Dark Solitons in the 1D Bose gas

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Lieb-Liniger model
Lieb-Liniger model

\[ H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1\leq j<k\leq N} \delta(x_j - x_k) \]

1D Bose gas with delta function potential (PBC)

Toy model

Realized in experiments

- Realization of the Tonks–Girardeau gas

- Observation of Out-of-equilibrium dynamics
Lieb-Liniger model

\[ H = - \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \leq j < k \leq N} \delta(x_j - x_k) \]

- Second quantization
  \[ L: \text{System size} \]

\[ [\Psi(x), \Psi^{\dagger}(y)] = \delta(x - y), \quad [\Psi(x), \Psi(y)] = [\Psi^{\dagger}(x), \Psi^{\dagger}(y)] = 0 \]
\[ \Psi(x)|0\rangle = 0, \quad \langle 0|\Psi^{\dagger}(x) = 0, \quad \langle 0|0\rangle = 1 \]

\[ H = \int_{0}^{L} dx \left[ \partial_x \Psi^{\dagger}(x) \partial_x \Psi(x) + c \Psi^{\dagger}(x) \Psi^{\dagger}(x) \Psi(x) \Psi(x) \right] \]

Exactly solved by Bethe ansatz
Thermodynamics is characterized by a single parameter $\gamma$.

- **Parameters**
  - $L$: System size
  - $N$: Number of particle
  - $c$: coupling constant
  - $n = N / L$: density
  - $\gamma = c / n$

The Hamiltonian for the Lieb-Liniger model is given by:

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{1 \leq j < k \leq N} \delta(x_j - x_k)$$
Lieb-Linger model
and
Bethe ansatz
**Bethe ansatz**

Quasi momentum: \( k_1, k_2, \cdots, k_N \)

**Bethe ansatz equation**

\[
e^{ik_j L} = \prod_{\ell \neq j}^{N} \frac{k_j - k_\ell + ic}{k_j - k_\ell - ic}, \quad j = 1, 2, \cdots, N
\]

**Bethe wave function**

\[
\varphi(x_1, \cdots, x_N) = \sum_{\sigma \in S_N} A_\sigma \exp \left[ i \sum_{j=1}^{N} k_{\sigma_j} x_j \right]
\]

\[
A_\sigma = (-1)^\sigma \prod_{j > \ell}^{N} \left[ k_{\sigma_j} - k_{\sigma_\ell} - ic \text{sign}(x_j - x_\ell) \right]
\]

If \( k_j = k_\ell \) then \( \varphi = 0 \)
Quasi momentum: \(k_1, k_2, \cdots, k_N\)

**Bethe ansatz equation**

\[
e^{i k_j L} = \prod_{\ell \neq j}^{N} \frac{k_j - k_\ell + ic}{k_j - k_\ell - ic}, \quad j = 1, 2, \cdots, N
\]

**Take logarithm**

\[
k_j L + \sum_{\ell=1}^{N} 2 \tan^{-1} \frac{k_j - k_\ell}{c} = 2\pi I_j, \quad j = 1, 2, \cdots, N
\]

\(I_j\) : Bethe quantum number

Easily solved numerically
Bethe ansatz

- Specify Bethe quantum number
- Solve Bethe ansatz equation
- Energy and momentum eigenvalues

- \[ \{ I_j \} = \{-2, -1, 1, 2, 3\} \]

\[ k_j L + \sum_{\ell=1}^{N} 2 \tan^{-1} \frac{k_j - k_\ell}{c} = 2\pi I_j \]

\[ k_j = \frac{2\pi}{L} I_j \]

\[ E = \sum_{j=1}^{N} k_j^2, \quad P = \sum_{j=1}^{N} k_j \]

One-to-one correspondence between \{I_j\} and \{k_j\}
Particle excitation (type I)

ex) $N = 5, \gamma = 1$

Hole excitation (type II)

Ground state

$(-2,-1,0,1,2)$

Ground State

Hole excitation (type II)
Determinant formula

Vector $|k\rangle$ Labeled by $k = (k_1, k_2, \cdots, k_N)$

Norm

$$\langle k|k \rangle = c^N \left( \prod_{j \geq \ell} \frac{k_{j \ell}^2 + c^2}{k_{j \ell}^2} \right) \det_N G(k)$$

