Inhomogeneous phases in strong-interaction physics

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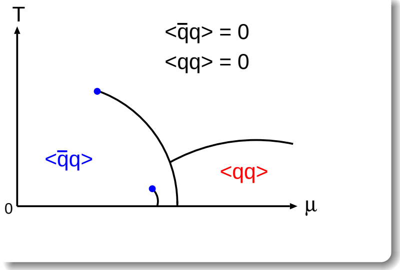
Michael Buballa

NFQCD, Kyoto, November 20, 2013

Motivation

QCD phase diagram (schematic):



- regions of interest:
  - hadronic phase
  - quark-gluon plasma
  - critical endpoint ?
  - color superconductors ?
  - nuclear matter liquid-gas transition

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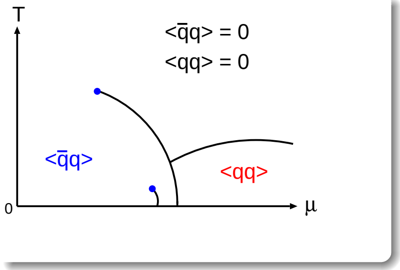

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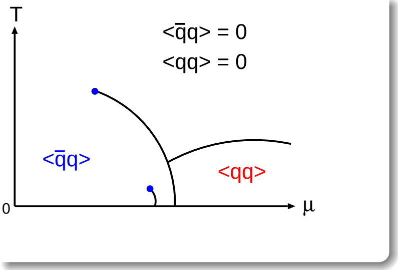

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- How about non-uniform phases ?

Inhomogeneous phases: (incomplete) historical overview

- 1960s:
  - spin-density waves in nuclear matter (Overhauser)
  - crystalline superconductors (Fulde, Ferrell, Larkin, Ovchinnikov)
- 1970s – 1990s:
  - p-wave pion condensation (Migdal)
  - chiral density wave (Dautry, Nyman)
  - Skyrme crystals (Goldhaber,Manton)
- after 2000:
  - 1+1 D Gross-Neveu model (Thies et al.)
  - crystalline color superconductors (Alford, Bowers, Rajagopal)
  - quarkyonic matter (Kojo, McLerran, Pisarski, ...)

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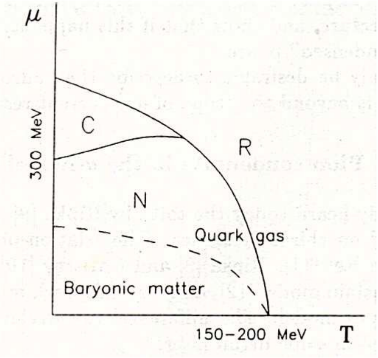

Broniowski et al. (1991)

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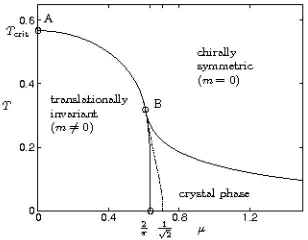

Thies, Urlichs (2003)

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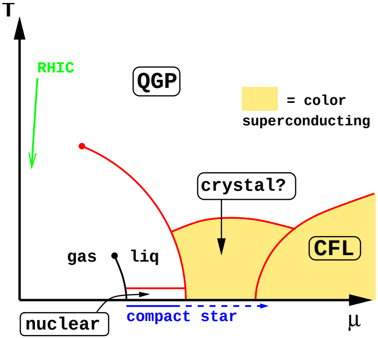

Alford (2003)

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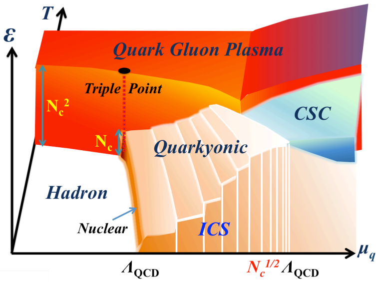

Kojo et al. (2011)

Outline

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1. Introduction
2. Inhomogeneous phases in NJL: formal setup
3. Applications:
   - 1-dimensional modulations
   - effect of vector interactions
   - 2-dimensional modulations
   - isospin-asymmetric matter
   - pion condensates
   - inhomogeneous color superconductivity
4. Chiral density waves with Dyson-Schwinger equations
5. Conclusions

Inhomogeneous phases in NJL

Model

- NJL model:

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + G_S\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2\right]$$

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$$\sigma(x) \to \langle\sigma(x)\rangle \equiv S(\vec{x}), \quad \pi_a(x) \to \langle\pi_a(x)\rangle \equiv P(\vec{x})\,\delta_{a3}$$

  - $S(\vec{x})$, $P(\vec{x})$ time independent classical fields
  - retain space dependence !

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- mean-field thermodynamic potential:

$$\Omega_{MF}(T,\mu) = -\frac{T}{V}\ln\int\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left(\int_{x\in[0,\frac{1}{T}]\times V}(\mathcal{L}_{MF} + \mu\bar\psi\gamma^0\psi)\right)$$

Mean-field model

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$$\mathcal{L}_{MF} = \bar{\psi}(x)\,\mathcal{S}^{-1}(x)\,\psi(x) - G_S\left[S^2(\vec{x}) + P^2(\vec{x})\right]$$

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  - $\mathcal{H}_{MF}$ hermitean $\Rightarrow$ can (in principle) be diagonalized (eigenvalues E_λ)
  - $\mathcal{H}_{MF}$ time-independent $\Rightarrow$ Matsubara sum as usual

Mean-field thermodynamic potential

- thermodynamic potential:

$$\Omega_{MF}(T,\mu;S,P) = -\frac{T}{V}\mathbf{Tr}\ln\left(\frac{1}{T}(i\partial_0 - \mathcal{H}_{MF} + \mu)\right) + \frac{G_S}{V}\int_V d^3x\,\left(S^2(\vec{x}) + P^2(\vec{x})\right)$$

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- general case: extremely difficult!

Periodic structures

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- crystal with a unit cell spanned by vectors $\vec{a}_i$, $i = 1, 2, 3$

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  - different momenta coupled by $M_{\vec{q}_k}$ $\Rightarrow$ $\mathcal{H}$ is nondiagonal in momentum space!
  - $\vec{q}_k$ discrete $\Rightarrow$ $\mathcal{H}$ is still block diagonal

Periodic structures: minimum free energy

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- general procedure:
  - choose a unit cell $\{\vec{a}_i\} \Rightarrow \{\vec{q}_k\}$
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$\rightarrow$ further simplifications necessary

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- 1 + 1D solutions known analytically: [M. Thies, J. Phys. A (2006)]

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- remaining task:
  - minimize w.r.t. 2 parameters: Δ, ν
  - (almost) as simple as CDW, but more powerful
  - $m \neq 0$: 3 parameters

Results

Phase diagram (chiral limit)

[D. Nickel, PRD (2009)]

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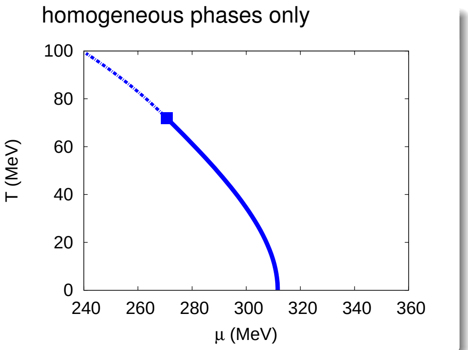

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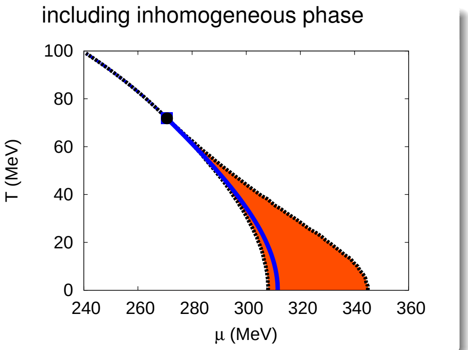

- 1st-order line completely covered by the inhomogeneous phase!
- all phase boundaries 2nd order (mean-field artifact?)
- critical point coincides with Lifshitz point

Mass functions and density profiles ($T = 0$)

