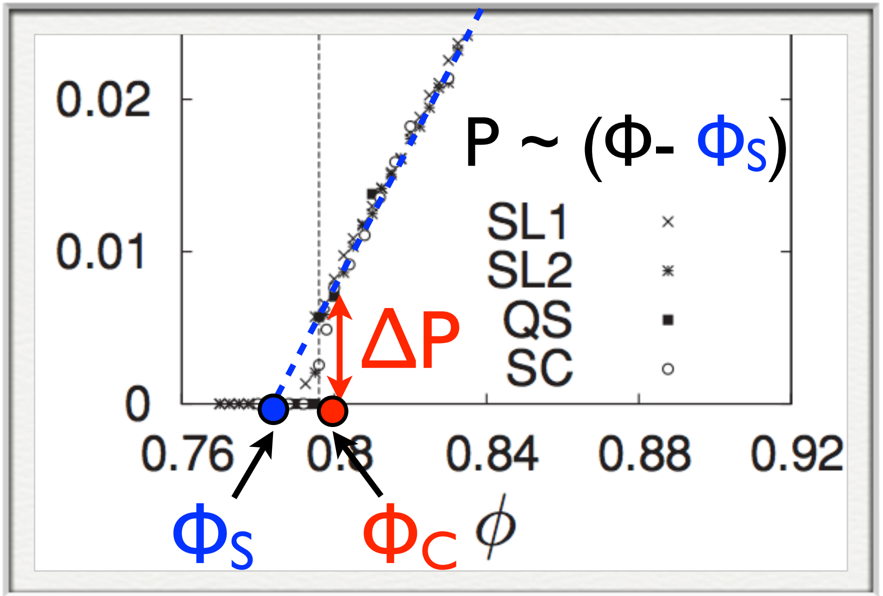
Frictional (μ = 2.0)
Discontinuous transition

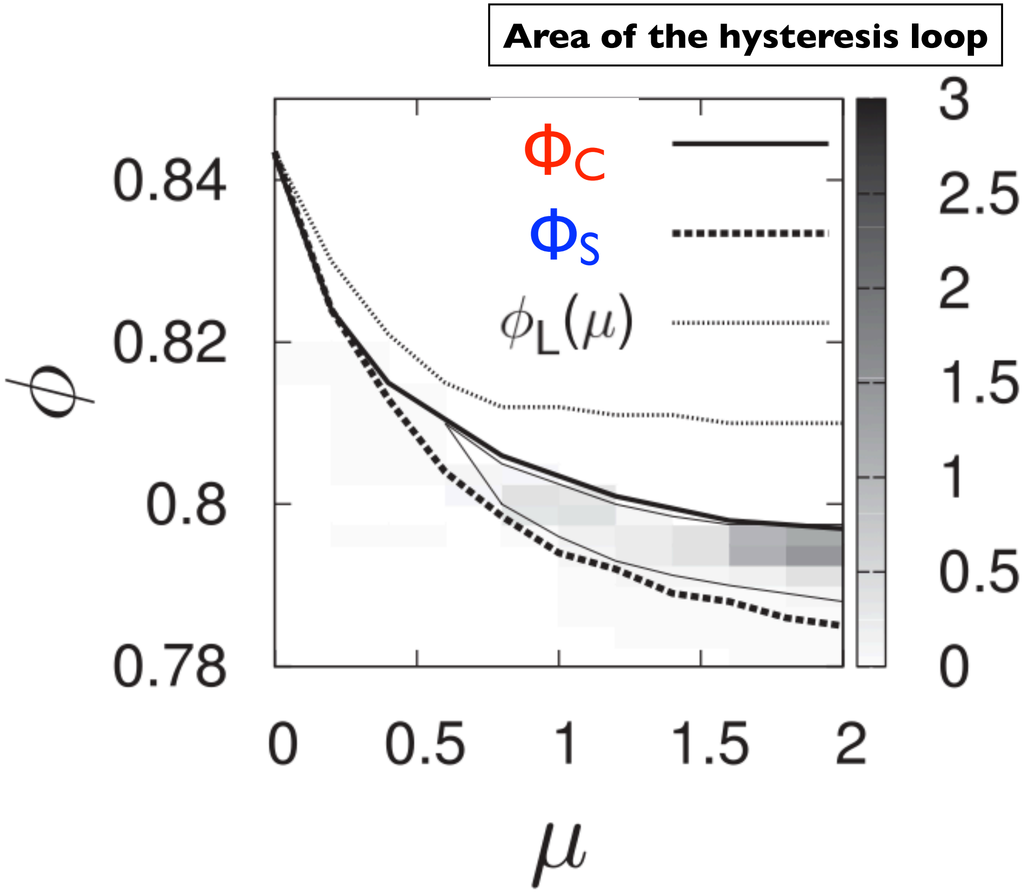
Effect of friction (type of the transition)

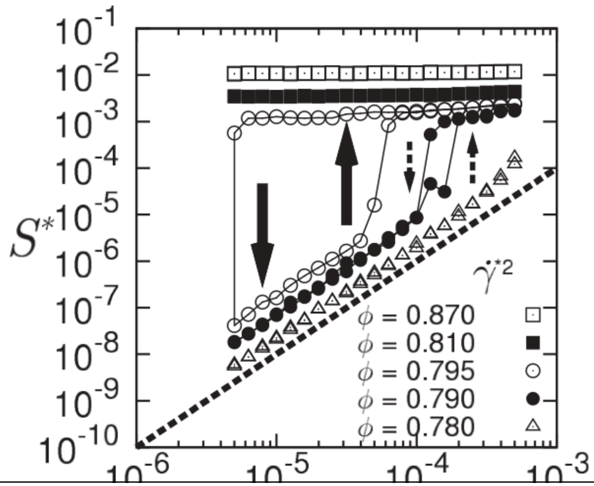




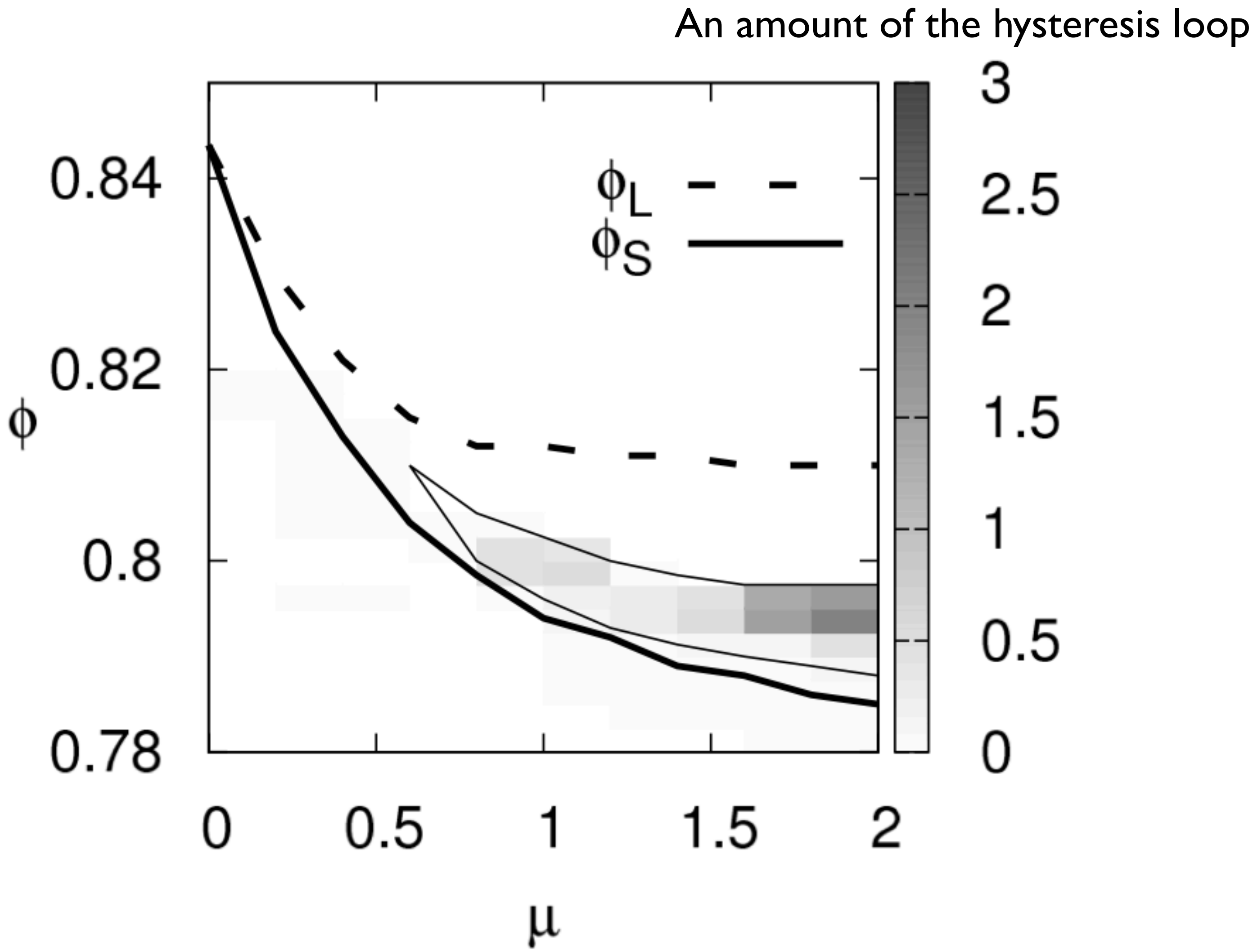
Phase diagram







Many critical densities

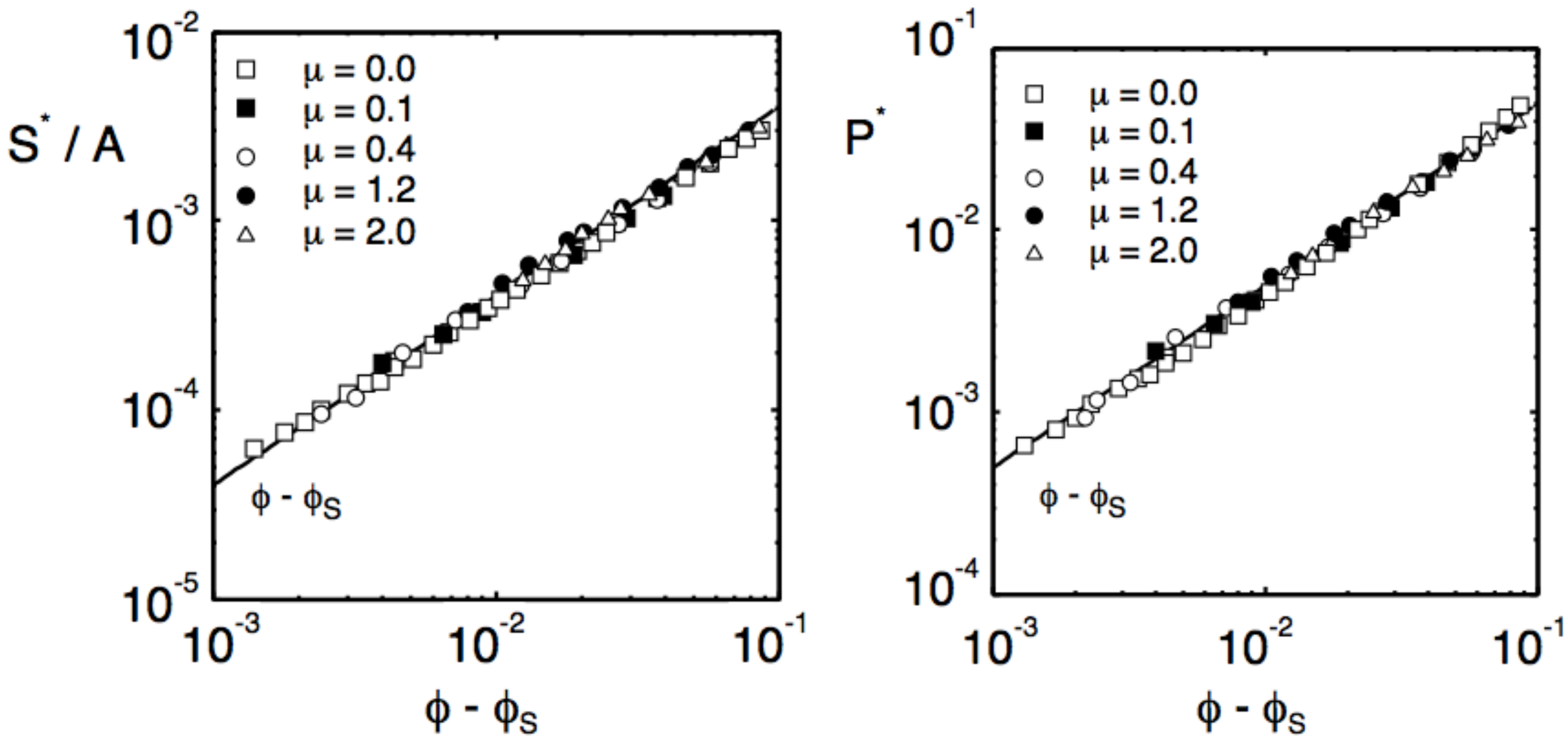


phase diagram

Scaling relations

Solid branch

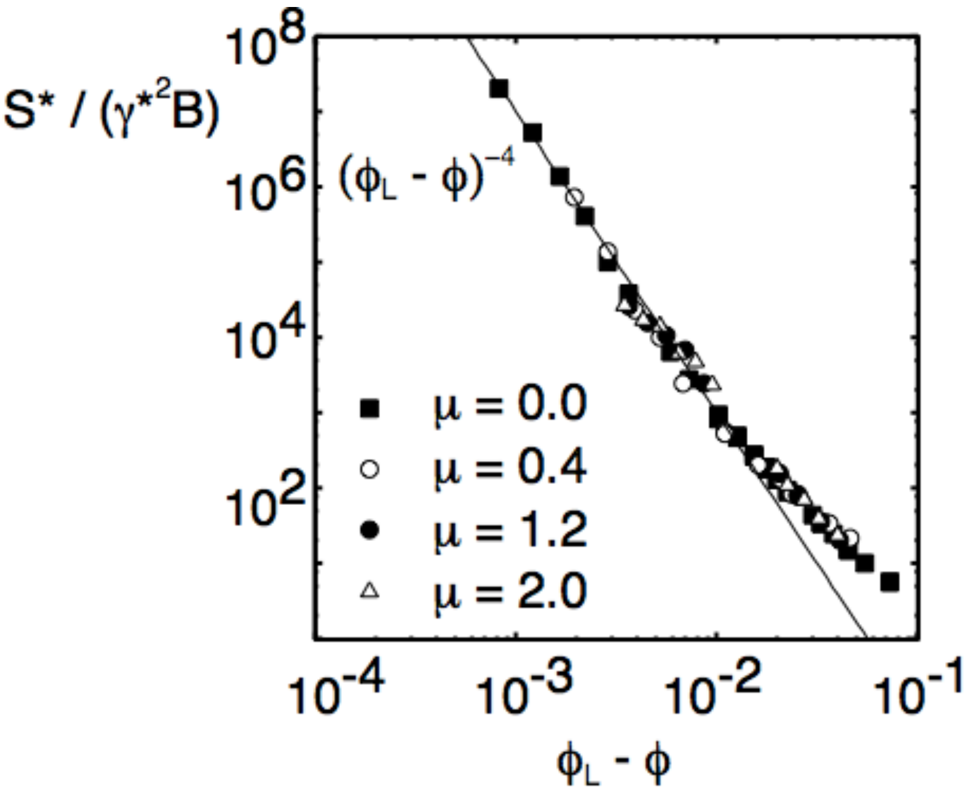
$P \sim (\phi - \phi_S)^{\Delta}$, $S \sim (\phi - \phi_S)^{\Delta}$,