Gaudin Matrix:

$$G(k)_{j \ell} = \delta_{j \ell} \left[ L + \sum_{\ell=1}^{N} K(k_{j \ell}) \right] - K(k_{j \ell})$$

$$K(k) = \frac{2c}{k^2 + c^2}$$

Form factor

\[ \langle k' | \hat{\psi}^\dagger \hat{\psi} | k \rangle = i^N (P_k - P_k') \left( \prod_{j, \ell=1}^{N} \frac{k_j - k_\ell + ic}{k'_j - k_\ell} \right) \det_{N-1} U \]

\[ \langle k' | \hat{\psi} | k \rangle = -\frac{i^N}{\sqrt{c}} \frac{\prod_{j, \ell=1}^{N} (k_j - k_\ell + ic)}{\prod_{j=1}^{N-1} \prod_{\ell=1}^{N} (k'_j - k_\ell)} \det_{N-1} U \]

\[ P_k = \sum_{j=1}^{N} k_j \]

\[ U_{j\ell} = \delta_{j\ell} \frac{V_j^+ - V_j^-}{i} + \frac{\prod_{a} (k'_a - k_j)}{\prod_{a \neq j} (k_a - k_j)} (K(k_j - k_\ell) - K(k_N - k_\ell)) \]

\[ V_j^\pm = \frac{\prod_{a} (k'_a - k_j \pm ic)}{\prod_{a=1}^{N} (k_a - k_j \pm ic)} \]
Nonlinear Schrödinger equation and Soliton
Nonlinear Schrödinger eq. and Soliton

2nd quantized Hamiltonian

\[ \mathcal{H} = \int_0^L dx \left[ \partial_x \hat{\psi}^\dagger \partial_x \hat{\psi} + c \hat{\psi}^\dagger \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \right] \]

Eq. of motion

\[ i \partial_t \hat{\psi} = [\hat{\psi}, \mathcal{H}] \]

\[ i \partial_t \hat{\psi} = -\partial_x \hat{\psi} + 2c \hat{\psi}^\dagger \hat{\psi} \hat{\psi} \]

Field operator \( \rightarrow \) \( c \)-number: \( \hat{\psi}(x, t) \rightarrow \phi(x, t) \)

\[ i \partial_t \phi = -\partial_x^2 \phi + 2c |\phi|^2 \phi \]

Inverse scattering method \( \Rightarrow \) Soliton solution

Exactly diagonalized by Quantum inverse scattering method
= Algebraic Bethe ansatz

L.A. Takhtajan, L.D. Faddeev (1979)

V.E. Zakharov, A.B. Shabat (1972)
Soliton solution

Nonlinear Schrödinger equation: \( i\partial_t \phi = -\partial_x^2 \phi + 2c|\phi|^2 \phi \)

\( c < 0 \): Bright Soliton

\[
\phi(x, t) = 2|c|^{-1/2} \eta \exp\left[4i\eta^2 t - iv^2 t/4 + ivx/2 + i\theta_0\right] \times \text{sech}\left[2\eta(x - x_0 - vt)\right]
\]

\( c > 0 \): Dark Soliton

\[
\phi(x, t) = n^{1/2} \left\{ 1 - \beta \text{sech}^2 \left[ (\beta nc)^{1/2} (x - vt) \right] \right\}^{1/2} \times \exp \left\{ \pm i \sin^{-1} \left( \frac{\beta^{1/2} \tanh \left[ (\beta nc)^{1/2} (x - vt) \right]}{\left\{ 1 - \beta \text{sech}^2 \left[ (\beta nc)^{1/2} (x - vt) \right] \right\}^{1/2}} \right) \right\}
\]

V.E. Zakharov, A.B. Shabat (1972)

Tsuzuki (1970)
QFT and Soliton

\[ c < 0 : \text{Bright Soliton} \]

\[ |N, P\rangle: N\text{-particle bound state with total momentum } P \]

Localized wave packet:

\[
|N, X, t\rangle = \int \frac{dP}{2\pi} e^{-iPX} e^{i(P^2/N)t} |N, P, t\rangle
\]

\[
\lim_{N \to \infty} \langle N, X, t|\hat{\psi}(x)|N + 1, X, t\rangle = \phi(x, t)|_{v=0, x_0=X}
\]

C.R. Nohl, Ann. Phys. 96, 234 (1976),

\[ c > 0 : \text{Dark Soliton} \]

It is shown that type II excitation and dark soliton have the same dispersion relation.

Construction of the Quantum Localized State and Classical-Quantum Correspondence
Construction of the Quantum Localized State

\[ |X\rangle = \frac{1}{\sqrt{N}} \sum_{p=0}^{N-1} \exp(-2\pi i p q / N) |P\rangle \]

\( p, q \in \{0, 1, 2, \cdots, N - 1\} \)

|P\rangle: Hole excitation with total momentum \( P = 2\pi p / L \)

|X\rangle: Wave packet localized at \( X = qL / N + L / 2 \)

Density operator: \( \rho(x) = \hat{\psi}^\dagger(x) \hat{\psi}(x) \)

\[ \langle X | \rho(x) | X \rangle \]

\( N = L = 1000 \)

\( c = 100 \)
Type II Excitation & Dark Soliton: Amplitude

Density operator: $\rho(x) = \hat{\psi}^\dagger(x)\hat{\psi}(x)$

$\langle X | \rho(x) | X \rangle$

$|\phi(x)|^2 = n \left( 1 - \beta \text{sech}^2 \left[ (\beta nc)^{1/2} x \right] \right)$