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- $M(z) = \sqrt{\nu}\Delta\,\mathrm{sn}(\Delta z|\nu) \quad\rightarrow\quad \begin{cases} \Delta\tanh(\Delta z) & \text{for}\quad \nu\rightarrow 1 \\ \sqrt{\nu}\Delta\sin(\Delta z) & \text{for}\quad \nu\rightarrow 0 \end{cases}$

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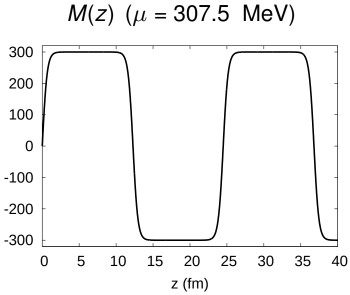

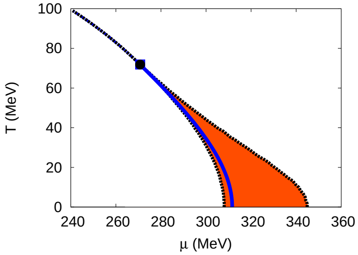

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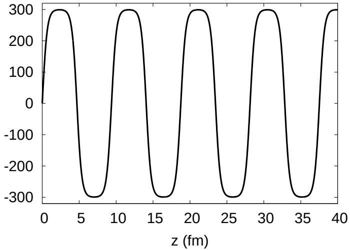
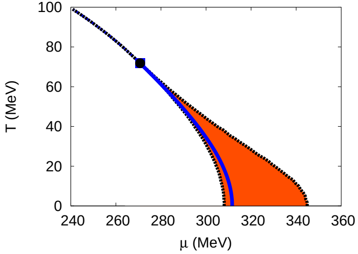
$M(z)$ (μ = 308 MeV)





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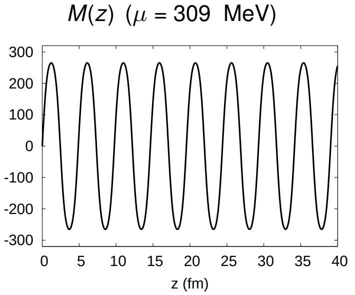

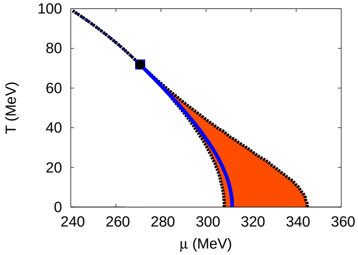

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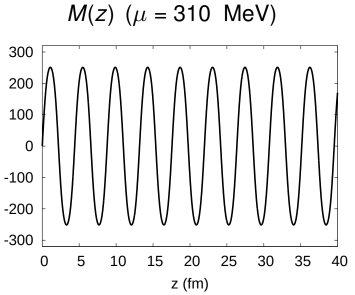

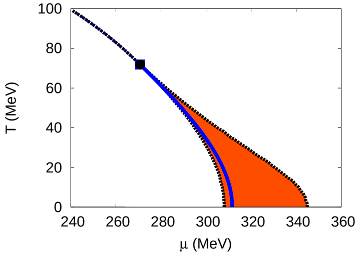

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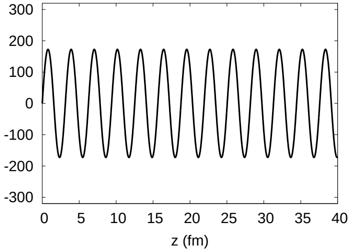
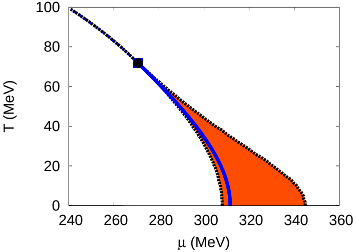
$M(z)$ ($\mu =$ 320 MeV)





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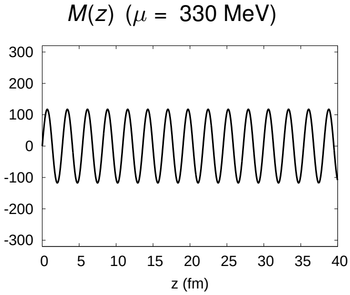

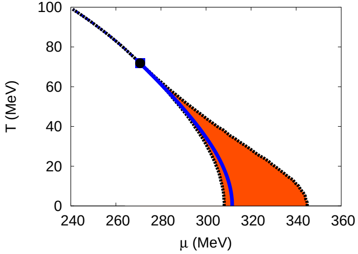

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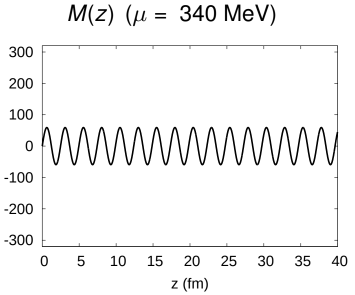

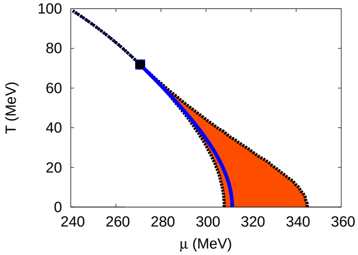

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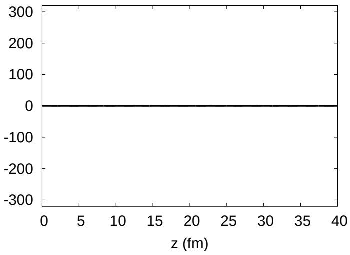
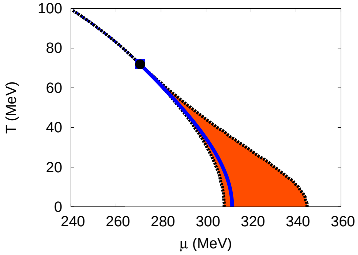
$M(z)$ (μ = 345 MeV)





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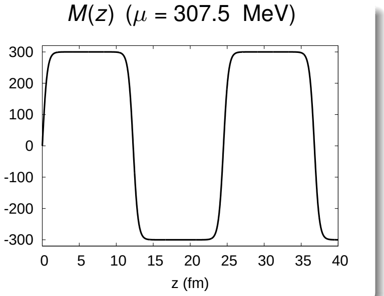

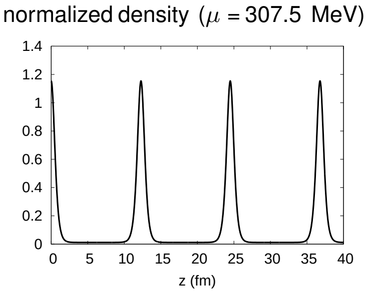

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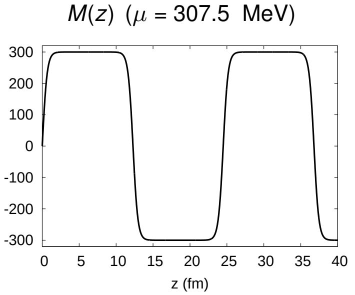

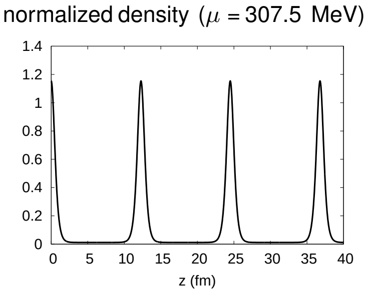

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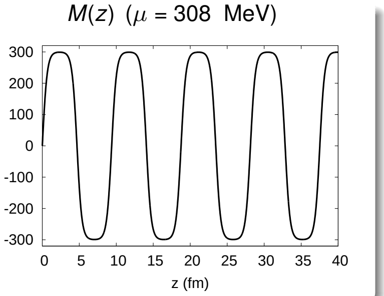

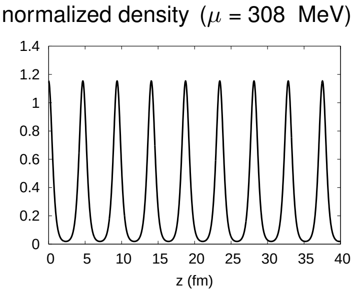

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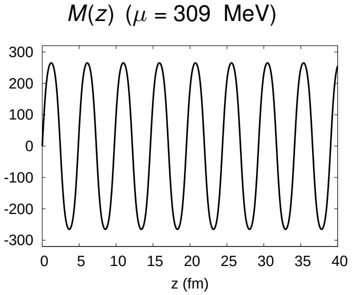