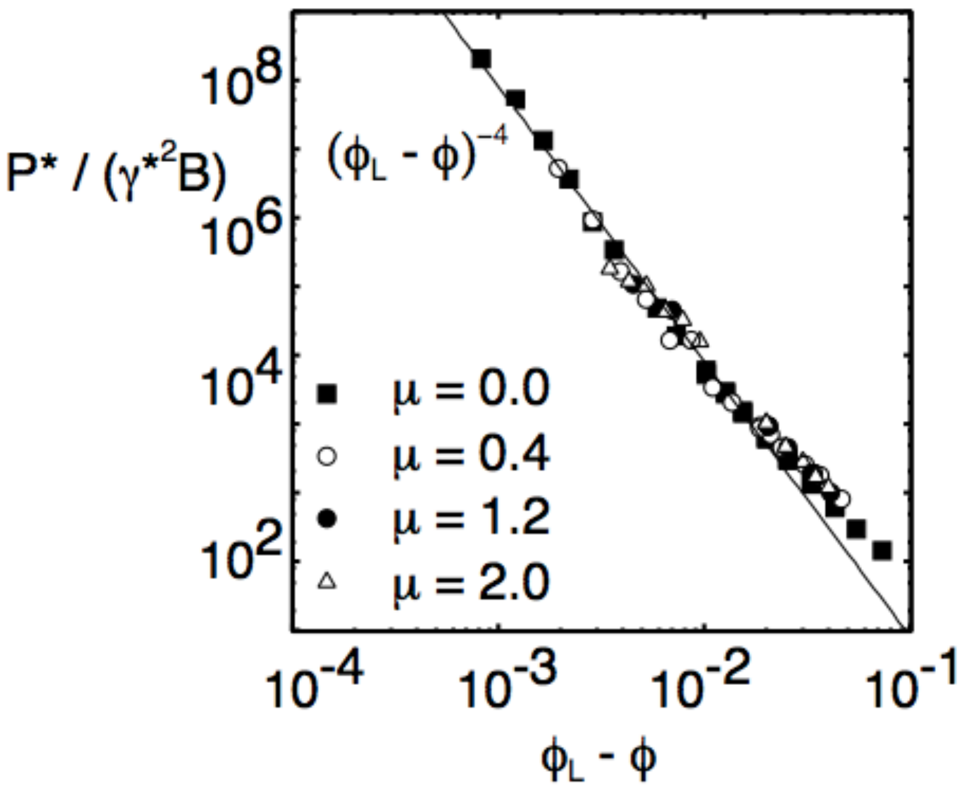


Scaling relations

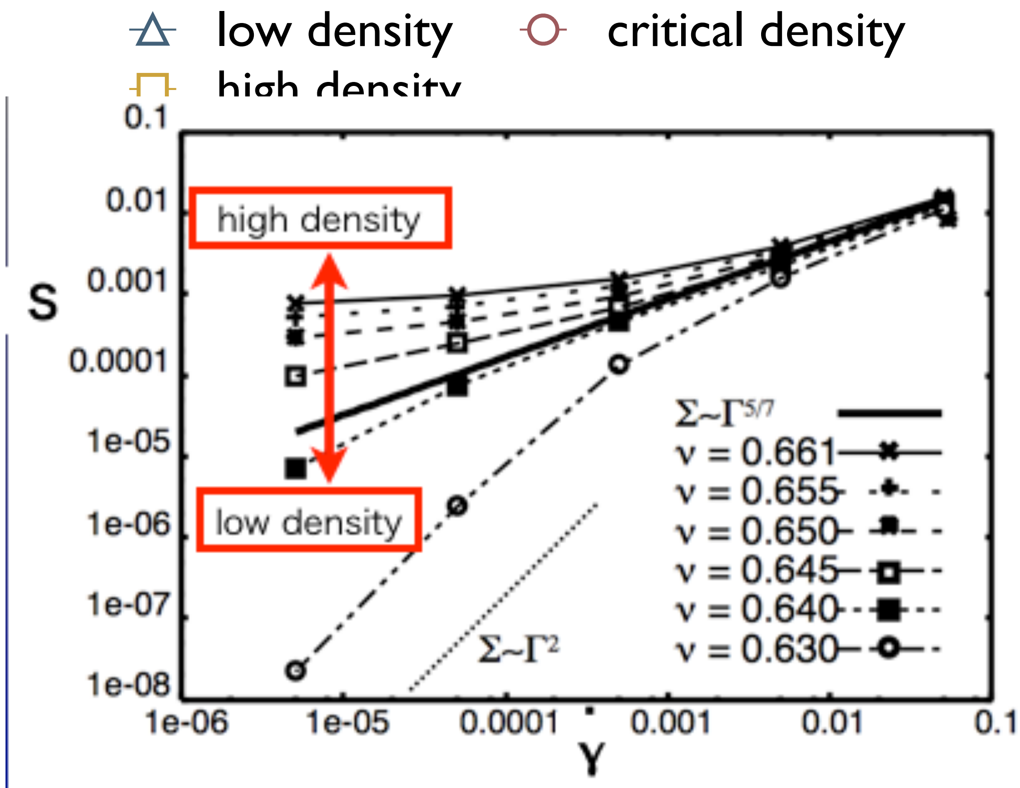
liquid branch

$P \sim \dot{\gamma}^2 (\phi - \phi_L)^{-4}$, $S \sim \dot{\gamma}^2 (\phi - \phi_L)^{-4}$





Granular rheology



0 70

Shear rate : γ2

Yield stress : S_Y

- low density : $S \propto \dot{\gamma}^2$

  Bagnold law

- critical density : $S \propto \dot{\gamma}^{y_\gamma}$

- high density : $S \to S_Y$

  Yield stress :

  $S_Y \propto (\Phi - \Phi_J)^{y_\Phi}$

Theory for exponents

PTP, PRE (2009)

$$T = |\Phi|^{x_\Phi} \mathcal{T}_\pm \left(\dot\gamma |\Phi|^{-\alpha}\right),$$

Kinetic energy

$$S = |\Phi|^{y_\Phi} \mathcal{S}_\pm \left(\dot\gamma |\Phi|^{-\alpha}\right),$$

Shear stress

$$P = |\Phi|^{y'_\Phi} \mathcal{P}_\pm \left(\dot\gamma |\Phi|^{-\alpha}\right),$$

Pressure

Assumption

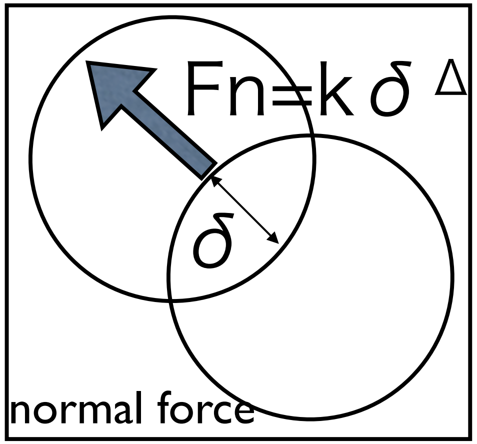
- S / P is constant.

  Coulomb's friction : Hatano (2007)
- P in high density region : $P \sim \Phi^{\Delta}$

  O'Hern, et al., (2003)
- Characteristic time : $P^{-1/2}$

  Wyart, et al. (2005)
- Low density region : collision frequency $\propto T^{1/2}$

  Kinetic theory



Δ-dependent critical exponents

$$x_\Phi = 2 + \Delta, \quad y_\Phi = \Delta, \quad y'_\Phi = \Delta, \quad \alpha = \frac{\Delta + 4}{2}$$

c.f. Hatano 2010, Teigh 2010 ($y_\Phi = \Delta + 0.5$)

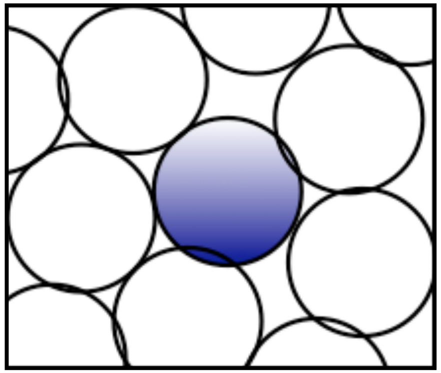
Derivation of exponents

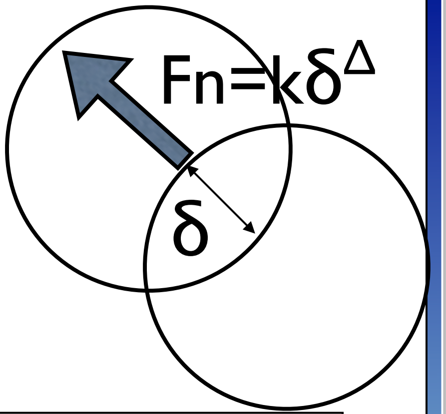
$\dot{\gamma} \to 0, \Phi > 0$ (high density region)

$$P \propto F_c(\Phi)$$

average force : $Fc(\Phi) \to k\,\delta(\Phi)^{\Delta}$

compression length : $\delta(\Phi) \propto \Phi$





$$P \sim \Phi^{\Delta} \qquad P \sim |\Phi|^{y'_\phi}$$

C. S. O'Hern, et al. (2003)

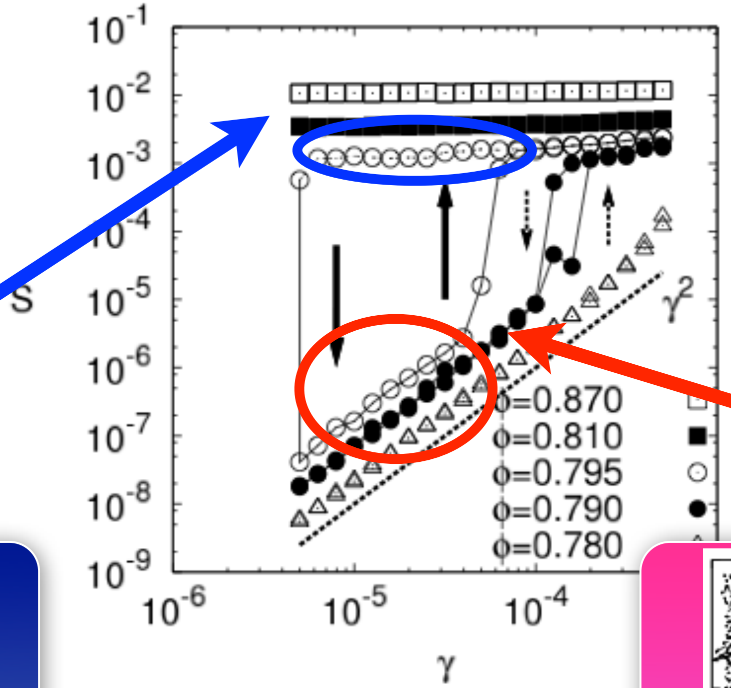
Assumption : S/P is constant.

Coulomb's law

$$S \sim |\Phi|^{y_\phi} \qquad P \sim |\Phi|^{y'_\phi}$$

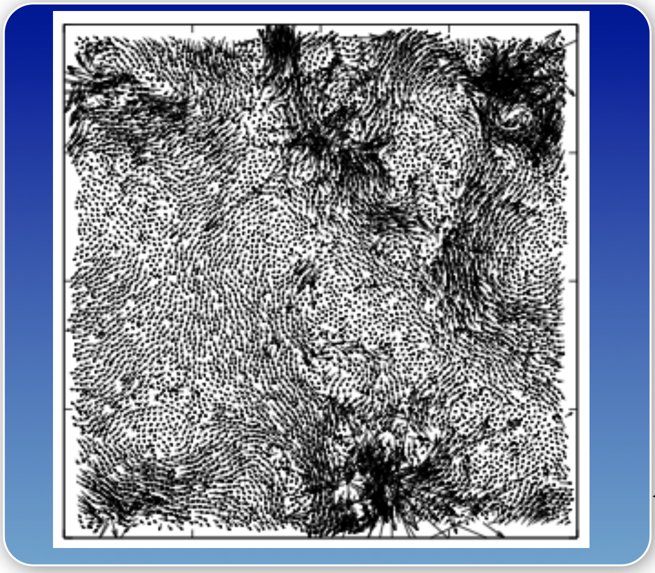
$$x_\Phi = 2 + \Delta, \quad y_\Phi = \Delta, \quad y'_\Phi = \Delta, \quad \alpha = \frac{\Delta + 4}{2}$$