$\beta = 1 - \left( \frac{v}{V_c} \right)^2 = 3/4$

$n = 1$

Density profile of $|X\rangle \Leftrightarrow$ Squared amplitude of dark soliton
Type II Excitation & Dark Soliton: Phase

Dark soliton is described by Elliptic function under PBC

\[ \text{Arg} \left[ \phi(x) \right] = vx/2 - \frac{1}{K} \sqrt{\frac{g_{sn} h_{sn}}{2f_{sn}}} \Pi \left( -\frac{2mK^2}{f_{sn}} ; \frac{2Kx}{L} \right|m \right) \]

\[ f_{sn} = \frac{cL^2}{4\pi} - 2K^2 + 2KE \]
\[ g_{sn} = f_{sn} + 2K^2 \]
\[ h_{sn} = f_{sn} + 2mK^2 \]

\( K, E, \Pi \): complete elliptic integral


\[ \text{Arg} \left[ \left\langle N - 1, X \mid \hat{\psi}(x) \mid N, X \right\rangle \right] / \pi \]

\[ \text{Arg} \left[ \phi(x) \right] / \pi \]

![Graphs showing Many-body and Mean-field comparisons for N=20 and N=500 with L=20 and L=500, respectively.](image)
Dynamics of Quantum Wave Packet
Time evolution of the quantum wave packet

\[ |X\rangle = \frac{1}{\sqrt{N}} \sum_{p=0}^{N-1} \exp(-2\pi ipq/N) |P\rangle \]

\[ |X(t)\rangle = \exp(-i\mathcal{H}t)|X\rangle = \frac{1}{\sqrt{N}} \sum_{p=0}^{N-1} \exp(-2\pi ipq/N) \exp(-iE_p t) |P\rangle \]

\(E_p\) is exactly obtained by Bethe ansatz

Time evolution is exactly calculated
Loschmidt echo

\[ |\langle X(t) | X(0) \rangle| \]

- \( N=1000, \ c=100 \)
- Analytic (Fresnel integral)
- \( N=20, \ c=100 \)

\[ c \to \infty, \ N \to \infty \]

\[ |\langle X(t) | X(0) \rangle| \quad \sim \quad t^{-1/2} \]

\[ \frac{1}{n\sqrt{2\pi t}} \sqrt{C(n\sqrt{2\pi t})^2 + S(n\sqrt{2\pi t})^2} \]

Fresnel integral:
\[ C(x) = \int_0^x \cos \left( \frac{\pi}{2} s^2 \right) \, ds, \quad S(x) = \int_0^x \sin \left( \frac{\pi}{2} s^2 \right) \, ds \]
Time evolution of the density operator

\[ \rho(x) = \hat{\psi}^\dagger(x) \hat{\psi}(x) \]

\[ \langle X(t) | \rho(x) | X(t) \rangle = \sum_{p,p'=0}^{N-1} e^{2\pi i (p-p')q/N} \times e^{i(P-P')x-i(E_p-E_{p'})t} \langle P' | \rho(0) | P \rangle \]

Determinant Formula

\[ \langle P' | \rho(0) | P \rangle = i^N (P - P') \left( \prod_{j,\ell=1}^{N} \frac{k_j - k_\ell + ic}{k'_\ell - k_\ell} \right) \det_{N-1} U(k, k') \]

\[ U(k, k')_{j,\ell} = 2\delta_{j\ell} \text{Im} \left[ \prod_{a=1}^{N} \frac{k'_a - k_j + ic}{k_a - k_j + ic} \right] \]

\[ + \prod_{a=1}^{N} \frac{(k'_a - k_j)}{\prod_{a\neq j}^{N} (k_a - k_j)} (K(k_j - k_\ell) - K(k_N - k_\ell)) \]

\[ K(k) = 2c/(k^2 + c^2) \]
Animation of the Dynamics
\[ N = 1000 \]

\[ c = 0.01 \]
$N = 1000$

$t = 0.0$

$c = 1$
\( N = 1000 \)

\( \rho(x,t) \)

\( c = 10 \)
\[ N = 1000 \]

\[ c = 100 \]
$N = 20, c = 100$

Short time Dynamics

\[ \rho(x,t) \]

\[ |\langle X(t) | X(0) \rangle| \]
$N = 20, \ c = 100$

Long time Dynamics

$|\langle X(t) | X(0) \rangle|$

Recurrence at $t \sim 33$?
$N = 20, \ c = 100$

Long time Dynamics

$| \langle X(t) | X(0) \rangle |$

Recurrence at $t \sim 33$?
Snap shots at $t = 0$ and $t \sim 33$

We observe recurrence phenomena
Kaminishi, et. al., arXive:1305.3412
1. We construct quantum wave packets by the discrete Fourier transform of the type II excitations.

2. Density profile of the quantum wave packet coincides with the squared amplitude of dark soliton.

3. Phase of the matrix element of the field operator between quantum wave packet coincides with that of dark soliton.

4. We can exactly calculate time evolution of the quantum wave packet.