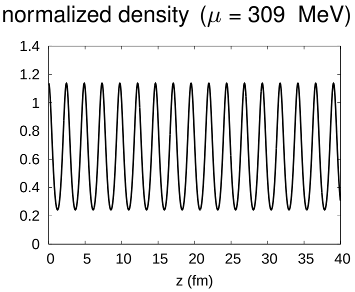

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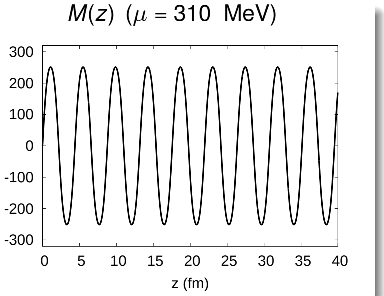

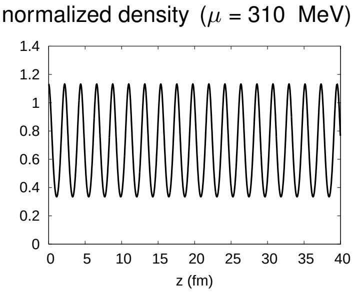

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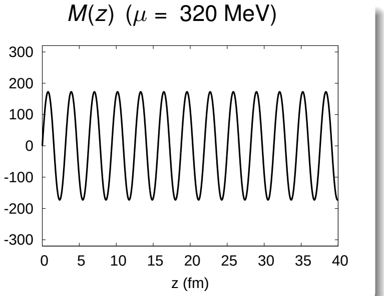

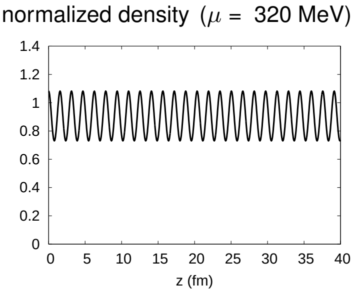

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Mass functions and density profiles ($T = 0$)

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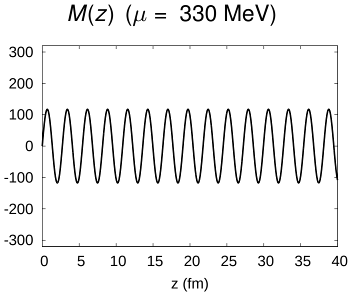

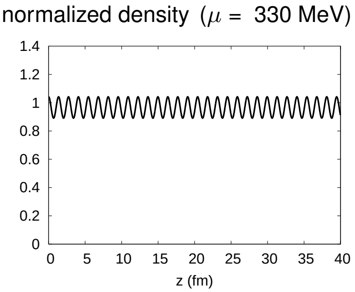

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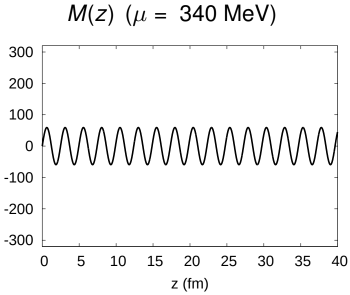

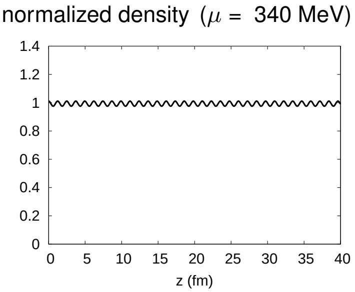

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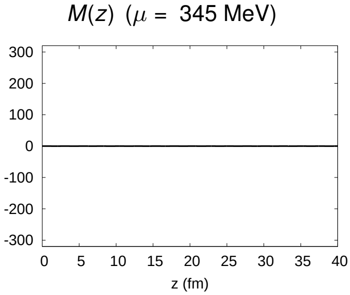

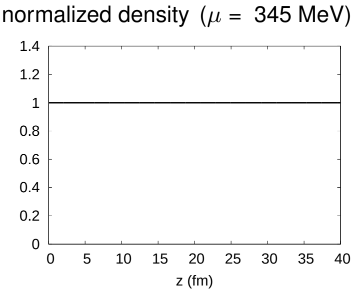

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Free energy difference

[D. Nickel, PRD (2009)]

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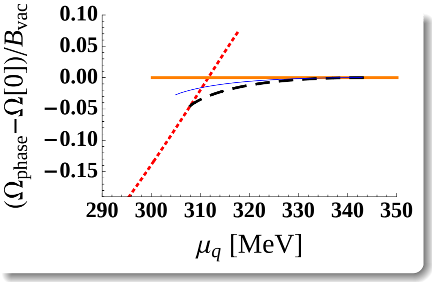

- homogeneous chirally broken
- Jacobi elliptic functions
- chiral density wave:
  $M_{CDW}(z) = M_1\, e^{iqz}$

- soliton lattice favored, when it exists
- $\delta\Omega_{Jacobi} \approx 2\delta\Omega_{CDW} \quad \Rightarrow \quad$ CDW never favored

Self-bound quark matter

[M.B., S. Carignano, PRD (2013)]

- homogeneous NJL at $T = 0$ with strong enough attraction:
  - 1st-order phase transition from vacuum to restored quark matter
  - $\Rightarrow$ phase coexistence of vacuum and dense matter
  - $\Rightarrow$ mechanically stable quark "droplets" in vacuum

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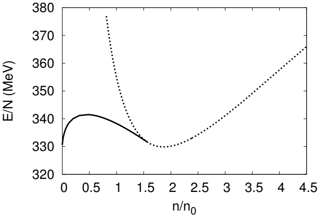
energy per particle



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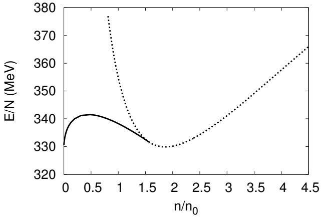

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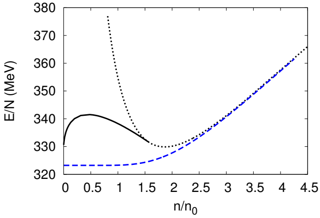

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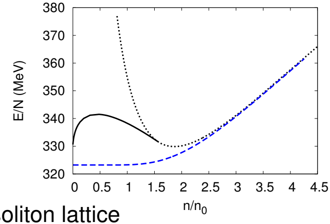

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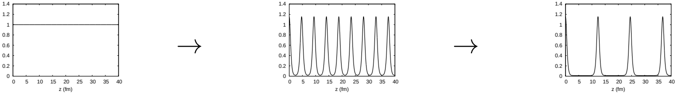

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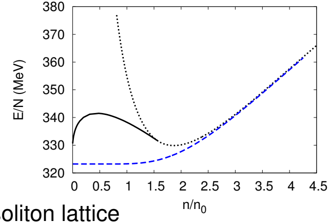

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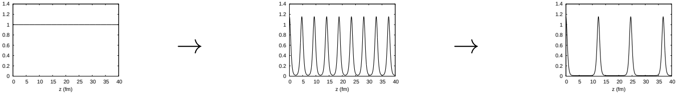

- Can one see this in a dynamical calculation?