Two branches

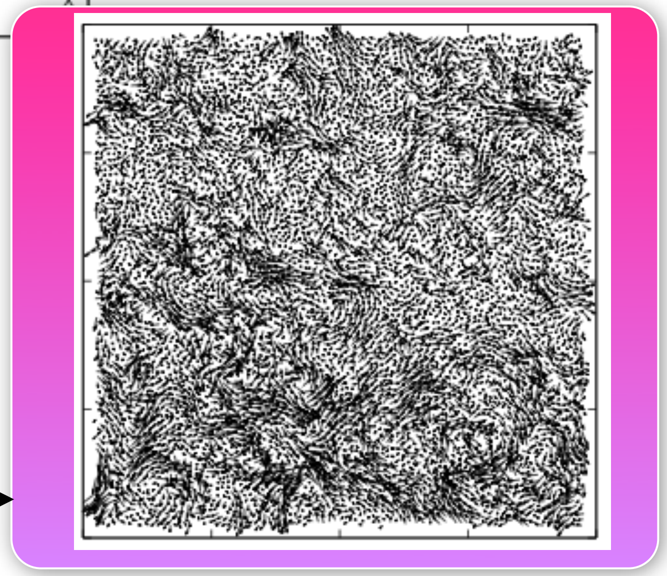


Solid branch

Liquid branch



different contact number



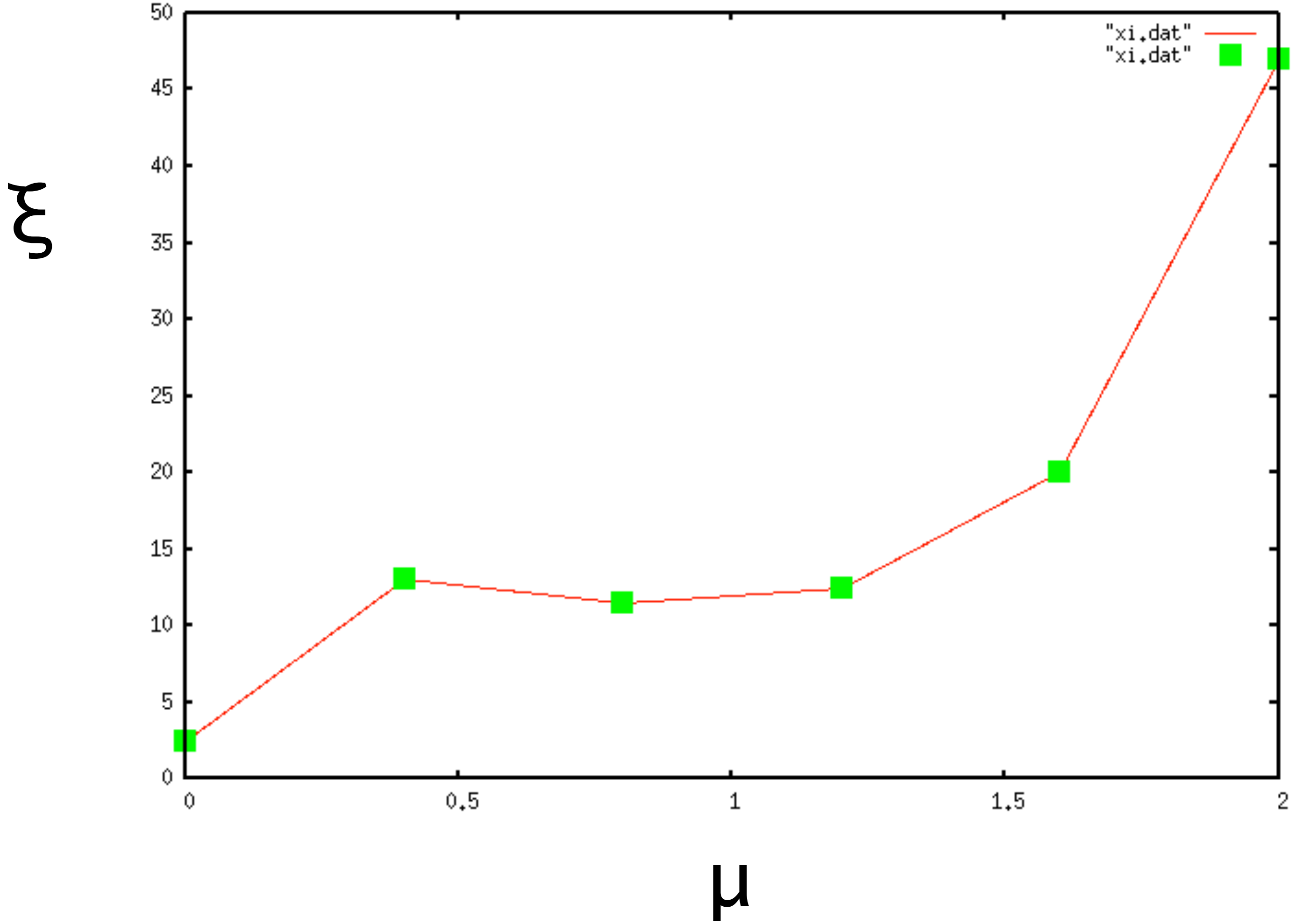
contact number > 3

contact number < 2

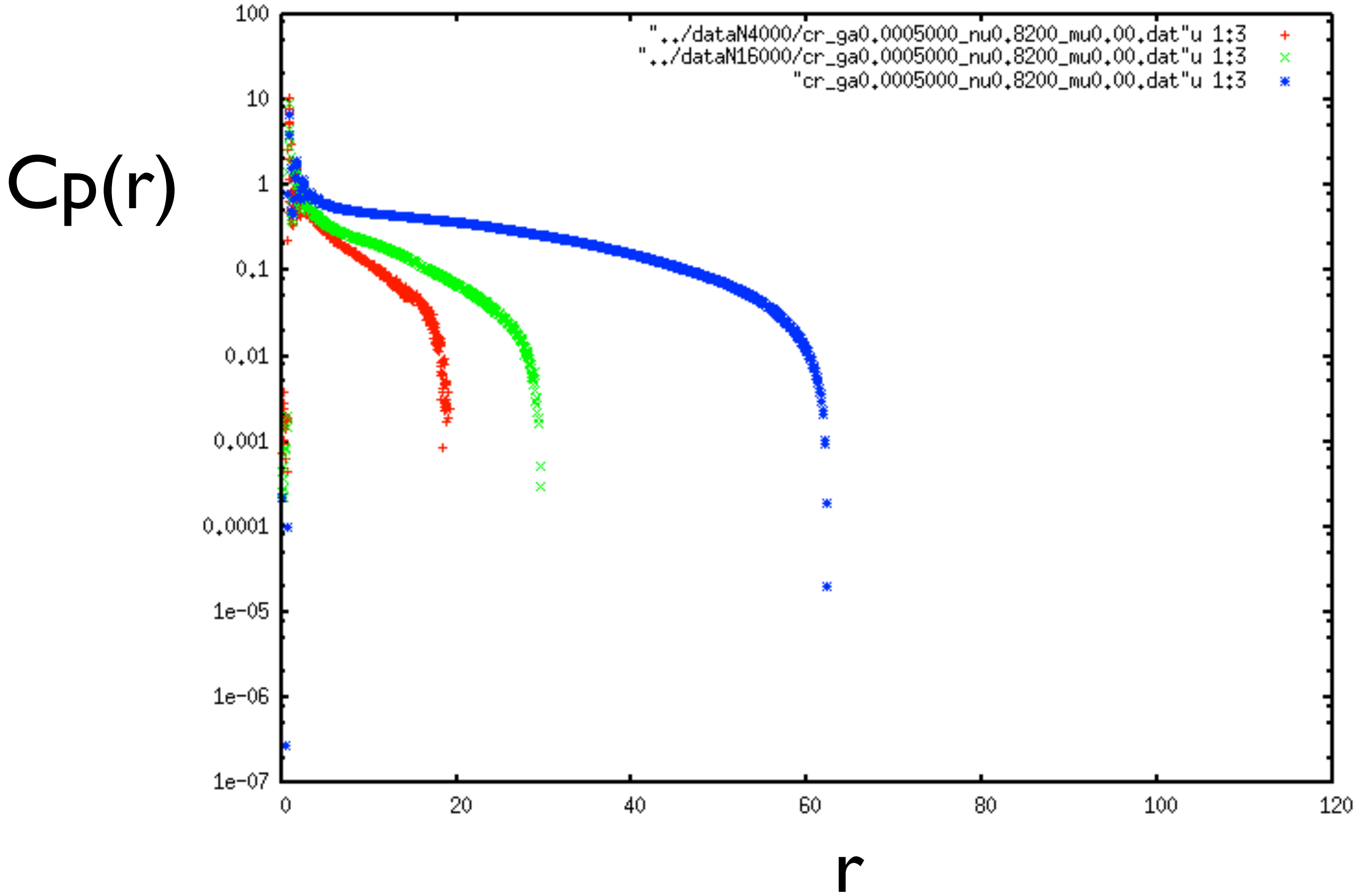
Exponents in other works

Author	y_Φ	$y_\gamma = \alpha / y_\Phi$	y_Φ'	x_Φ	α	system	critical point	shear rate	Number of particles
Olsson & Titel 2007	1.2 = Δ+0.2 (Δ=1)	0.413			2.9	foam	0.8415 (diameters 1:1.4)		1024
Hatano 2008	1.2 = Δ+0.2 (Δ=1)	0.63 (Δ=1)	1.2 = Δ+0.2 (Δ=1)	2.5 (Δ=1)	1.9 (Δ=1)	granular	0.646 (diameters 1:1.4)	$10^{-4} \sim 10^{0}$	1000
Otsuki, Hayakawa, 2009	Δ	2Δ / (Δ+4)	Δ	Δ+2	(Δ+4) / 2	granular	0.648 (diameters 1:1.4)	$5 \times 10^{-7} \sim 5 \times 10^{-5}$	4000
Tighe et al. 2010	Δ+0.5	1/2				foam	0.8423 (diameters 1:1.4)	$10^{-5} \sim 10^{-1}$	1210
Hatano 2010	1.5 = Δ+0.5 (Δ=1)	0.6 (Δ=1)	1.5 = Δ+0.5 (Δ=1)	3.3 (Δ=1)	2.5 (Δ=1)	granular	0.6473 (diameters 1:1.4)	$10^{-8} \sim 10^{-2}$	4000
Nordstrom et al. 2010	2.1 = Δ+0.6 (Δ=1.5)	0.48 (Δ=1.5)			4.1 (Δ=1.5)	foam	0.635		
Olsson & Titel 2010	1.08 = Δ +0.08 (Δ=1)	0.28 (Δ=1)	1.08 = Δ +0.08 (Δ=1)		3.85 (Δ=1)	foam	0.84347 (diameters 1:1.4)	$10^{-8} \sim 10^{-6}$	

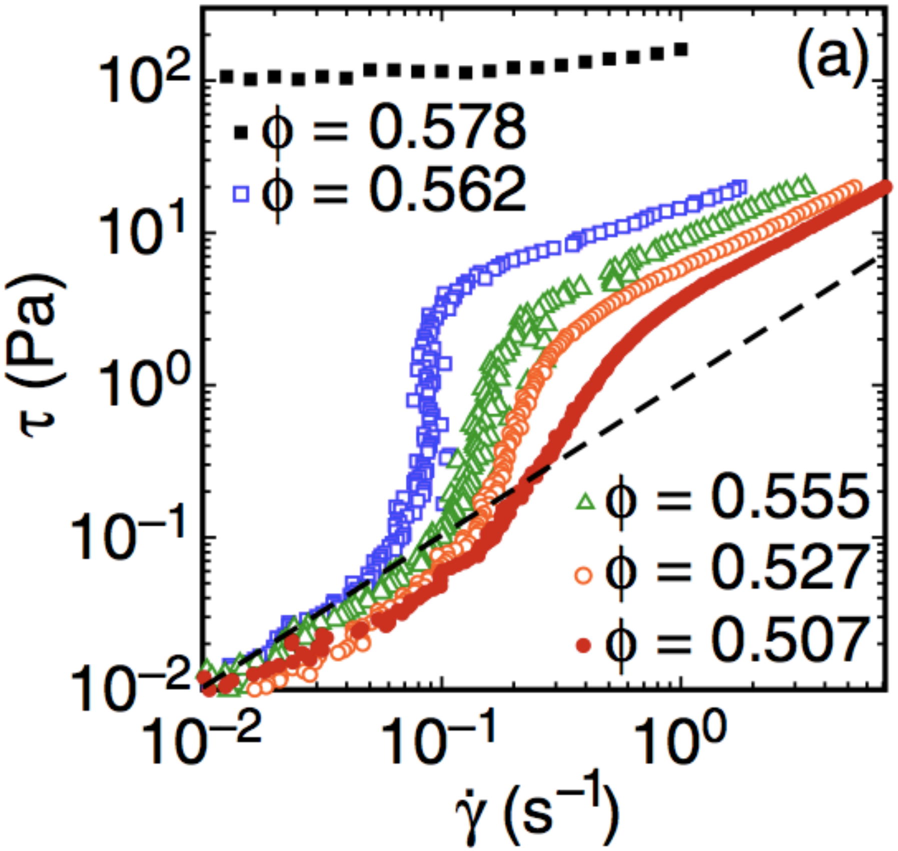
Correlation length



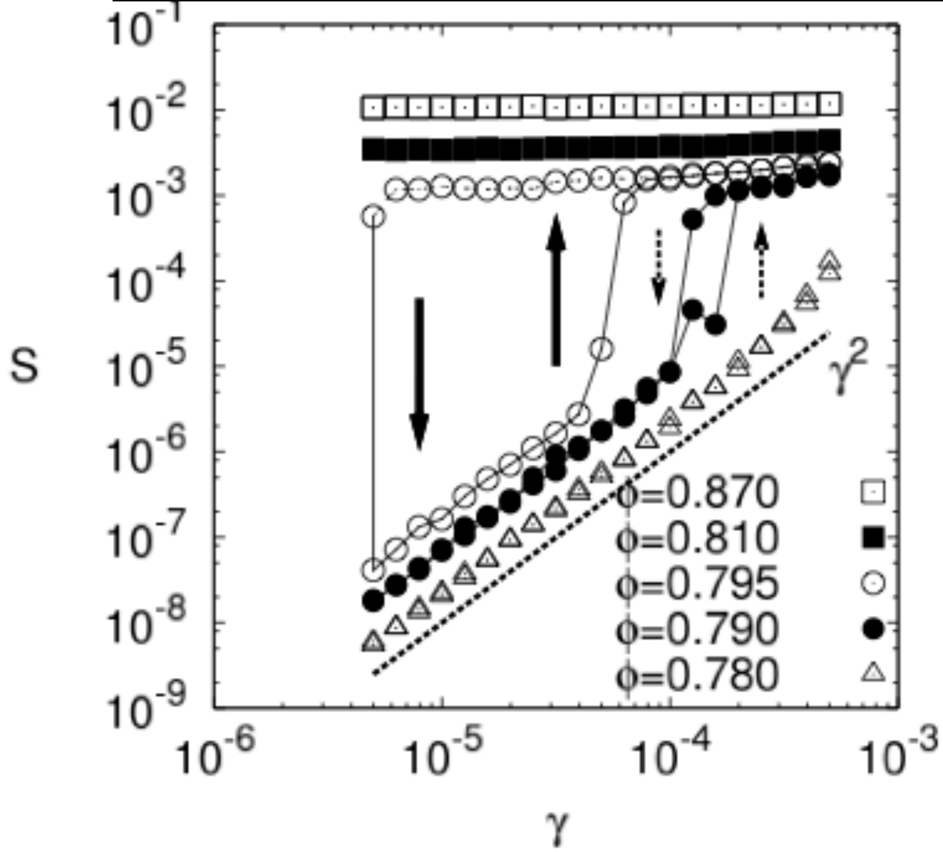
Correlation function



glass spheres in oil



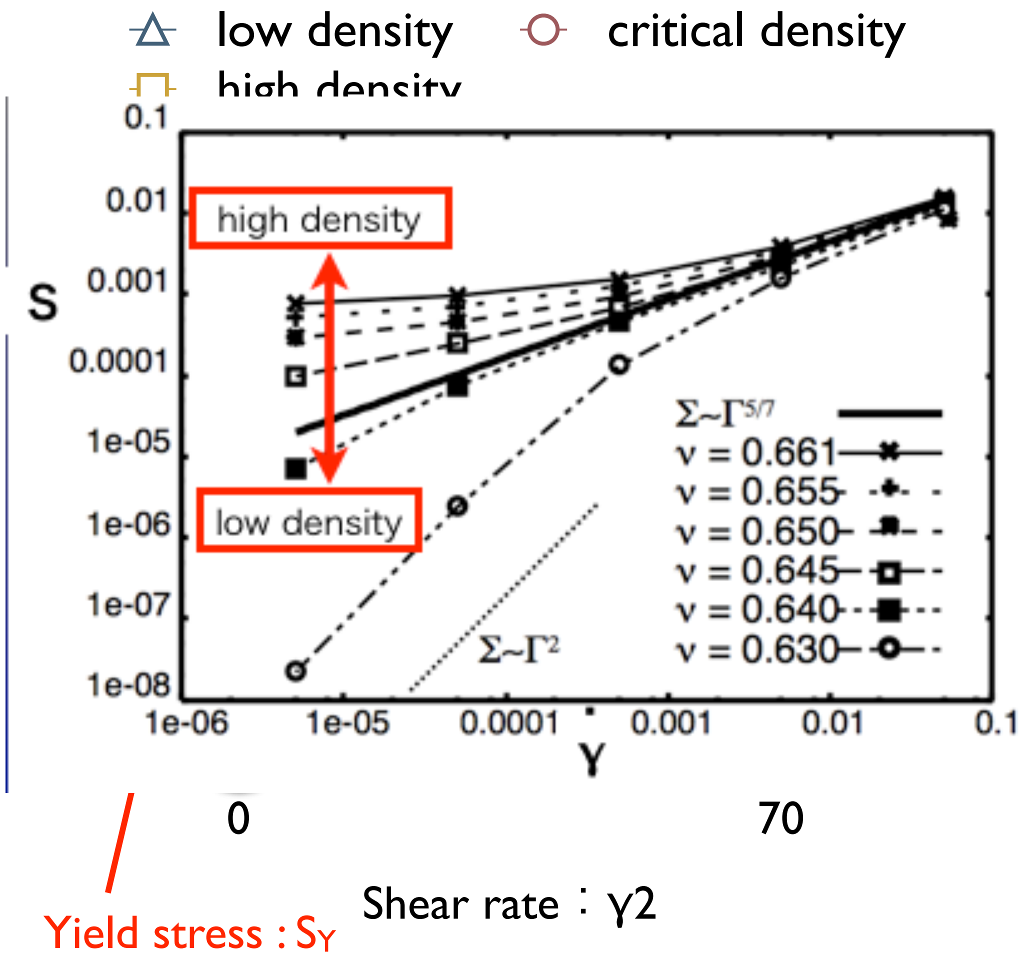
Sheared granular material



Eric Brown, et al. (2010)
Hysteresis loop appears in this system
[private communication]

Related experiment

Granular rheology



- low density : $S \propto \dot{\gamma}^2$

  Bagnold law

- critical density : $S \propto \dot{\gamma}^{y_\gamma}$

- high density : $S \to S_Y$

  Yield stress :

  $S_Y \propto (\Phi - \Phi_J)^{y_\Phi}$

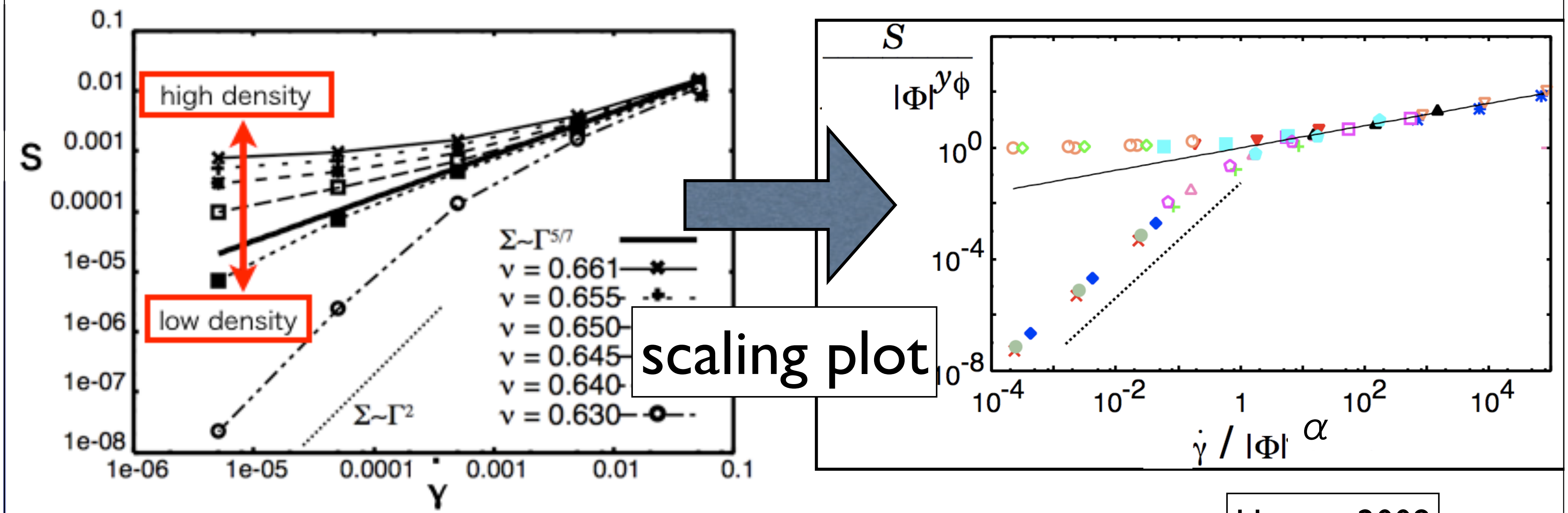
Critical scaling

$$S = |\Phi|^{y_\Phi} S_\pm \left(\frac{\dot{\gamma}}{|\Phi|^{\alpha}} \right)$$

y_Φ, α : Critical exponents

S : Shear stress, $\dot{\gamma}$: Shear rate

$$\Phi \equiv \phi - \phi_J$$



Hatano, 2008

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$$S = |\Phi|^{y_\Phi} \mathcal{S}_\pm \left(\dot\gamma |\Phi|^{-\alpha}\right),$$

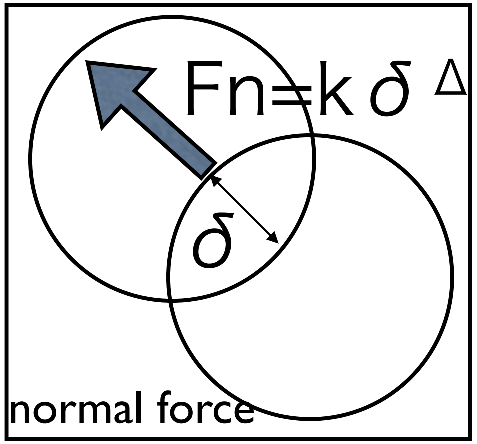
Shear stress

$$P = |\Phi|^{y'_\Phi} \mathcal{P}_\pm \left(\dot\gamma |\Phi|^{-\alpha}\right),$$

Pressure

Assumption

- S / P is constant.
  Coulomb's friction : Hatano (2007)
- P in high density region : $P \sim \Phi^{\Delta}$
  O'Hern, et al., (2003)
- Characteristic time : $P^{-1/2}$
  Wyart, et al. (2005)
- Low density region : collision frequency $\propto T^{1/2}$
  Kinetic theory



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c.f. Hatano 2010, Teigh 2010 ($y_\Phi = \Delta + 0.5$)

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$\dot{\gamma} \to 0, \Phi > 0$ (high density region)

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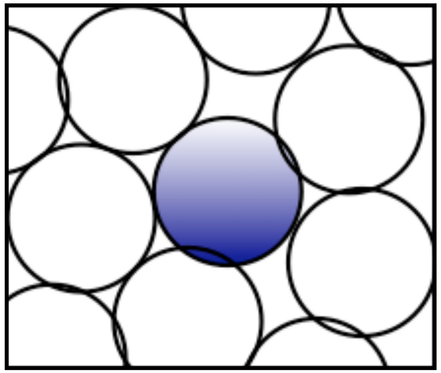
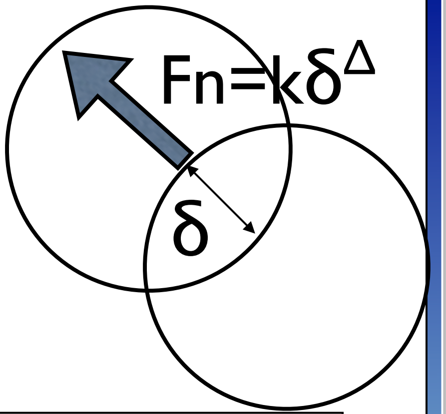
compression length : $\delta(\Phi) \propto \Phi$

$$P \sim \Phi^{\Delta} \qquad P \sim |\Phi|^{y'_\phi}$$





C. S. O'Hern, et al. (2003)

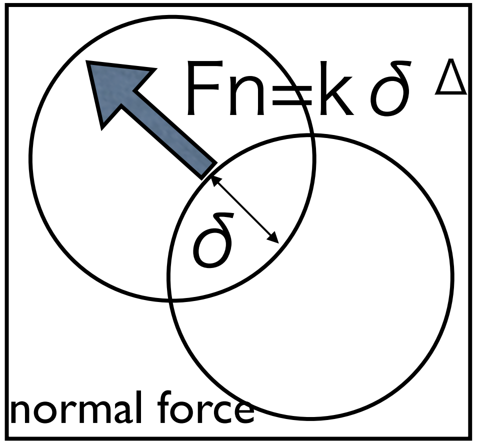
Assumption : S/P is constant.

Coulomb's law

$$S \sim |\Phi|^{y_\phi} \qquad P \sim |\Phi|^{y'_\phi}$$

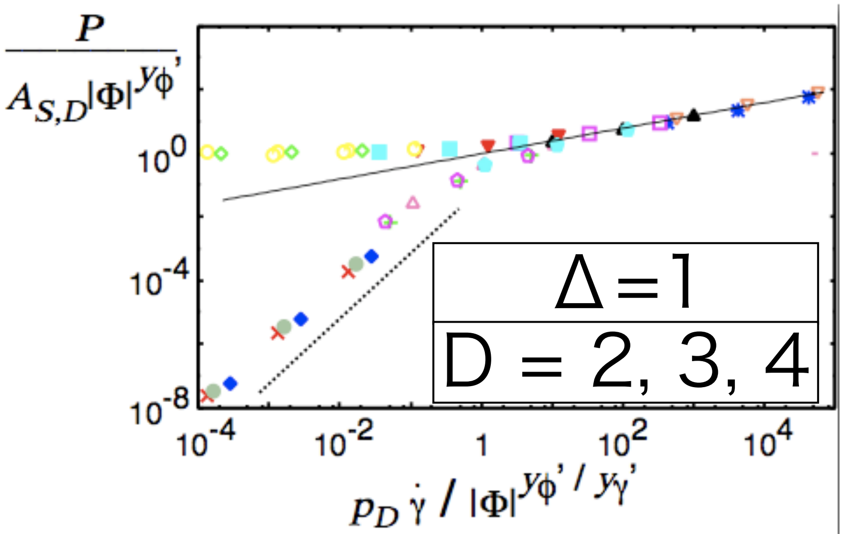
$$x_\Phi = 2 + \Delta, \quad y_\Phi = \Delta, \quad y'_\Phi = \Delta, \quad \alpha = \frac{\Delta + 4}{2}$$

Validity of the theory

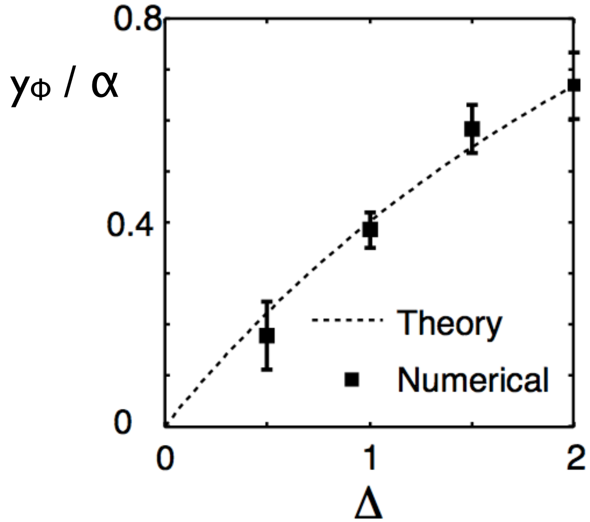


$$x_\Phi = 2 + \Delta, \quad y_\Phi = \Delta, \quad y'_\Phi = \Delta, \quad \alpha = \frac{\Delta + 4}{2}$$

$$P = |\Phi|^{y'_\Phi} \mathcal{P}_\pm \left(\dot{\gamma} |\Phi|^{-\alpha} \right),$$



Scaling plot for P



Δ-dependence of exponent

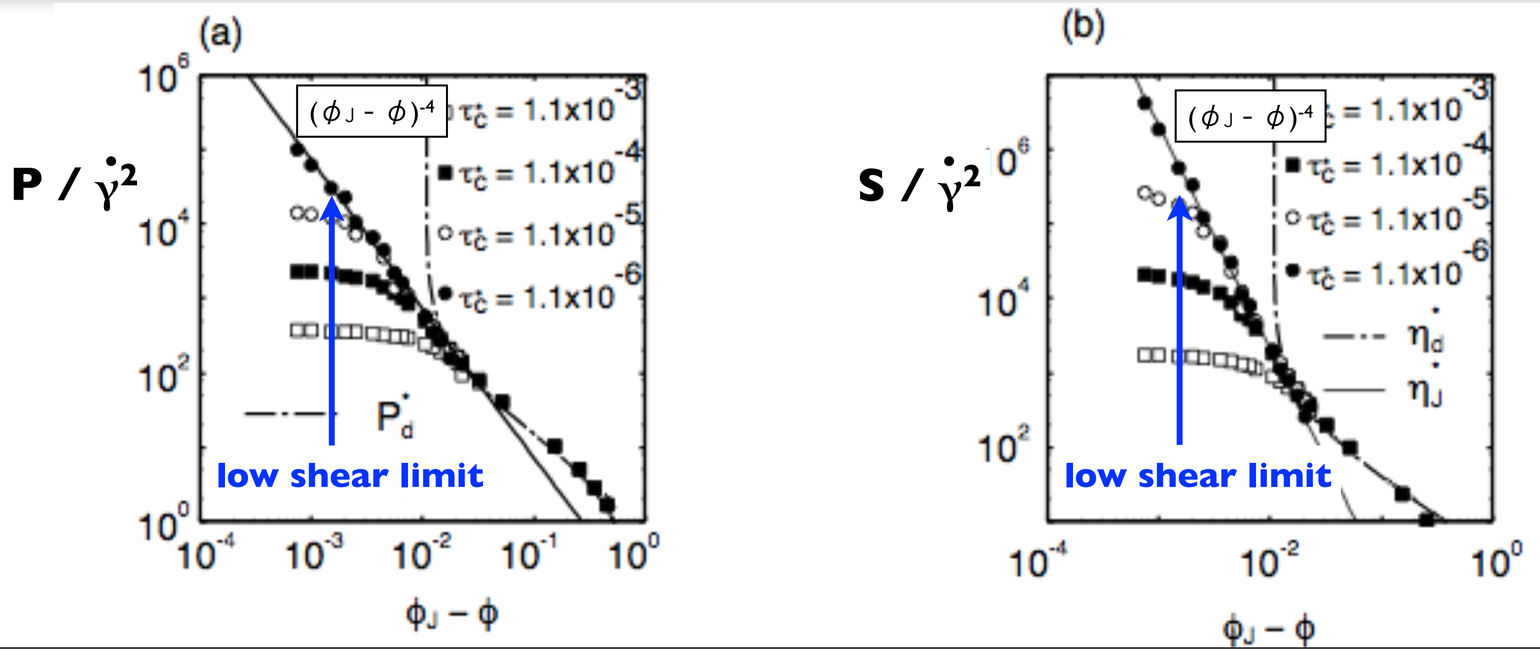
Scaling law

- high density region($\phi > \phi_J$) + low shear limit($\dot{\gamma} \to 0$)

$$P \sim (\phi - \phi_J)^{\Delta}, \quad S \sim (\phi - \phi_J)^{\Delta}$$

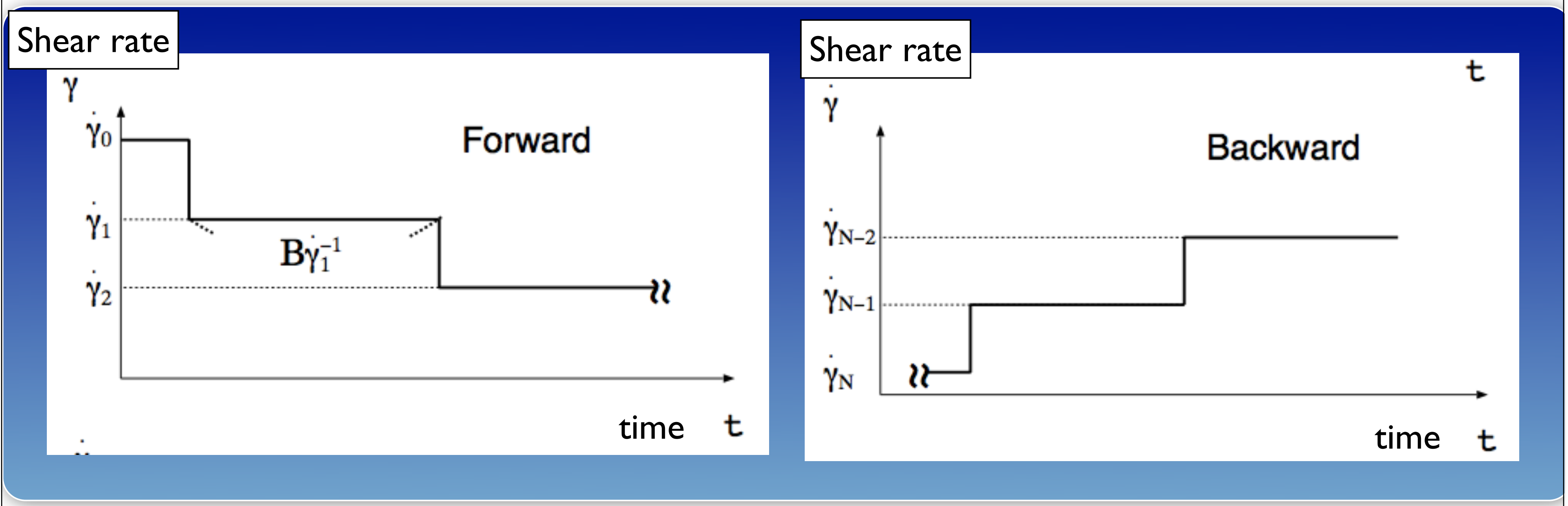
- low density region($\phi < \phi_J$) + low shear limit($\dot{\gamma} \to 0$)

$$P \sim \dot{\gamma}^2 (\phi_J - \phi)^{-4}, \quad S \sim \dot{\gamma}^2 (\phi_J - \phi)^{-4}$$

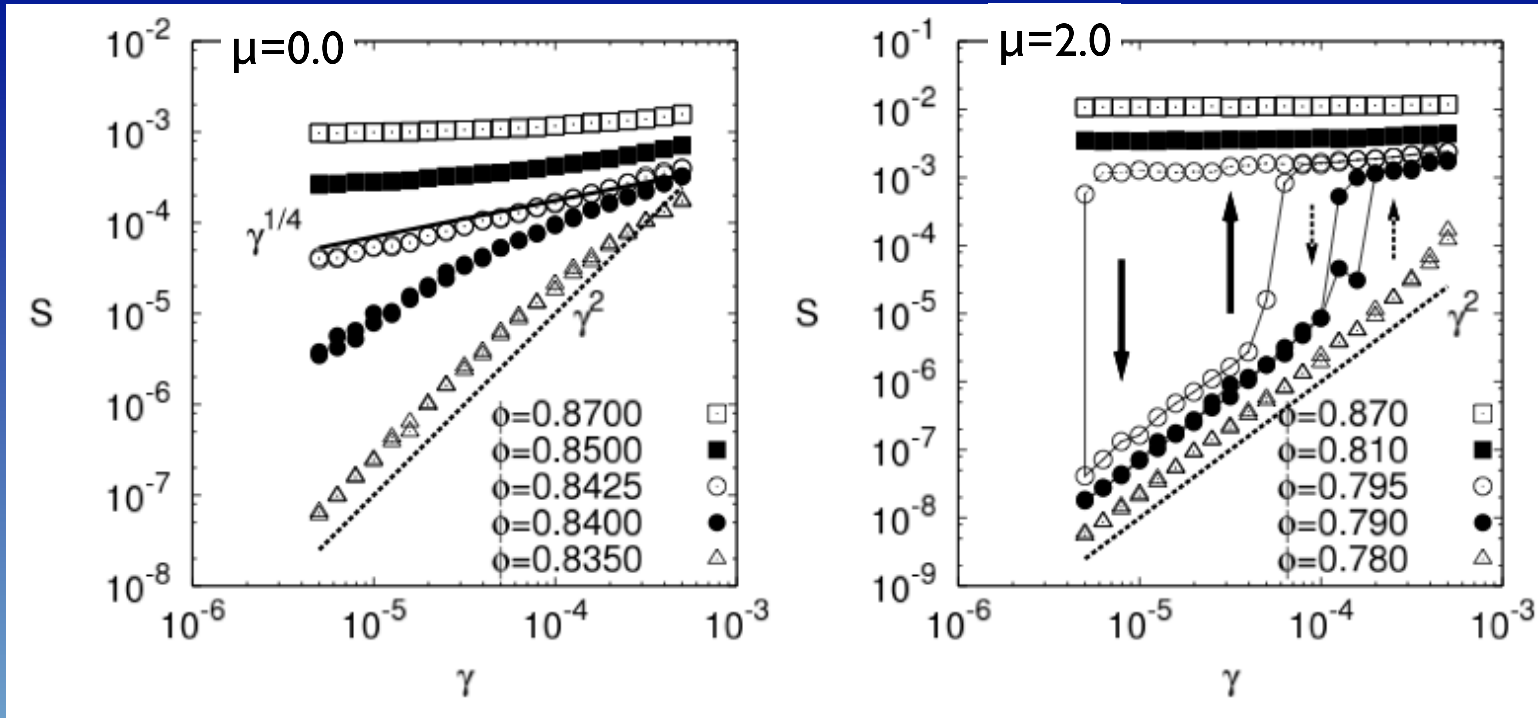


Protocol

- We sequentially change shear rate.