Including vector interactions

[S. Carignano, D. Nickel, M.B., PRD (2010)]

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  - $\bar{\psi}\gamma^\mu\psi \to \langle\bar{\psi}\gamma^\mu\psi\rangle \equiv n(\vec{x})\,\delta^{\mu 0}$ (*density*!)
  - $\langle\bar{\psi}\gamma^3\psi\rangle$ possible for inhomogeneous phases ? $\to$ will first be neglected
- mean-field Hamiltonian: $\mathcal{H}_{MF} - \mu = \mathcal{H}_{MF}|_{G_V=0} - \tilde{\mu}(\vec{x})$
  - $\tilde{\mu}(\vec{x}) = \mu - 2G_V\, n(\vec{x})$ "shifted chemical potential"
- further approximation: $n(\vec{x}) \to \langle n\rangle = const. \Rightarrow \tilde{\mu} = const.$
  - questionable in the inhomogeneous phase at low μ and T
  - ok near the restored phase (including the Lifshitz point)
  - advantage: known analytic solutions can still be used
  - additional parameter: $\tilde{\mu}$, fixed by constraint $\frac{\partial\Omega_{MF}}{\partial\tilde{\mu}} = 0$

Phase diagram

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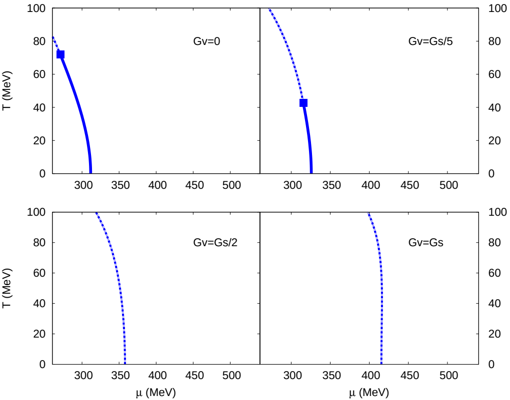

- homogeneous phases: strong G_V-dependence of the critical point

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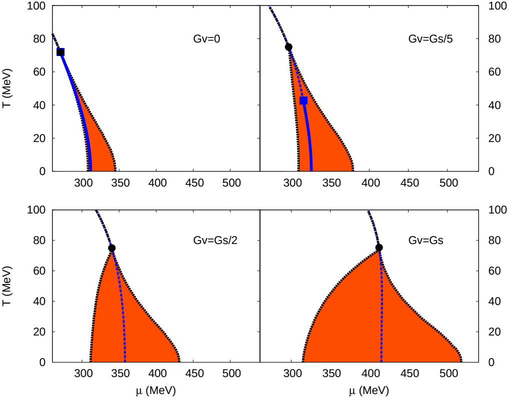

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- inhomogeneous regime: stretched in μ direction, Lifshitz point at constant T

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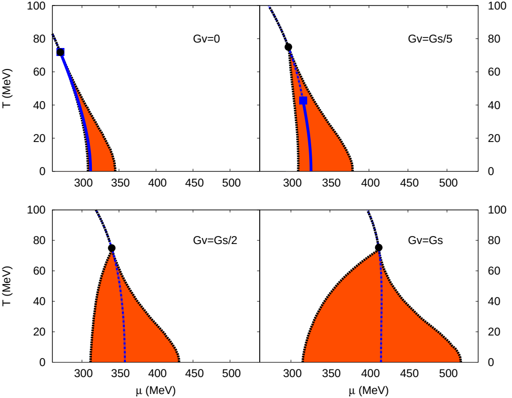
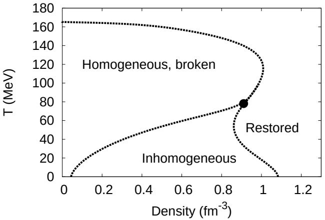


T-$\langle n \rangle$ phase diagram:



- independent of G_V!

- homogeneous phases: strong G_V-dependence of the critical point
- inhomogeneous regime: stretched in μ direction, Lifshitz point at constant T

Susceptibilities

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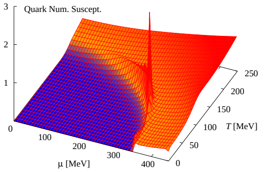
- signature of the critical point: divergent susceptibilities
- e.g., quark number susceptibility:

$$\chi_{nn} = -\frac{\partial^2 \Omega}{\partial \mu^2} = \frac{\partial n}{\partial \mu}$$

homogeneous phases only:



[K. Fukushima, PRD (2008)]

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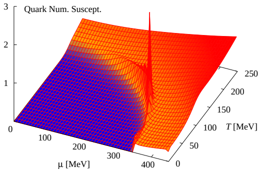

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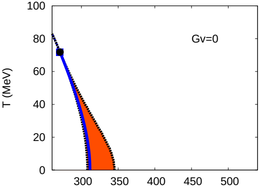

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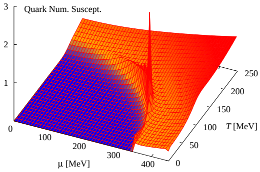

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  $\rightarrow$ no qualitative change

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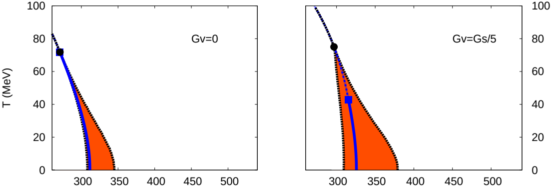

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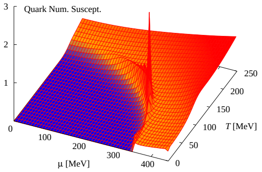

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- $G_V = 0$:
  CP = Lifshitz point
  $\rightarrow$ no qualitative change
- $G_V > 0$:
  no CP $\rightarrow$ no divergence

Susceptibilities

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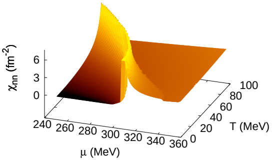

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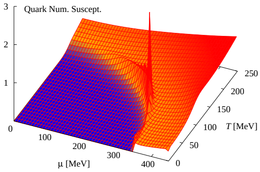

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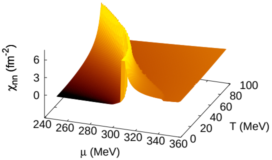

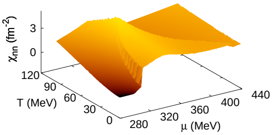

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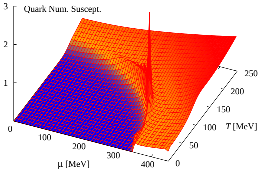

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- $\underline{G_V > 0:}$
  no divergence

Chiral density wave

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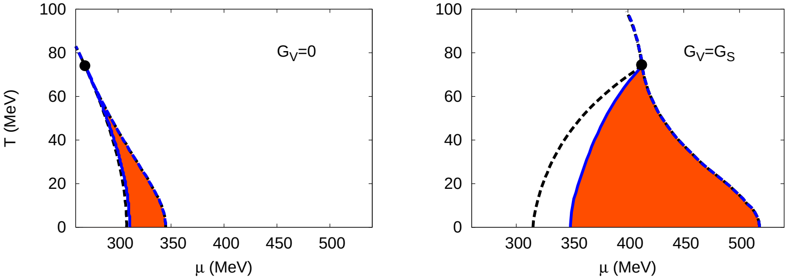
- How much can we trust the approximation $\tilde{\mu} = \mu - 2G_V \langle n \rangle$?
- Chiral density wave: $M(z) = \Delta\, e^{iqz} \quad \Rightarrow \quad n(z) = const.$



- CDW $\rightarrow$ restored and Lifshitz point agree with soliton solution
- chirally broken $\rightarrow$ CDW: 1st order and at higher μ
- exact phase boundary somewhere in between

Improved approximation

[M. Schramm, MSc Thesis, 2013]

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- modified ansatz:
  - mass function: $M(z) = M\cos(qz)$
  - shifted chemical potential: $\tilde{\mu}(z) = \tilde{\mu}_0 + \tilde{\mu}_1 \cos(2qz)$
  - spacelike vector condensate: $\langle\bar{\psi}\gamma^3\psi\rangle = 0$ (unbroken parity)

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- brute-force numerical diagonalization
- solve gap equations: $\frac{\partial\Omega}{\partial M} = \frac{\partial\Omega}{\partial q} = \frac{\partial\Omega}{\partial \tilde{\mu}_0} = \frac{\partial\Omega}{\partial \tilde{\mu}_1} = 0$
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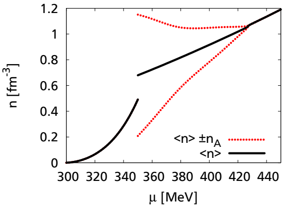
- minimize w.r.t. M and q
- density:

$$n(z) = \frac{\mu-\tilde{\mu}_0}{2G_V} - \frac{\tilde{\mu}_1}{2G_V}\cos(2qz)$$
$$= \langle n\rangle + n_A\cos(2qz)$$