Shear stress



- Similar behavior to the frictionless case

low density $S \propto \dot{\gamma}^2$ critical density $S \sim \dot{\gamma}^{y_\gamma}$ high density $S(\gamma) \to S_Y$

- Hysteresis loop appears around the critical point

Scaling law

frictionless

- high density region($\phi > \phi_J$) + low shear limit($\dot{\gamma} \to 0$)

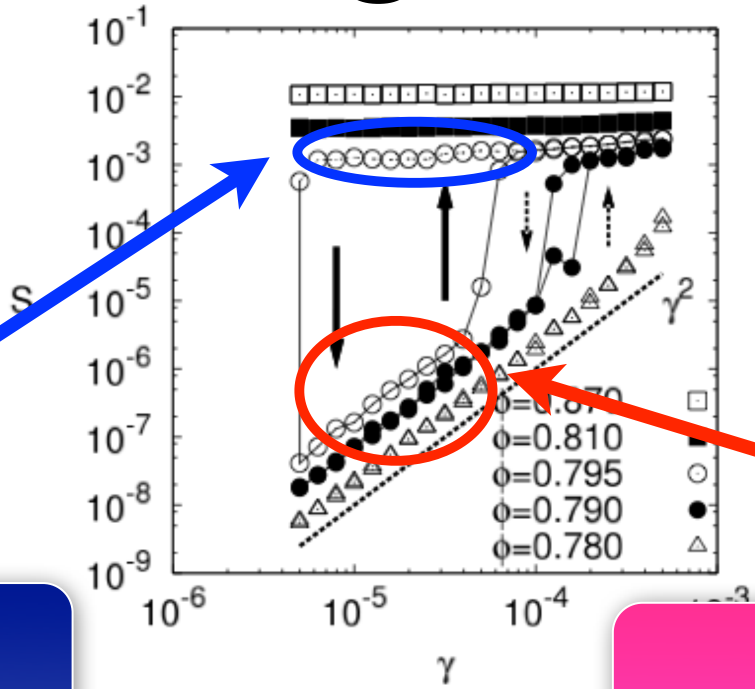
$P \sim (\phi - \phi_J)^{\Delta}$, $S \sim (\phi - \phi_J)^{\Delta}$,

- low density region($\phi < \phi_J$) + low shear limit($\dot{\gamma} \to 0$)

$P \sim \dot{\gamma}^2 (\phi_J - \phi)^{-4}$, $S \sim \dot{\gamma}^2 (\phi_J - \phi)^{-4}$

Scaling laws

frictional



Solid branch

liquid branch

$P \sim (\phi - \phi_S)^{\Delta}$,

$S \sim (\phi - \phi_S)^{\Delta}$,

critical densities

$\phi_S(\mu)$, $\phi_L(\mu)$

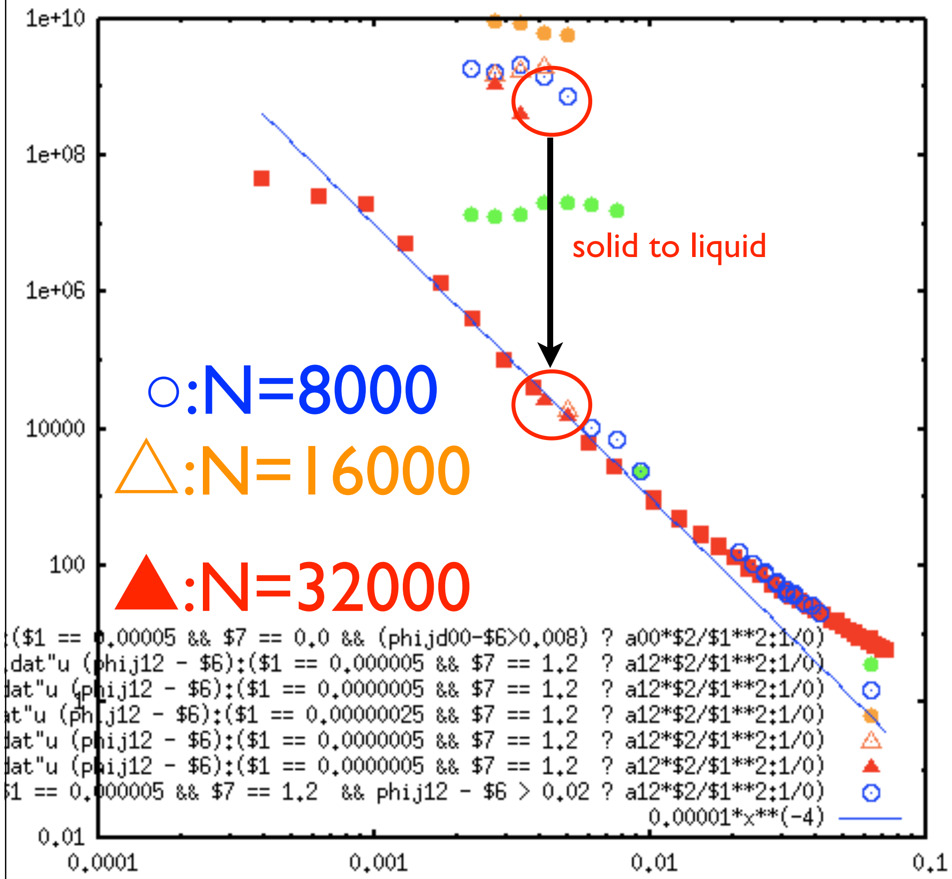
$P \sim \dot{\gamma}^2 (\phi - \phi_L)^{-4}$,

$S \sim \dot{\gamma}^2 (\phi - \phi_L)^{-4}$

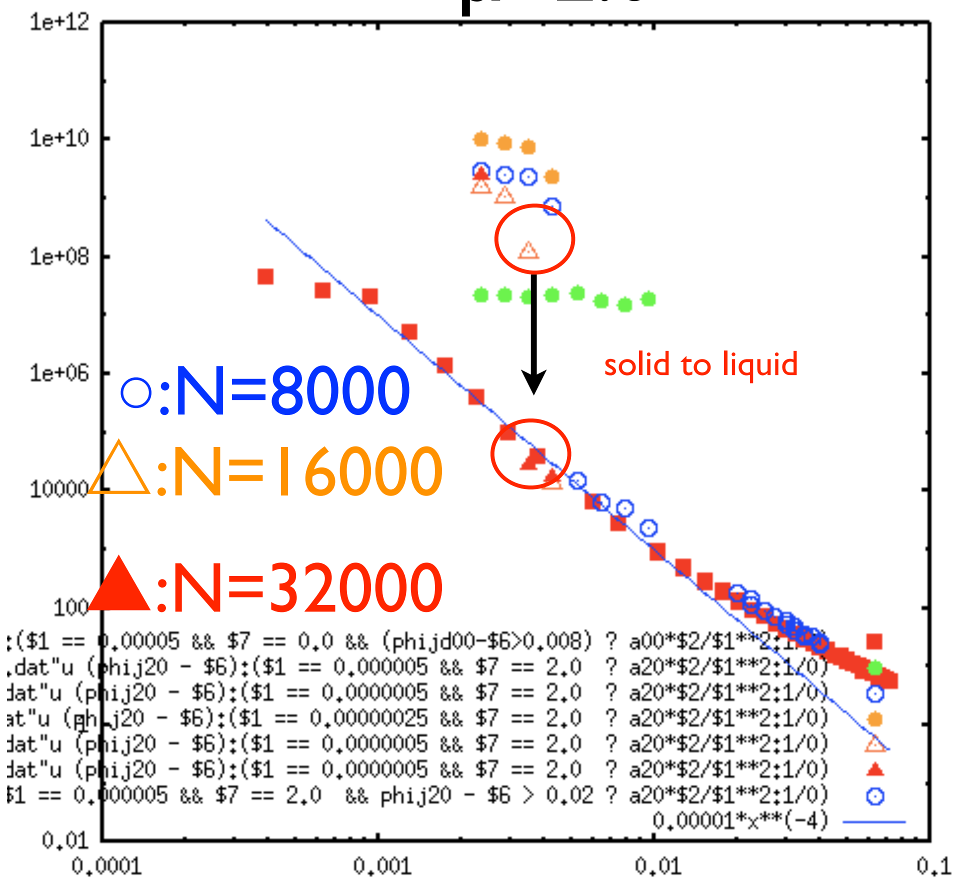
c.f. Somfai, et al. (2007), Silbert (2010) [unsheared case]

Finite-size effect

μ=1.2

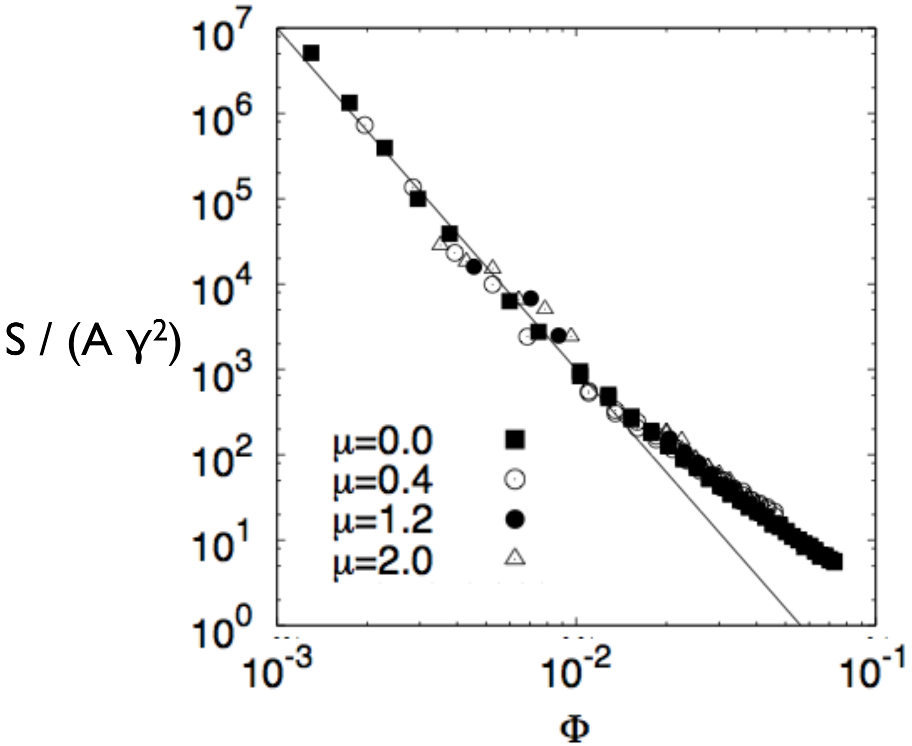


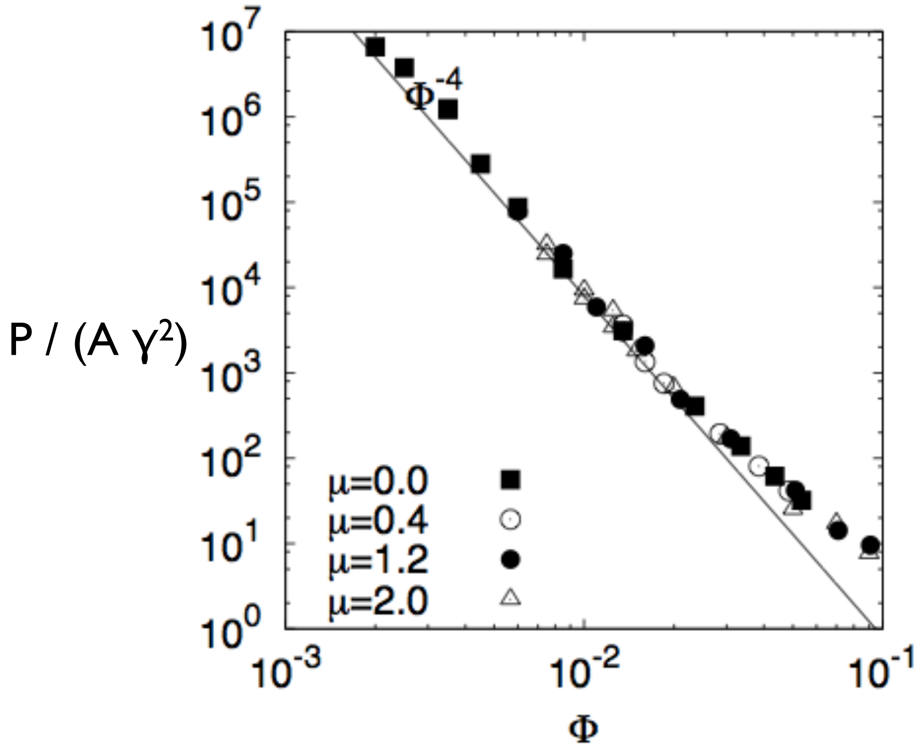
μ=2.0



Scaling in the liquid branch

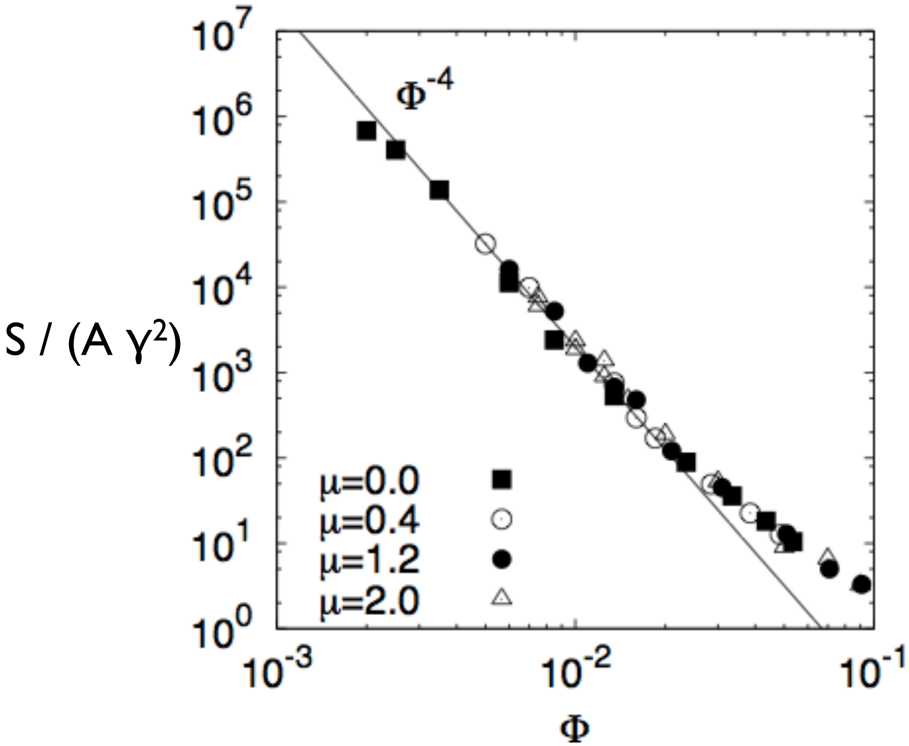
$S \sim \dot{\gamma}^2 (\phi - \phi_L)^{-4}$ $P \sim \dot{\gamma}^2 (\phi - \phi_L)^{-4}$,

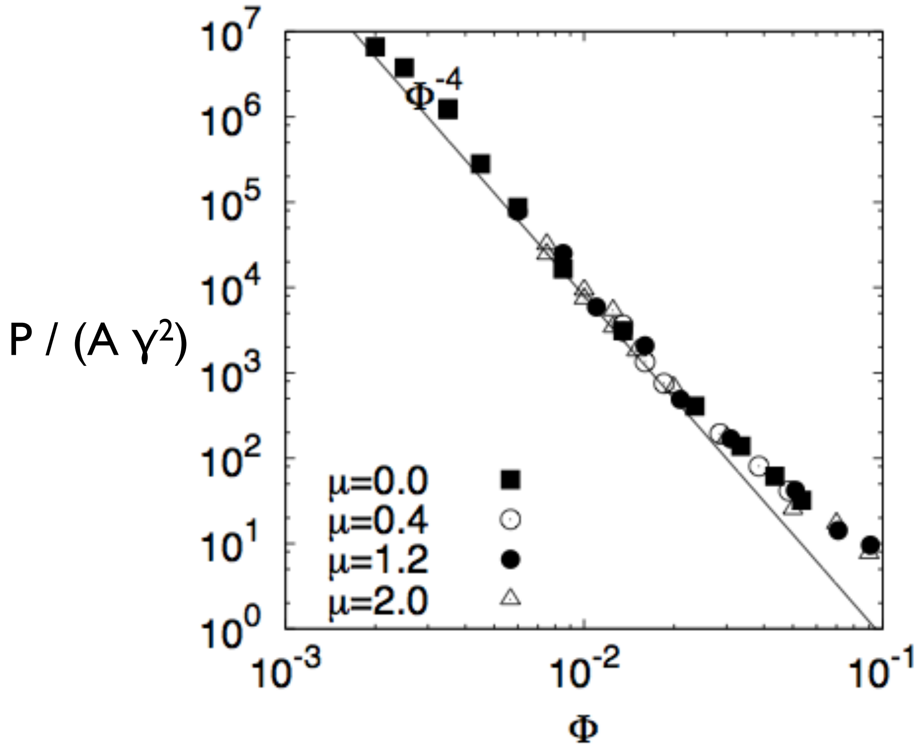




Scaling in the liquid branch

$S \sim \dot{\gamma}^2 (\phi - \phi_L)^{-4}$ $P \sim \dot{\gamma}^2 (\phi - \phi_L)^{-4}$,





Critical exponents

$$T = |\Phi|^{x_\Phi} \mathcal{T}_\pm \left(\dot\gamma |\Phi|^{-\alpha} \right),$$

Temperature

$$S = |\Phi|^{y_\Phi} \mathcal{S}_\pm \left(\dot\gamma |\Phi|^{-\alpha} \right),$$

Shear stress

$$P = |\Phi|^{y'_\Phi} \mathcal{P}_\pm \left(\dot\gamma |\Phi|^{-\alpha} \right),$$

Pressure

$$\omega = |\Phi|^{z_\Phi} \mathcal{W}_\pm \left(\dot\gamma |\Phi|^{-\alpha} \right),$$

Characteristic frequency

$$\omega \equiv \frac{\dot\gamma S}{nT}$$

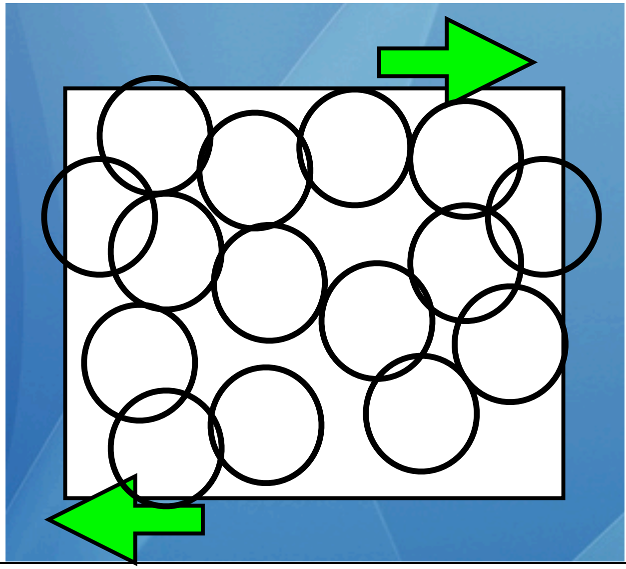
n : number density

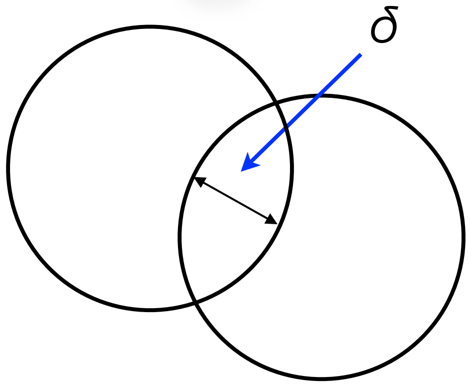
ω characterizes the dissipation of the energy

$$\frac{Dn}{2} \frac{d}{dt} T = \dot\gamma S - n \omega T$$

D : dimension

Model (frictionless grains)