Results

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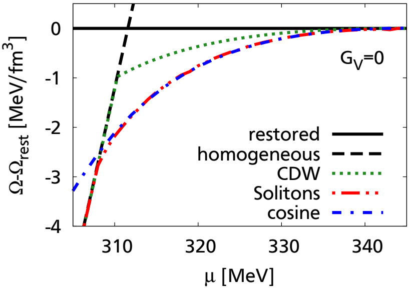
free energies:



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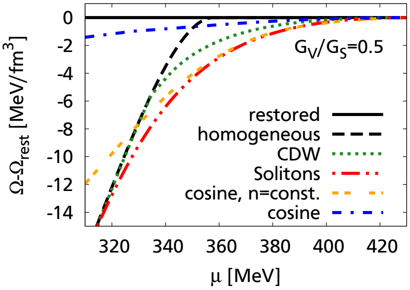

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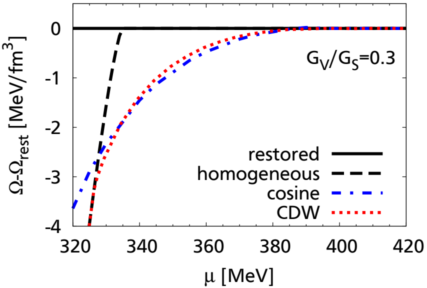

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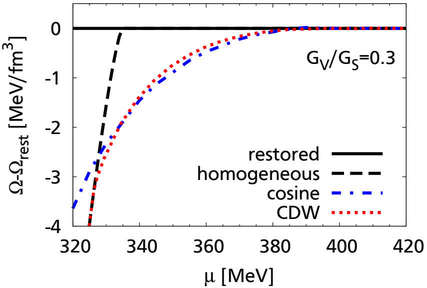
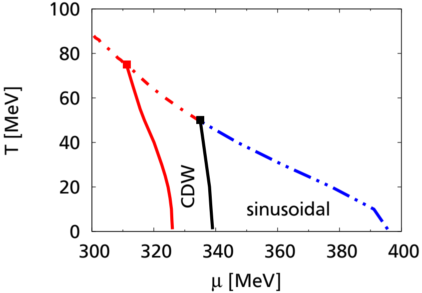


phase diagram:



Two-dimensional modulations

[S. Carignano, M.B., PRD (2012)]

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- no known analytical solutions

  $\rightarrow$ brute-force numerical diagonalization of $\mathcal{H}$ for a given ansatz

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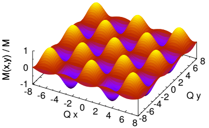
- consider two shapes:

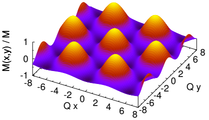
  - square lattice (“egg carton”)

    $$M(x, y) = M \cos(Qx) \cos(Qy)$$



  - hexagonal lattice

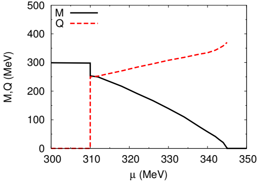
    $$M(x, y) = \frac{M}{3}\left[2\cos(Qx)\cos\left(\frac{1}{\sqrt{3}}Qy\right) + \cos\left(\frac{2}{\sqrt{3}}Qy\right)\right]$$



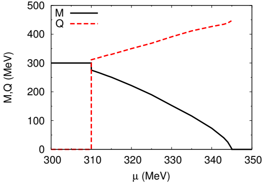
- minimize both cases numerically w.r.t. M and Q

Two-dimensional modulations: results

- amplitudes and wave numbers:
  - egg carton:



  - hexagon:



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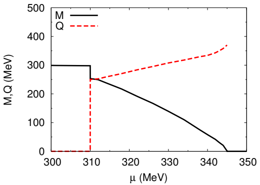

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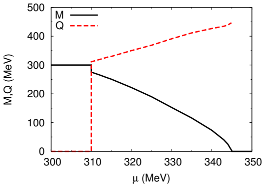
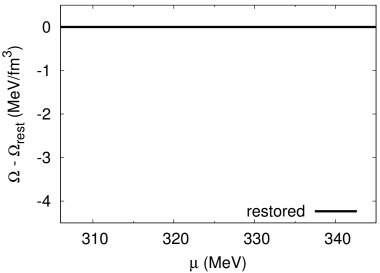


free-energy gain at $T = 0$:



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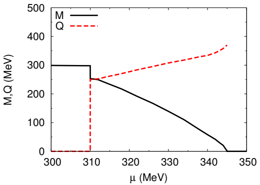

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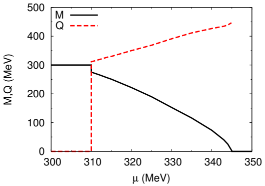

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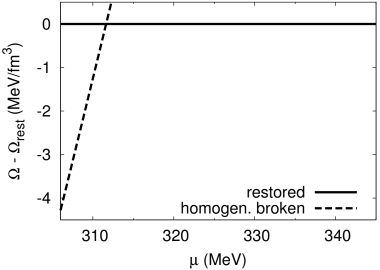

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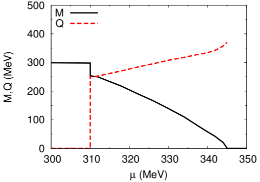

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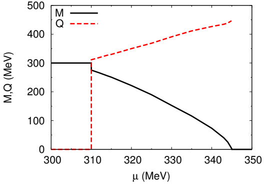

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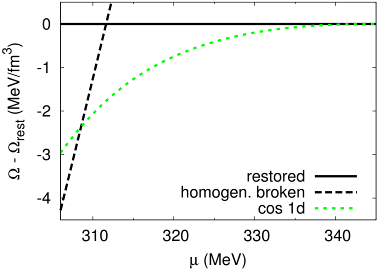

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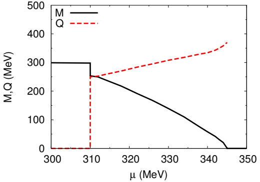

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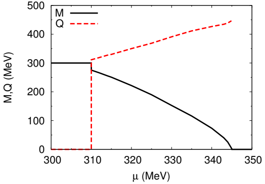

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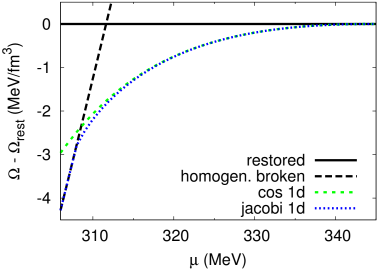

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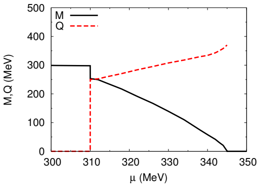

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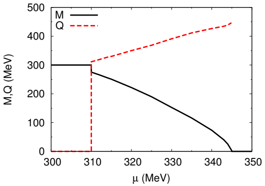

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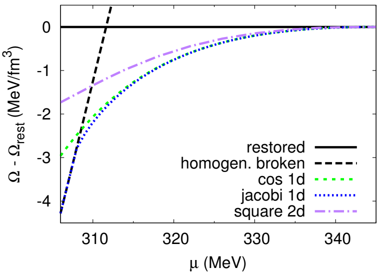

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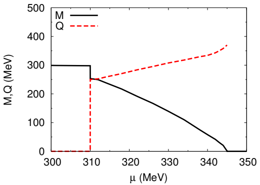

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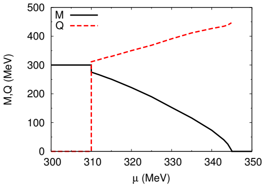

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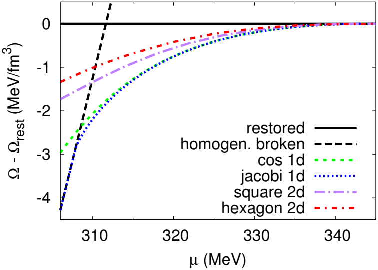

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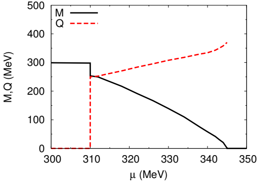

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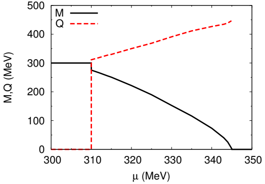

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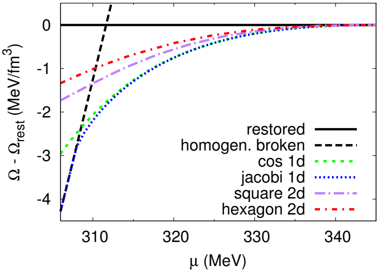

- 2d not favored over 1d in this regime

Rectangular lattice

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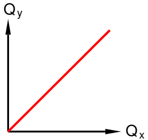
- generalization:

$$M(x, y) = M \cos(Q_x x) \cos(Q_y y)$$

  - one-dim cosine: $Q_x = 0$ or $Q_y = 0$
  - egg carton: $Q_x = Q_y$



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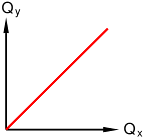

- Is the egg-carton solution
  a saddle point or a local minimum?