Δ=1(Linear model)

Δ=3/2(Hertz model)

Interaction Force：$F=k\delta^{\Delta}$

Compressed Length：δ

The exponent for the interaction：Δ

Dissipative force between the contacting particles

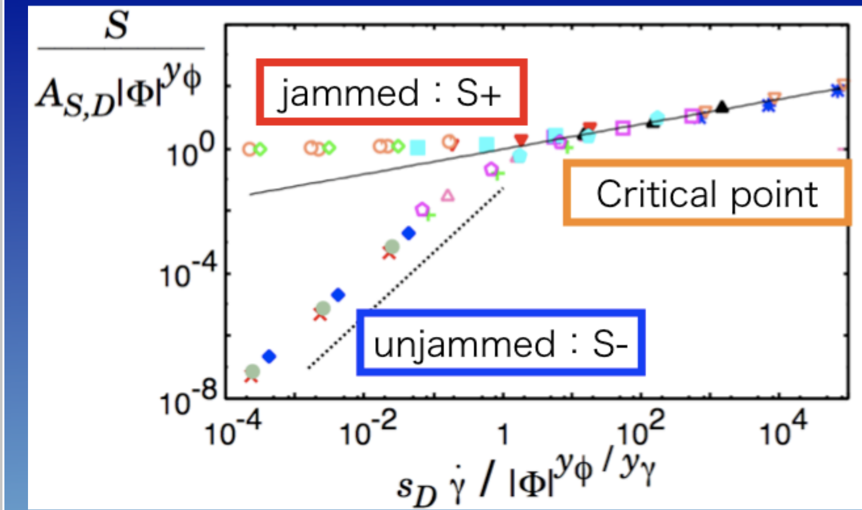
Scaling function

$$T = |\Phi|^{x_\Phi} \mathcal{T}_\pm \left(\dot{\gamma}|\Phi|^{-\alpha}\right),$$

$$S = |\Phi|^{y_\Phi} \mathcal{S}_\pm \left(\dot{\gamma}|\Phi|^{-\alpha}\right),$$

$$P = |\Phi|^{y'_\Phi} \mathcal{P}_\pm \left(\dot{\gamma}|\Phi|^{-\alpha}\right),$$

$$\omega = |\Phi|^{z_\Phi} \mathcal{W}_\pm \left(\dot{\gamma}|\Phi|^{-\alpha}\right),$$



Scaling properties of S, T, P, ω

$$T \sim \dot{\gamma}|\Phi|^{x_\phi - \alpha}, \quad S \sim |\Phi|^{y_\phi}, \quad P \sim |\Phi|^{y'_\phi}, \quad \omega \sim |\Phi|^{z_\phi},$$

jammed

$$\omega = \frac{\dot{\gamma} S}{nT}$$

1st eq.

$$\alpha = x_\phi - y_\phi + z_\phi.$$

Pressure & shear stress

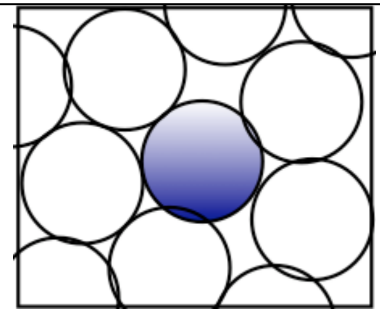
For $\dot{\gamma} \to 0$, $\Phi > 0$ (jammed phase)

$$P \propto F_c(\Phi)$$

average force : $F_c(\Phi) \to k\,\delta(\Phi)^{\Delta}$

compressed length : $\delta(\Phi) \to (\sigma / D\phi_J)\Phi$

$$P \sim \Phi^{\Delta} \qquad P \sim |\Phi|^{y'_\phi}$$



C. S. O'Hern, et al. (2003)

Assumption : S/P is independent of Φ

$$S \sim |\Phi|^{y_\phi} \quad P \sim |\Phi|^{y'_\phi}$$

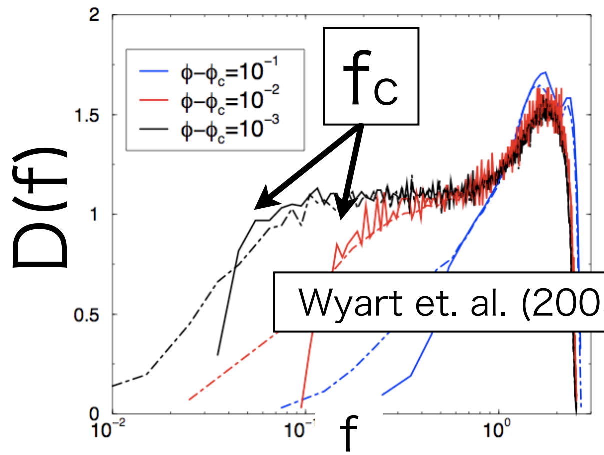
Coulomb friction

T. Hatano (2007)

2nd eq. $y'_\phi = \Delta$

3rd eq. $y_\Phi = y'_\Phi$

Characteristic frequency



For $\dot{\gamma}\to 0$, $\Phi > 0$ (jammed phase)

D(f) : Density of state

f : frequency, fc : cut-off frequency

$$f_c \sim \sqrt{P}\,,\ \omega \sim |\Phi|^{z_\phi}$$

Assumption : ω is scaled by fc.

For $\dot{\gamma}\to 0$, $\Phi < 0$ (unjammed phase)

vc : characteristic velocity

ℓ (Φ) : mean free path

$v_c^2 \sim T/m$

$\ell(\Phi) \to (\sigma/D\phi_J)|\Phi|$

$$\omega \propto v_c/l(\Phi) \propto \sqrt{T}/|\Phi|$$

4th eq.

$$z_\Phi = y'_\Phi/2$$

final eq.

$$x_\phi = 2z_\phi + 2$$

Critical exponents

$$\alpha = x_\phi - y_\phi + z_\phi.$$

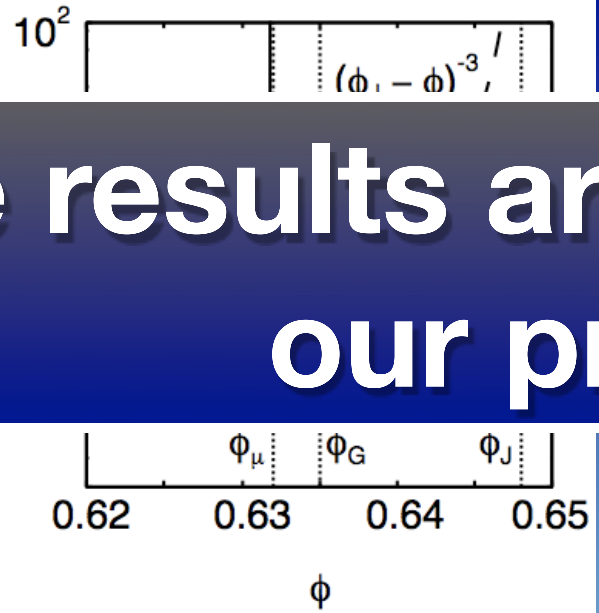
$$y'_\phi = \Delta$$

$$z_\Phi = y'_\Phi/2 \qquad y_\Phi = y'_\Phi \qquad x_\phi = 2z_\phi + 2$$

$$x_\Phi = 2+\Delta, \quad y_\Phi = \Delta, \quad y'_\Phi = \Delta, \quad z_\Phi = \frac{\Delta}{2}, \quad \alpha = \frac{\Delta+4}{2}$$

The exponents depend on Δ.

Previous works



Our theory :

$$\mu/\sqrt{T} \sim (\phi_J - \phi)^{-3}$$

The results are consistent with our prediction.

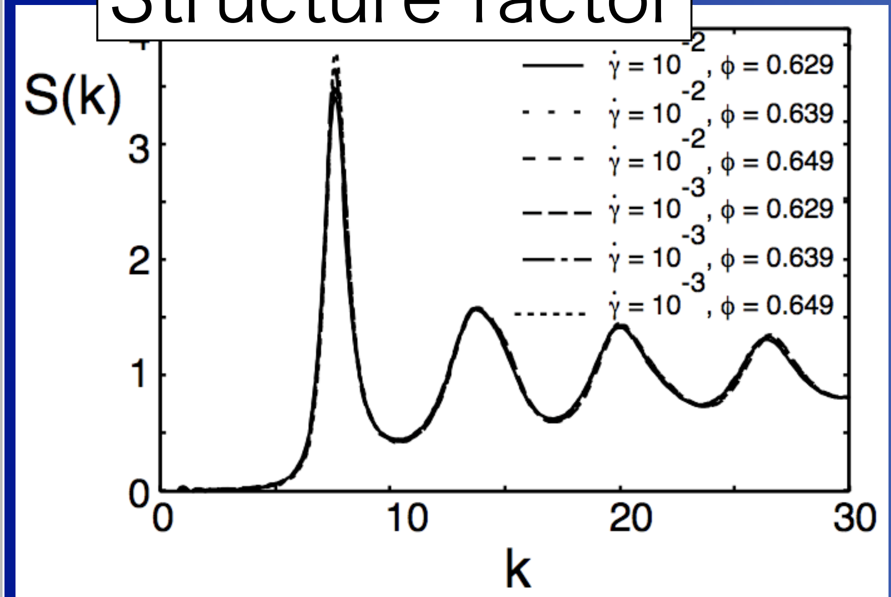
R. Garcia-Rojo, et al. , PRE (2006)
3-dimensional elastic particles

Singular behavior around $\Phi = \Phi_G < \Phi_J$

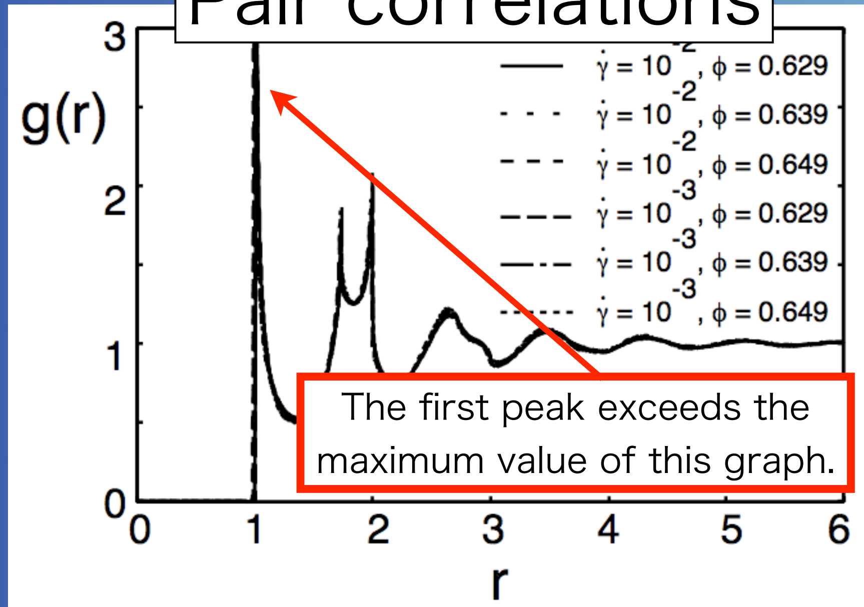
L. Berthier and T. A. Witten, EPL (2009)

Pair correlation functions

Structure factor



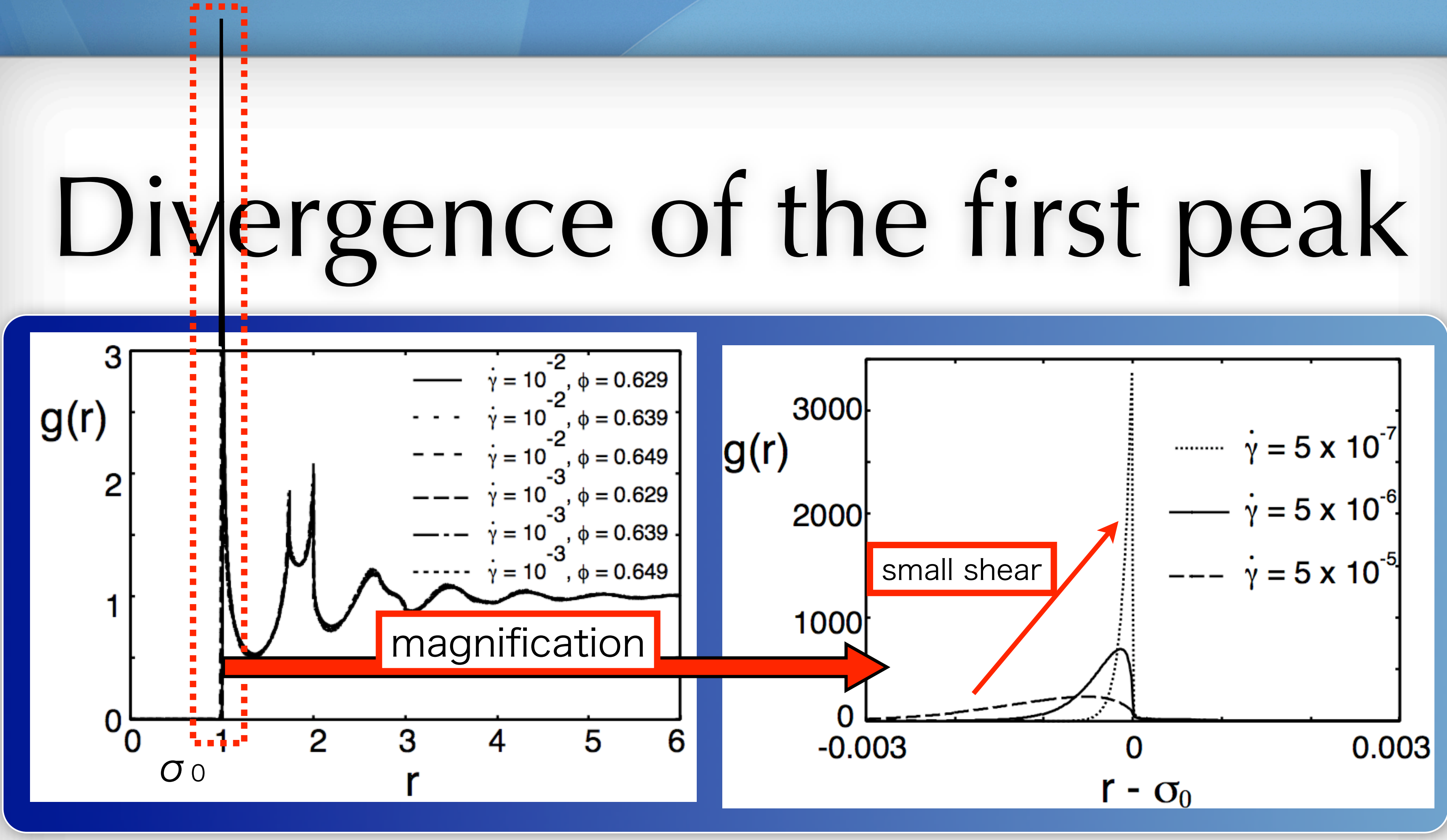
Pair correlations



D=3, mono-disperse, $\Delta=1$

- S(k) does not show any critical behaviors.
- The first peak of g(r) changes drastically near Φ_J.

Divergence of the first peak



- The first peak diverges as the shear rate gets smaller.

Scaling of the first peak

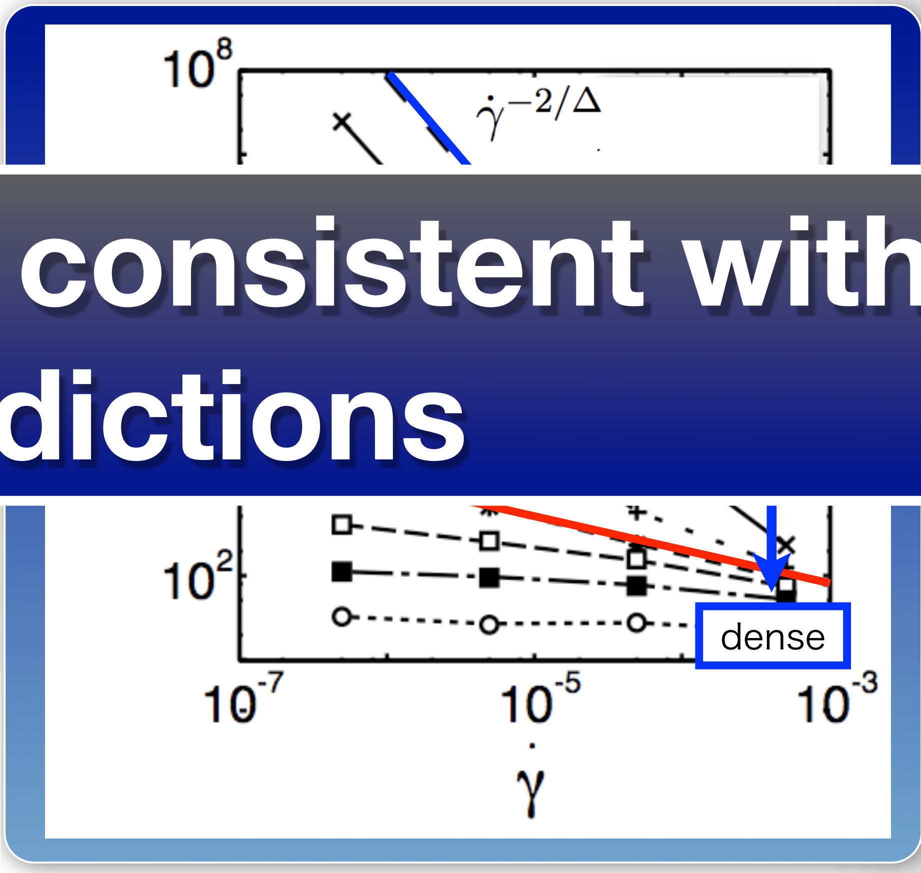
coordination number : Z
Pressure : P

$$Z \sim \int_0^{\sigma_0} dr r^{D-1} g(r),$$

The results are consistent with our predictions

$$g_0 \sim P^{-1/\Delta}$$

$$g_0 \sim \begin{cases} \dot{\gamma}^{-2/\Delta}|\Phi|^{4/\Delta} & \text{for } \phi < \phi_J \\ \dot{\gamma}^{-2/(\Delta+4)} & \text{for } \phi \sim \phi_J \\ |\Phi|^{-1} & \text{for } \phi > \phi_J \end{cases}$$

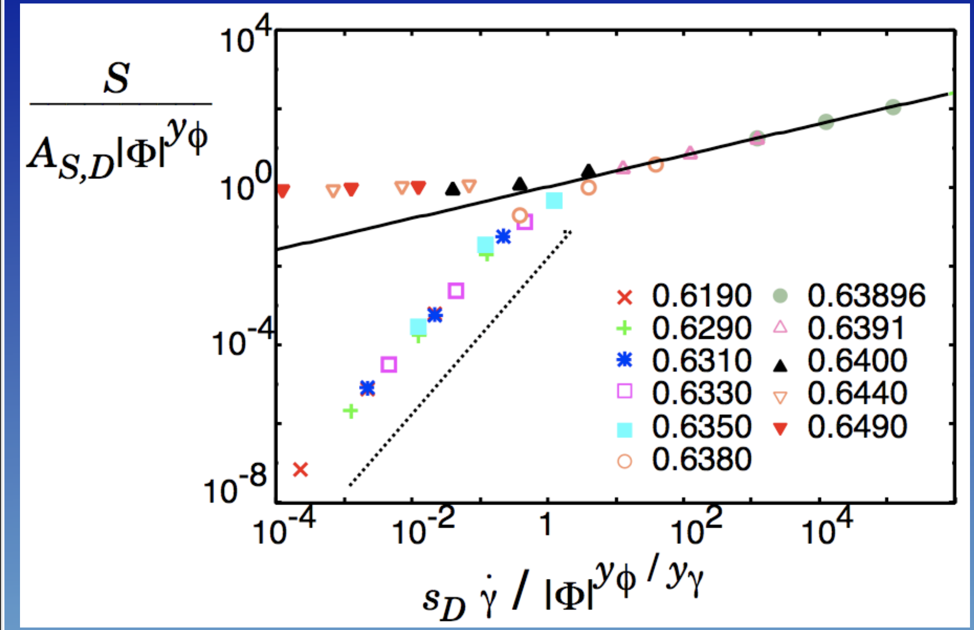


D=3, mono-disperse, Δ=1

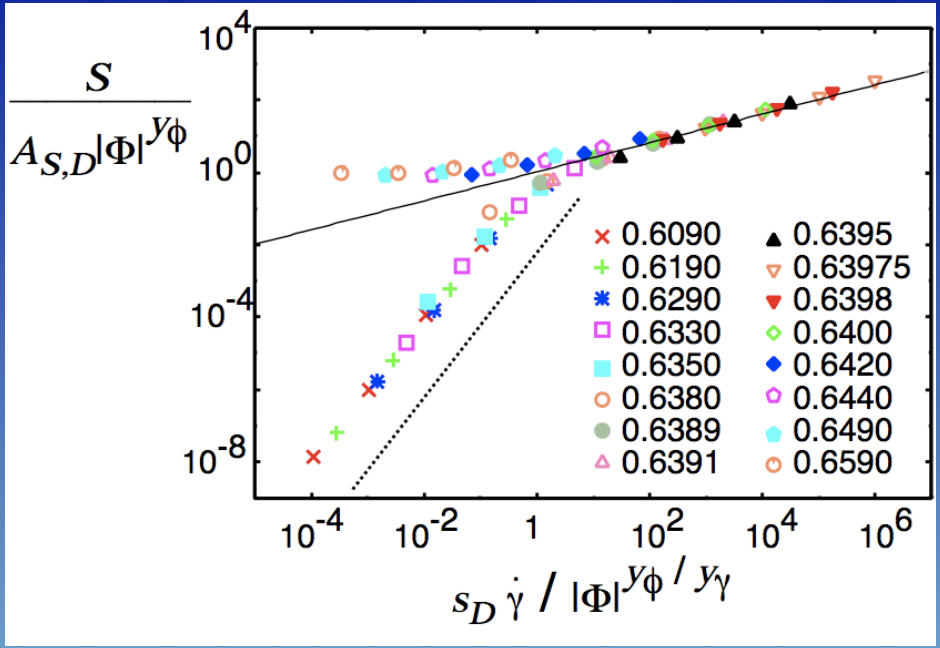
System size

D=3, mono-disperse, Δ=1

N=2000



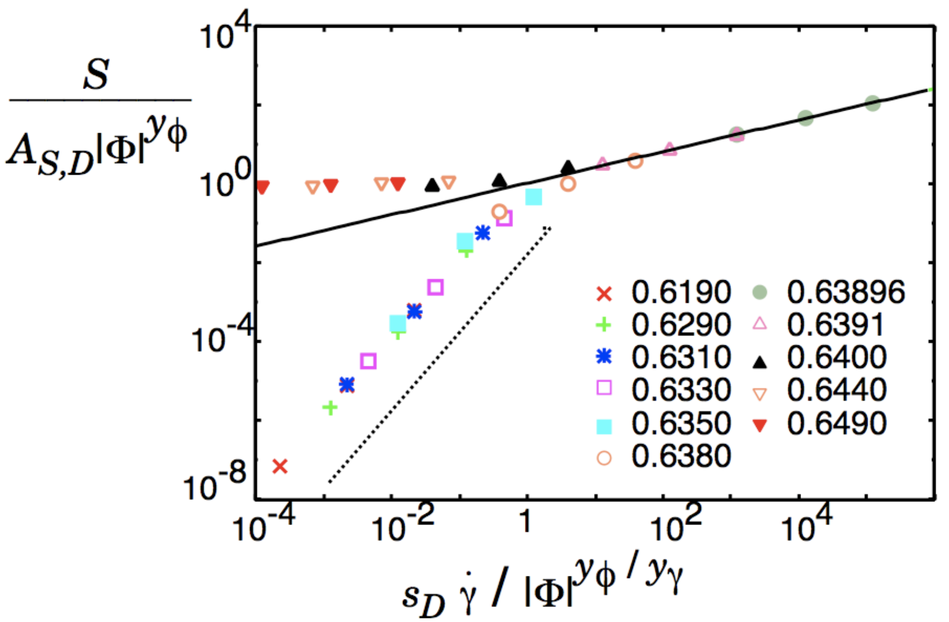
N=20000



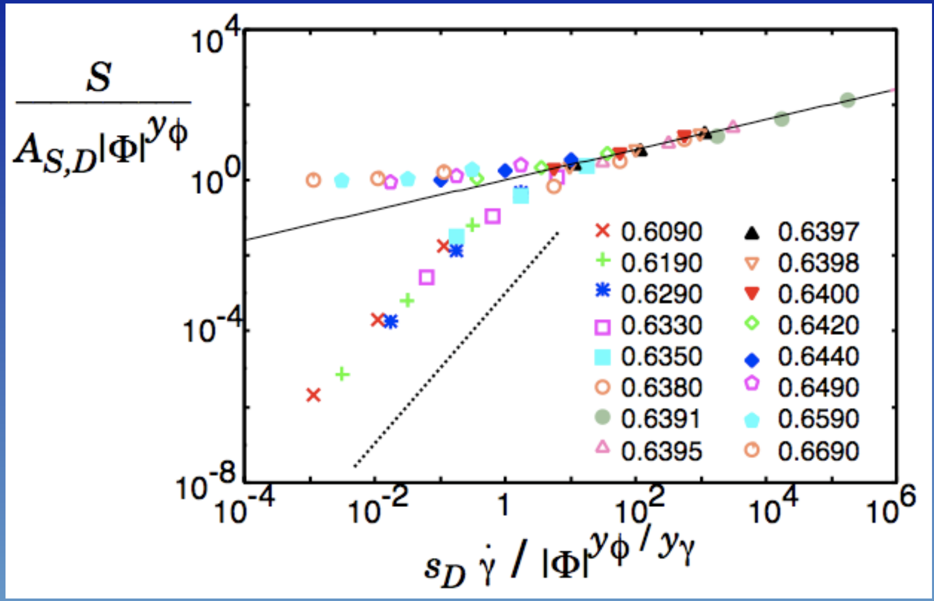
Inelasticity

D=3, mono-disperse, Δ=1

e=0.0035



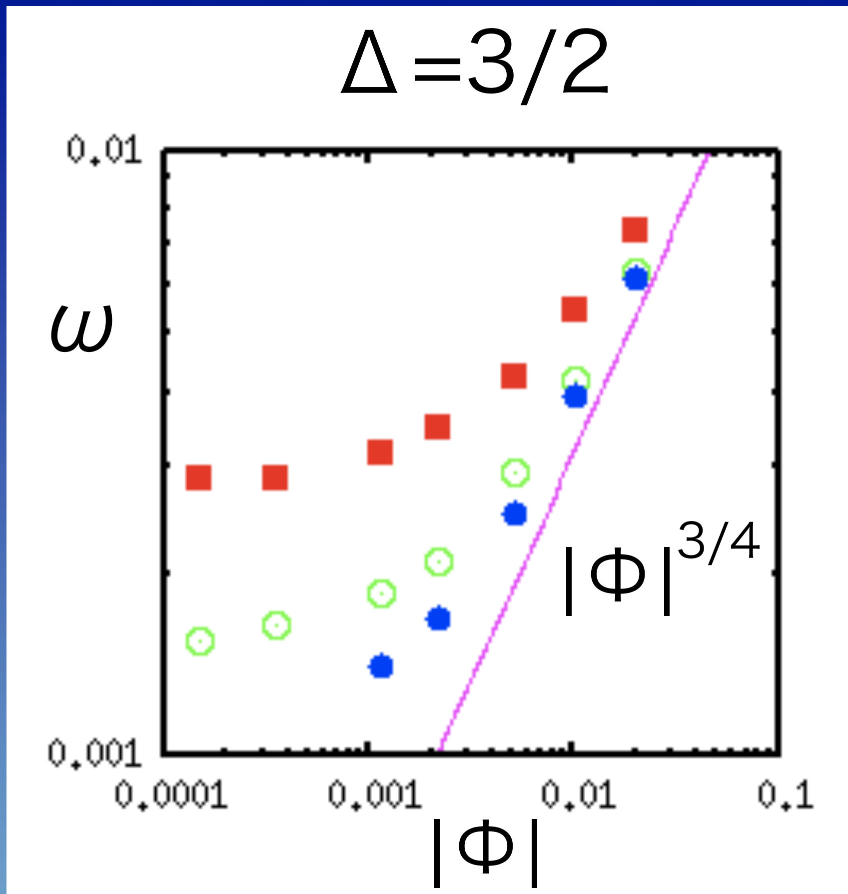
e=0.95



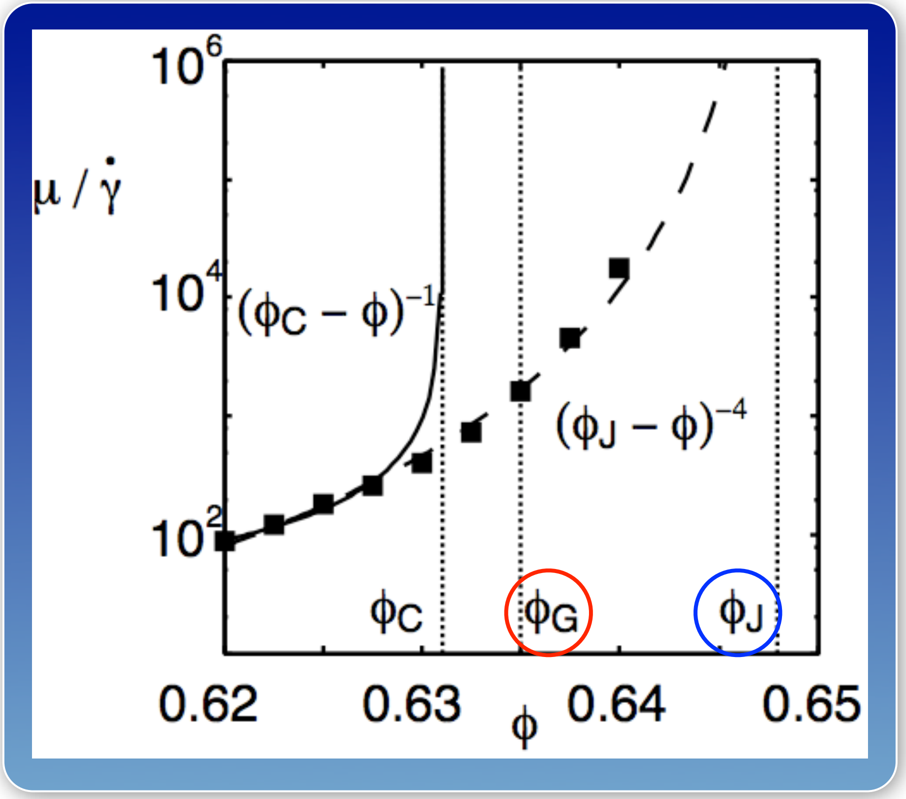
Φ-dependence

Jammed phase

$$\omega \sim |\Phi|^{\Delta/2}$$



Point G ?



Berthier and Witten (2008)

Equilibrium simulation

Point G ?

ϕ_G=0.635, ϕ_J=0.642

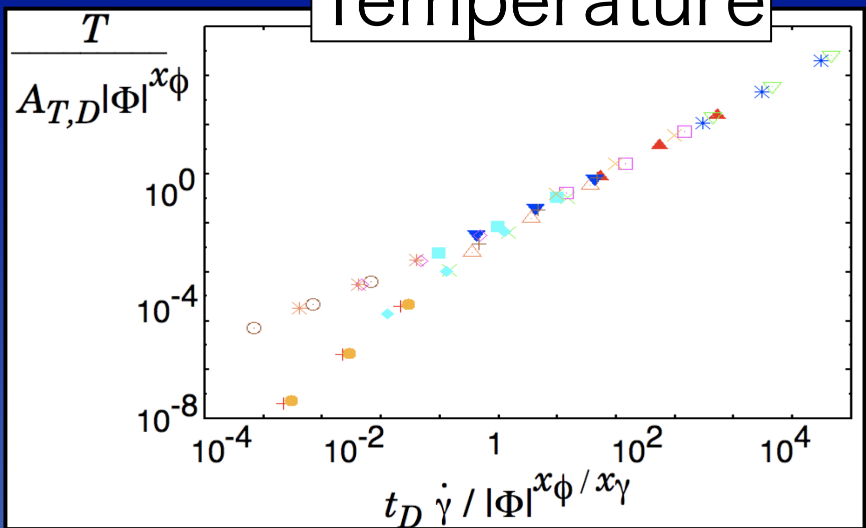
There is no singularity other than point J.