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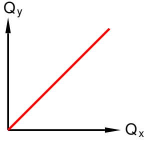
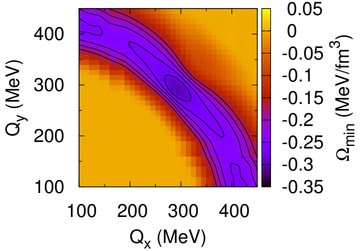
  - one-dim cosine: $Q_x = 0$ or $Q_y = 0$
  - egg carton: $Q_x = Q_y$



- free energy:

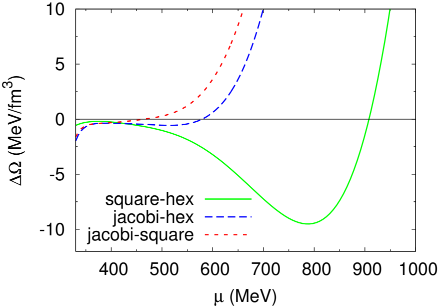


- Is the egg-carton solution a saddle point or a local minimum?

$\Rightarrow$ local minimum!

Two-dimensional modulations: higher densities

- higher chemical potentials:



- favored phase:

  one-dim $\rightarrow$ square $\rightarrow$ hexagon

Two-dimensional modulations: higher densities

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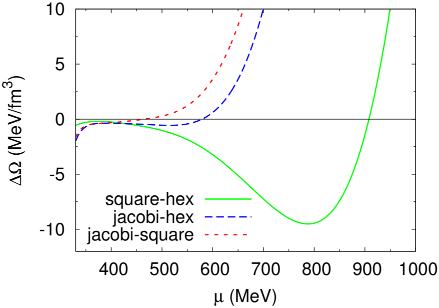
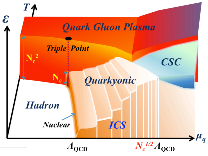
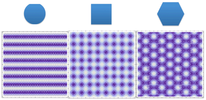


- "interweaving chiral spirals"



[Kojo et al., NPA (2012)]



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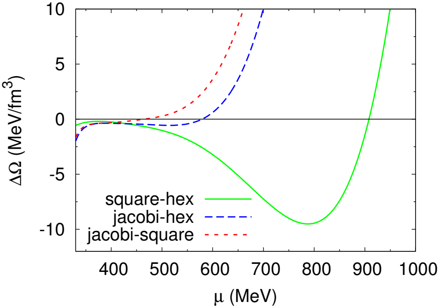

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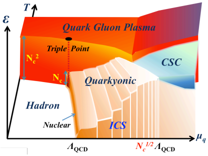

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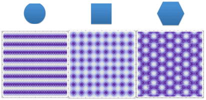
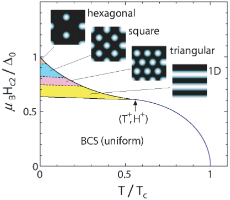


- two-dim supercond. in a magnetic field:



[Matsuda & Shimahara, J. Phys. Soc. Jpn. (2007)]

Isospin-asymmetric matter

[D. Nowakowski et al., work in progress]

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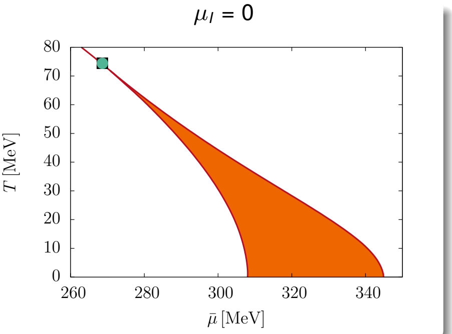

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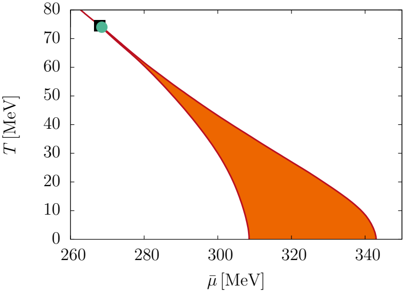
  - inhomogeneous region shrinks
  - CP and LP split

μ_I = 20 MeV



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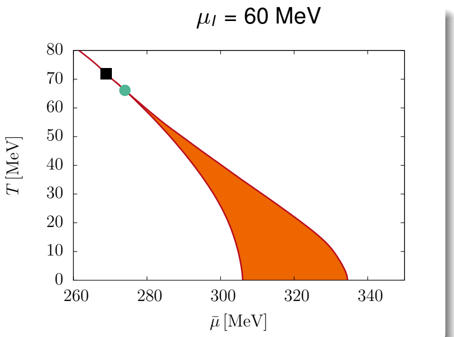

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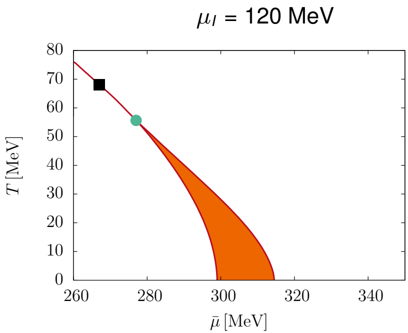

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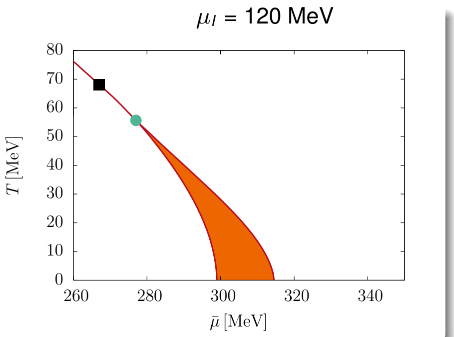

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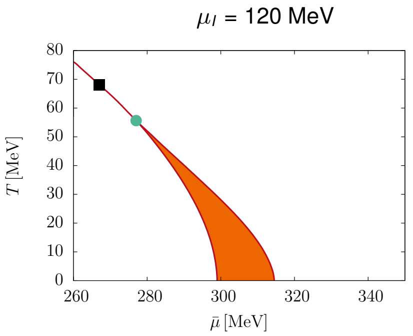

- technical difficulty: We can only describe overall periodic systems

  $\Rightarrow$ $q_u/q_d = m/n$ = rational number (for 1D modulations)

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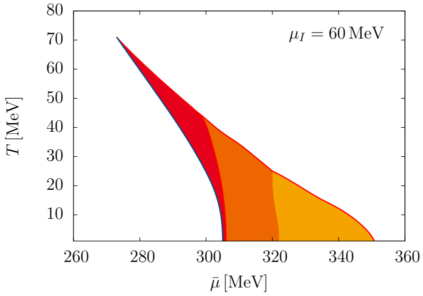

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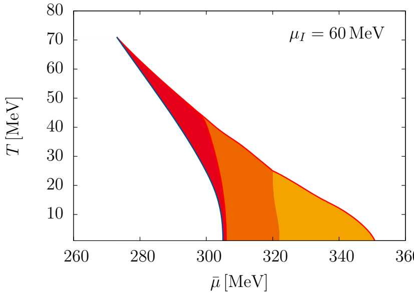

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Pion condensation

TECHNISCHE UNIVERSITÄT DARMSTADT

- nonzero isospin chemical potential only:

  $\mu_u = +\frac{\mu_I}{2}, \quad \mu_d = -\frac{\mu_I}{2}$

- charged pion condensation:

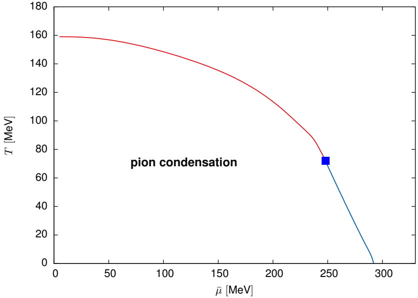
  $\mu_I > m_\pi \quad \rightarrow \quad \langle \bar{u}\, i\gamma_5\, d \rangle \neq 0$

- + quark chemical potential:

  $\mu_u = \frac{\mu_I}{2} + \bar{\mu}, \quad \mu_d = -\frac{\mu_I}{2} + \bar{\mu}$

  $\rightarrow$ stressed pion condensation

homogeneous phases only



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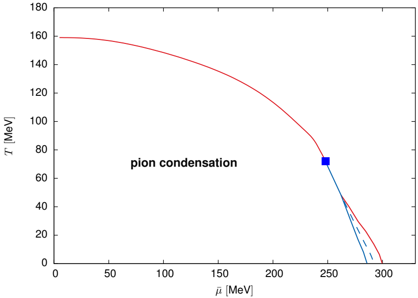
- allowing for inhomogeneous condensates:

  analogous phase structure as before

including inhomogeneous phase



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Inhomogeneous CSC: NJL-model results

[D. Nickel, M.B., PRD (2008)]

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- general gap function with one-dimensional modulation: $\Delta(z) = \sum_k \Delta_k \, e^{ikqz}$
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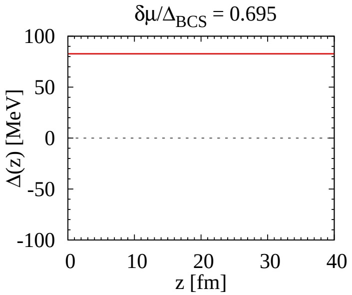

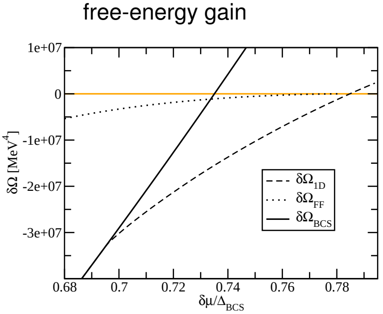

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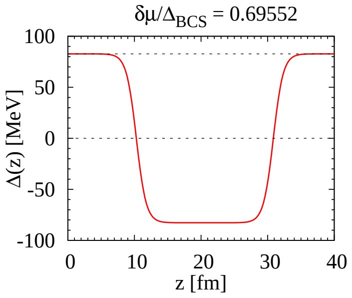
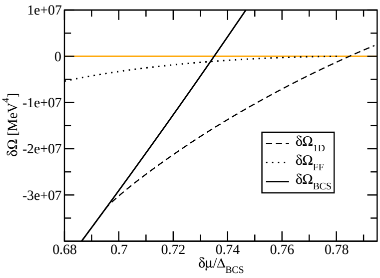


free-energy gain



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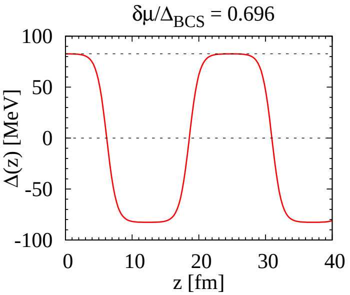

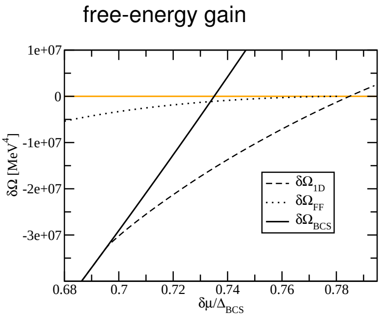

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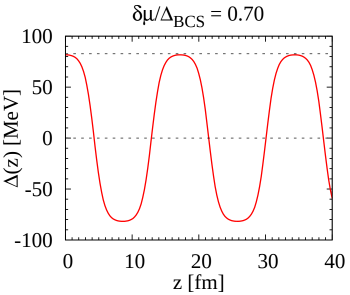

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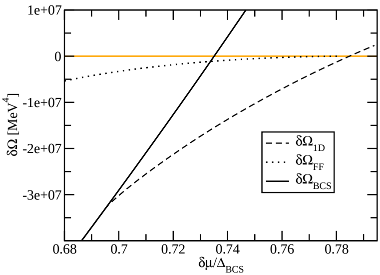

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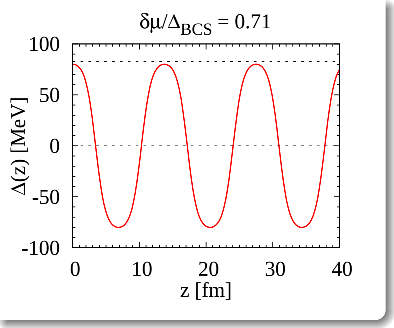

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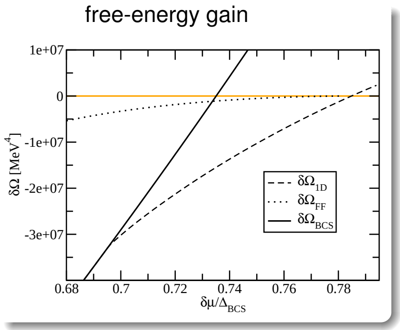

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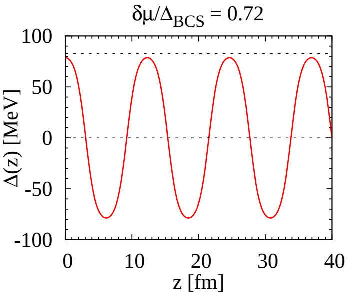

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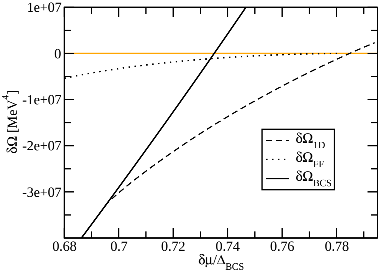

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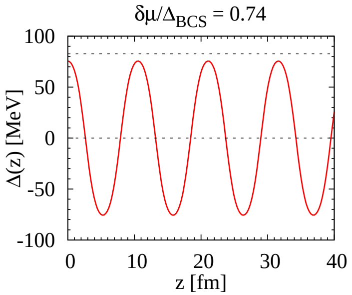

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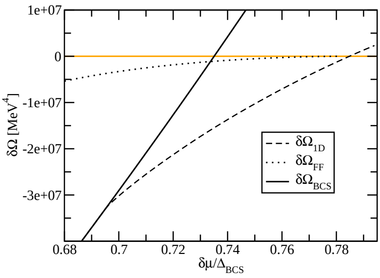

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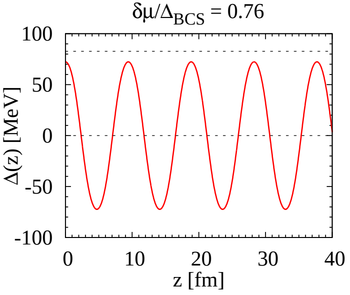

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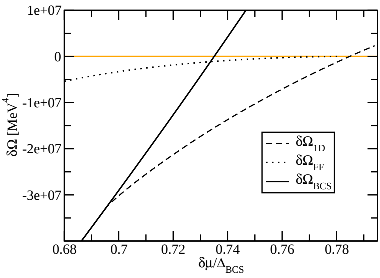

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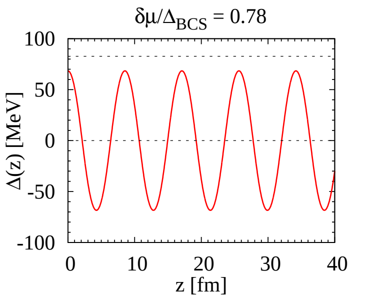