Simulation(Δ=3/2)

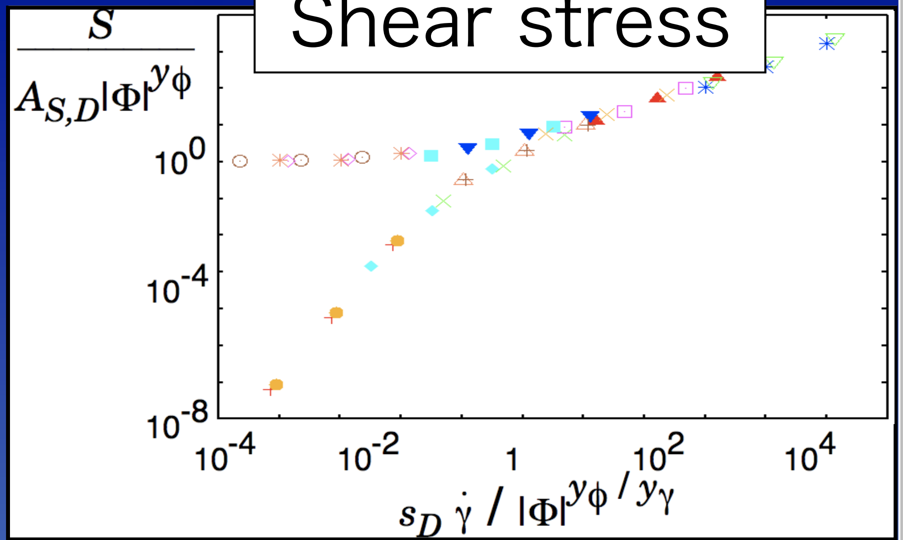
Dimension：D=2, 3, 4, Interaction：F=k δ^Δ

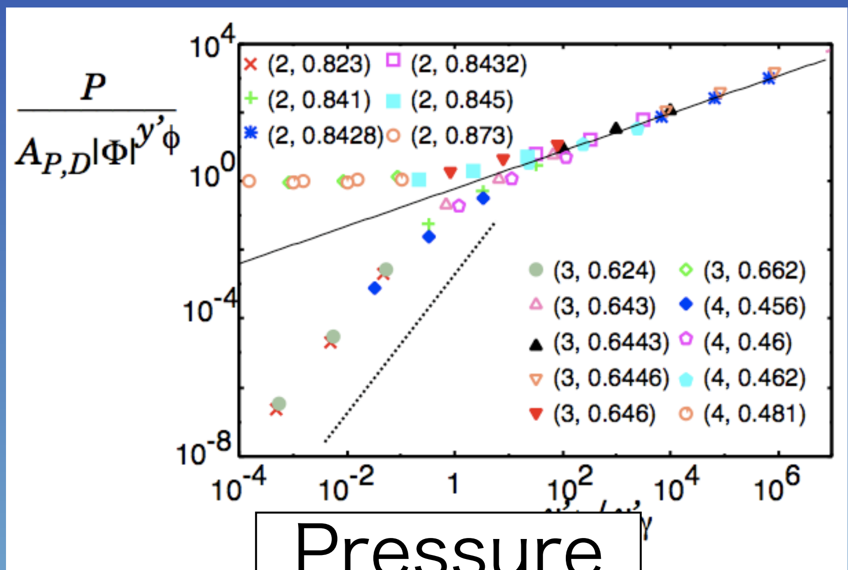
Particle's size：σ, 0.9σ, 0.8σ, 0.7σ

Temperature

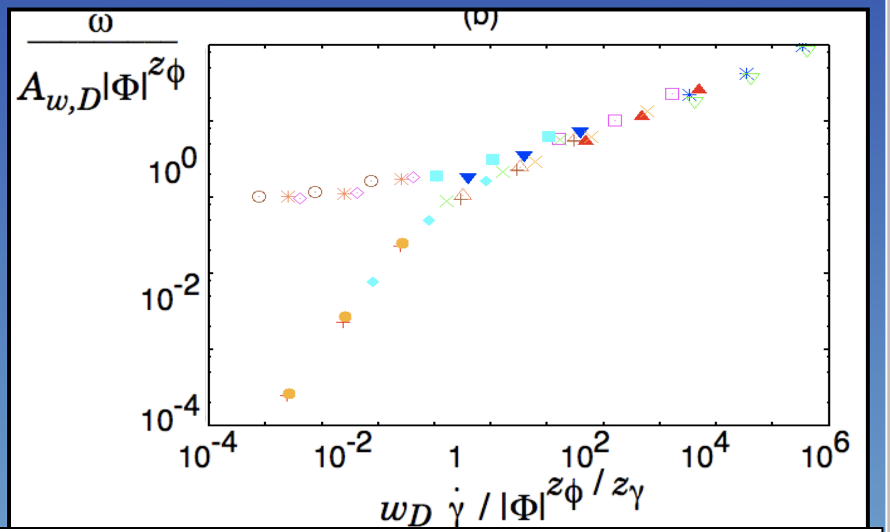


Shear stress





Pressure



Characteristic frequency

Theory for g(r)

$$g(\boldsymbol{r}) = \frac{V}{N^2}\left\langle \sum_i \sum_{j\neq i} \delta(\boldsymbol{r} - \boldsymbol{r}_{ij}) \right\rangle,$$

$$\begin{aligned} \bar{g}(r) &= \int \frac{d\Omega}{S_D} g(\boldsymbol{r}) \\ &= \frac{1}{S_D r^{D-1} n} \left\langle \frac{1}{N} \sum_i \sum_{j\neq i} \delta(r - r_{ij}) \right\rangle, \end{aligned}$$

$$\begin{aligned} Z &= \frac{1}{N}\int_0^{\sigma_0} dr \left\langle \frac{1}{2} \sum_i \sum_{j\neq i} \delta(r - r_{ij}) \right\rangle \\ &= \frac{S_D n}{2} \int_0^{\sigma_0} dr r^{D-1} \bar{g}(r) \end{aligned}$$

$$\begin{aligned} Z &\simeq \frac{S_D n}{2} \int_{\sigma_0 - h_0}^{\sigma_0} r^{D-1} g_0 \\ &= \frac{S_D n \sigma_0^{D-1}}{2} g_0 h_0 \{1 + O(h_0)\}, \end{aligned}$$

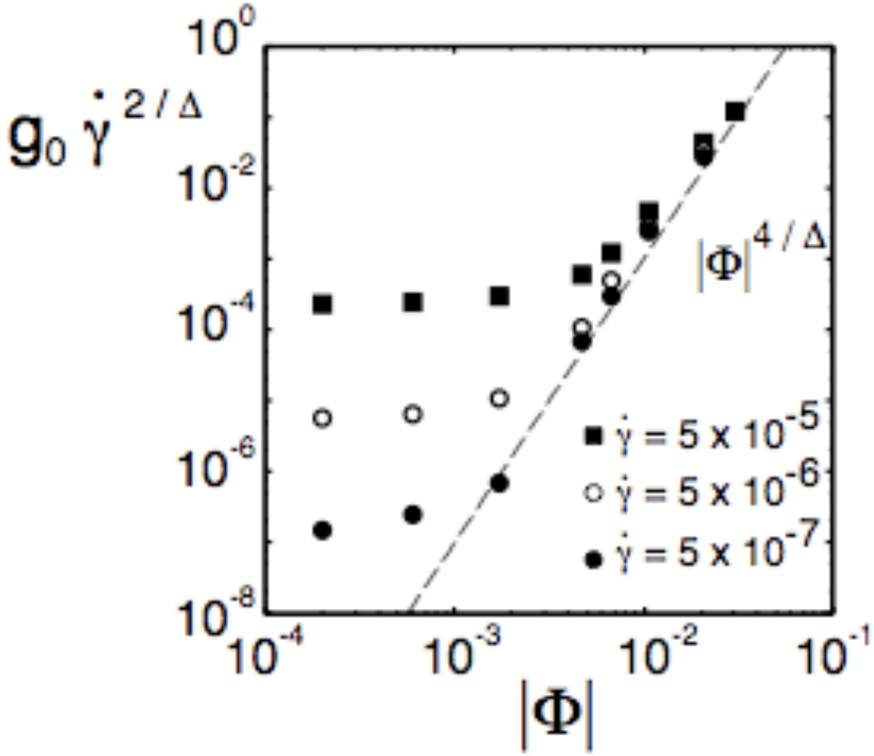
$$\begin{aligned} P &\simeq \frac{1}{2DV} \left\langle \sum_i \sum_{j\neq i} r_{ij} f_{\rm el}(r_{ij}) \Theta(\sigma_0 - r_{ij}) \right\rangle \\ &= \frac{1}{2DV} \int_0^{\infty} dr r f_{\rm el}(r) \Theta(\sigma_0 - r) \left\langle \sum_i \sum_{j\neq i} r_{ij} \delta(r - r_{ij}) \right\rangle \\ &= \frac{S_D n^2}{2} \int_0^{\sigma_0} dr r^D f_{\rm el}(r) \bar{g}(r), \end{aligned}$$

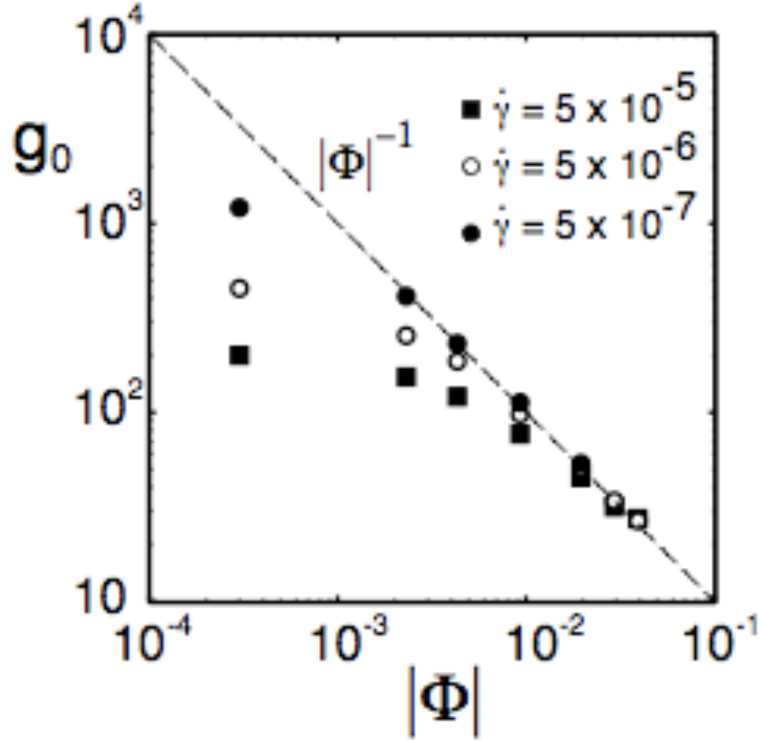
$$\begin{aligned} P &\simeq \frac{S_D n^2}{2} \int_{\sigma_0 - h_0}^{\sigma_0} dr r^D k(\sigma_0 - r)^{\Delta} g_0, \\ &= \frac{S_D n^2 k \sigma_0^D}{2} h_0^{\Delta+1} g_0 \{1 + O(h_0)\}, \end{aligned}$$

$g_0 h_0 \sim const.$

$$g_0 \sim P^{-1/\Delta}$$

g0 vs Φ





$$g_0 \sim \begin{cases} \dot{\gamma}^{-2/\Delta}|\Phi|^{4/\Delta} & \text{for } \phi < \phi_J \\ \dot{\gamma}^{-2/(\Delta+4)} & \text{for } \phi \sim \phi_J \\ |\Phi|^{-1} & \text{for } \phi > \phi_J \end{cases}$$

general force

$$F(r) = k(r - \sigma_0)^{\Delta}$$

$$x_\Phi = 2 + \Delta, \quad y_\Phi = \Delta, \quad y'_\Phi = \Delta, \quad z_\Phi = \frac{\Delta}{2}, \quad \alpha = \frac{\Delta + 4}{2}$$

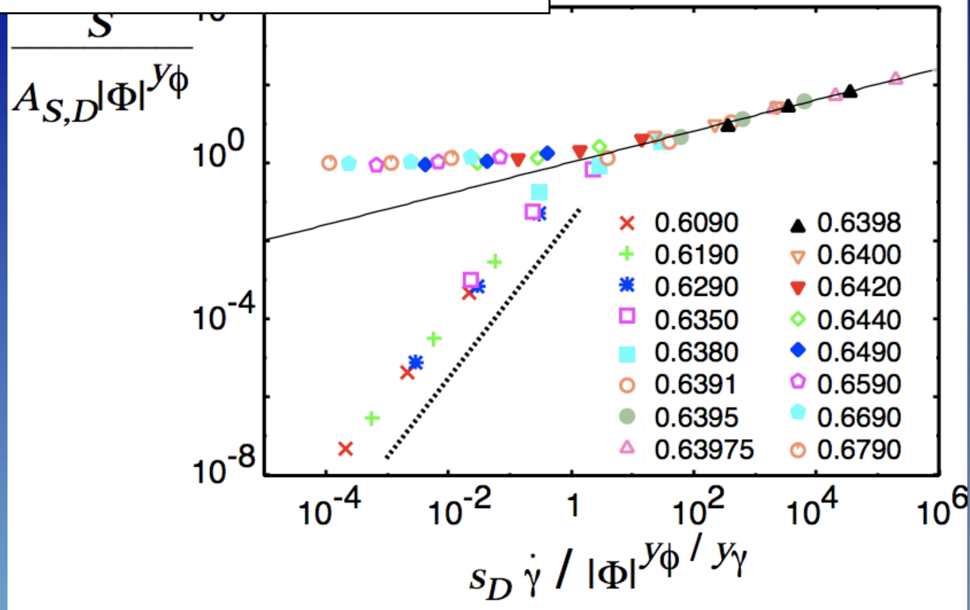
general case : $\lim_{r \to \sigma_0} F(r) \sim (r - \sigma_0)^{\Delta}$

D=3, repulsive Lennard-Jones

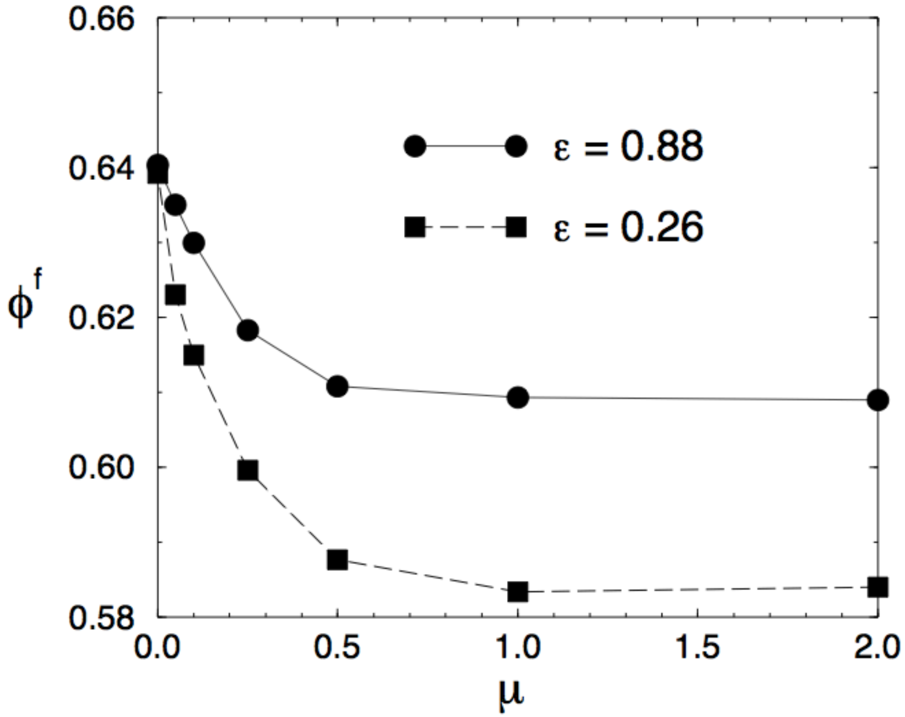
$$F(r) = \epsilon \left\{ \left(\frac{\sigma_0}{r}\right)^{13} - \left(\frac{\sigma_0}{r}\right)^{7} \right\} \quad \text{for } r < \sigma_0$$

$\Delta = 1$

The exponents are estimated with $\Delta =$.

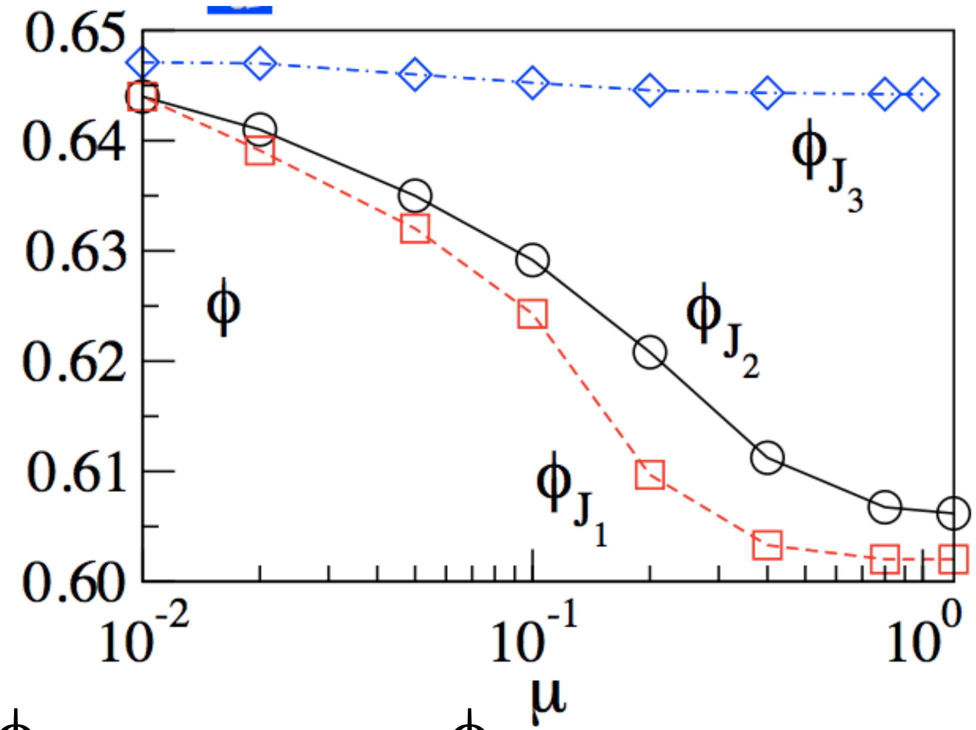


Φ_J for static granular packing



Silbert, et al. (2001)

Stress control simulation



Ciamarra, et al. (2010)

ϕ_{J1} is related with ϕ_J

ϕ_{J2} is related with ϕ_L

Discussion : previous works