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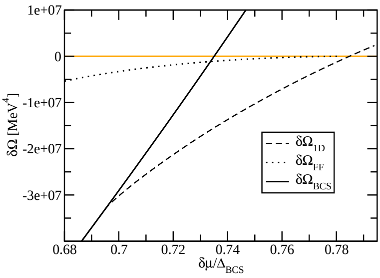

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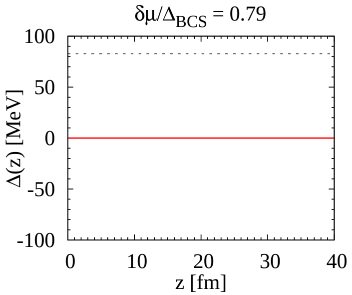

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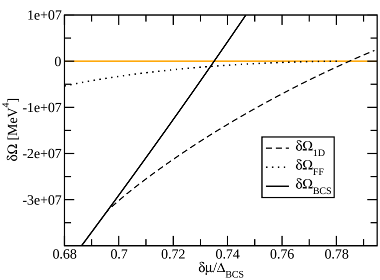

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Comparison

condensate	favored by	stressed by
$\langle \bar{q}q \rangle$	(vacuum)	$\bar{\mu}$
$\langle q^T \mathcal{O} q \rangle$	$\bar{\mu}$	$\delta\mu$
$\langle \bar{u}\, i\gamma_5\, d \rangle$	$\delta\mu$	$\bar{\mu}$

- similar structures in the $T - \mu_{stress}$ phase diagram!

Dyson-Schwinger studies

[D. Müller, M.B, J. Wambach, PLB 2013]

Dyson-Schwinger studies

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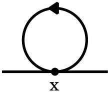
- NJL: local selfenergy
  - homogeneous: $\Sigma(x,y) \propto \delta(x-y) \;\rightarrow\; \Sigma(p,p') = const.$
  - crystalline: $\Sigma = \sigma(\vec{x})\delta(x-y)$
  - periodicity in $\vec{x}$ $\rightarrow$ reciprocal lattice in $\vec{q}$
  - CDW: $\sigma(\vec{x}) = Me^{i\vec{Q}\cdot\vec{x}} \quad \rightarrow \quad \Sigma(\vec{p},\vec{p}') \sim M\delta(\vec{p}-\vec{p}'-\vec{Q})$



Dyson-Schwinger studies

[D. Müller, M.B, J. Wambach, PLB 2013]

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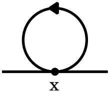
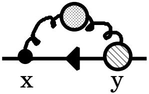
- NJL: local selfenergy
  - homogeneous: $\Sigma(x,y) \propto \delta(x-y) \;\rightarrow\; \Sigma(p,p') = const.$
  - crystalline: $\Sigma = \sigma(\vec{x})\delta(x-y)$
  - periodicity in $\vec{x} \;\rightarrow\;$ reciprocal lattice in $\vec{q}$
  - CDW: $\sigma(\vec{x}) = Me^{i\vec{Q}\cdot\vec{x}} \quad\rightarrow\quad \Sigma(\vec{p},\vec{p}') \sim M\delta(\vec{p}-\vec{p}'-\vec{Q})$



- QCD: non-local selfenergy
  - homogeneous: $\Sigma(x,y) = \Sigma(\vec{x}-\vec{y})$

    $\rightarrow \quad S^{-1}(p) = -i\omega_n\gamma_4\, C(p) + i\vec{p\!\!/}\, A(p) + B(p)$
  - crystalline: $\Sigma = \Sigma(\vec{x},\vec{y})$, depending on relative and c.m. coordinates



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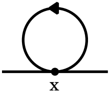

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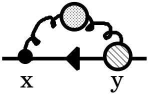

  - crystalline: $\Sigma = \Sigma(\vec{x}, \vec{y})$, depending on relative and c.m. coordinates
  - CDW ansatz: $B(p, p') = \frac{1}{2}\left(B(p) + B(p')\right)\delta(p - p' + Q)$

Chiral density wave

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- quark DSE: $S^{-1}(p, p') = Z_2 \left(S_0^{-1}(p)\delta(p - p') + \Sigma(p, p') \right)$

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$$B(p,p') = \tfrac{1}{2}\left(B(p) + B(p')\right)\delta(p-p'+Q), \qquad \vec{Q} \sim \vec{e}_z$$

- selfconsistency: 10 Dirac components

$$\begin{aligned}
&S^{-1}(p,p') \\
&= \Big[-i(\omega_n + i\mu)\gamma_4 C(p) - ip_3\gamma_3 E(p) - i\vec{\not p}_\perp \, A(p) \\
&\quad - i(\omega_n + i\mu)\gamma_5\gamma_4 C_5(p) - ip_3\gamma_5\gamma_3 E_5(p) - i\gamma_5 \vec{\not p}_\perp \, A_5(p) \Big] \delta(p-p') \\
&\quad + \left(B(p,p') - i\gamma_4\gamma_3 F(p,p') - i\gamma_4 \frac{\vec{\not p}_\perp}{|p_\perp|} G(p,p') - i\gamma_3 \frac{\vec{\not p}_\perp}{|p_\perp|} H(p,p') \right) \frac{(1-\gamma_5)}{2} \delta(p-p'+Q) \\
&\quad + \left(B(p,p') + i\gamma_4\gamma_3 F(p,p') + i\gamma_4 \frac{\vec{\not p}_\perp}{|p_\perp|} G(p,p') + i\gamma_3 \frac{\vec{\not p}_\perp}{|p_\perp|} H(p,p') \right) \frac{(1+\gamma_5)}{2} \delta(p-p'-Q)
\end{aligned}$$

Gap equations

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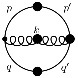
- effective action:



$$\Gamma_{eff} = \mathrm{Tr} \ln S^{-1} - \mathrm{Tr}(1 - Z_2 S_0^{-1} S) + \frac{g^2}{2} \mathrm{Tr} \left(S \Gamma_0^a D_{\mu\nu}^{ab} S \Gamma_\nu^b \right)$$

  - dressed gluon propgator $D_{\mu\nu}^{ab}$:
    quenched lattice results + quark polarization in HTL-HDL approximation
  - model ansatz for the dressed quark-gluon vertex Γ_ν^b

Gap equations

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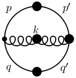

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$$\frac{\partial \Gamma_{eff}}{\partial S(p,p')} = 0 \quad \to \quad S^{-1}(p,p') = Z_2(S_0^{-1}(p,p') + \Sigma(p,p'))$$

$$\frac{d\Gamma_{eff}}{dQ} = 0$$

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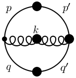

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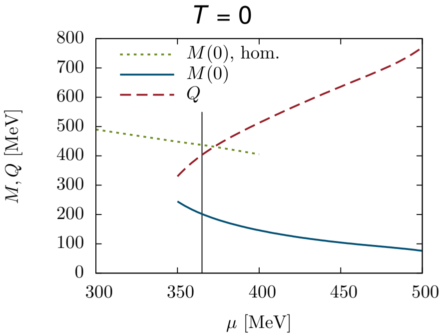
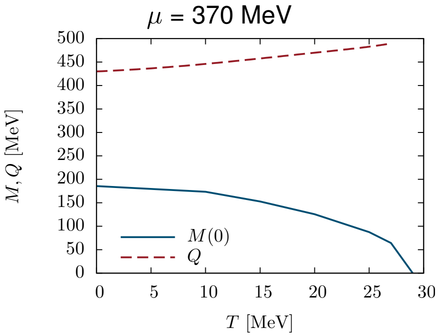
- Solve both equations simultaneously!

Results

- Mass $M(0) = \left|\frac{B(0)}{C(0)}\right|$ and wave number Q:





Phase diagram (spinodal lines)

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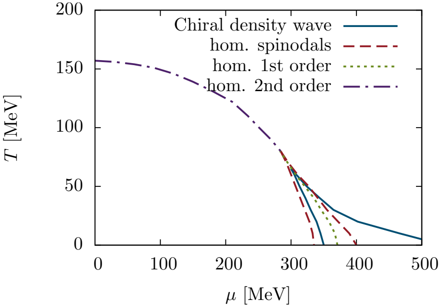
DSE [D. Müller et al. (2013)]



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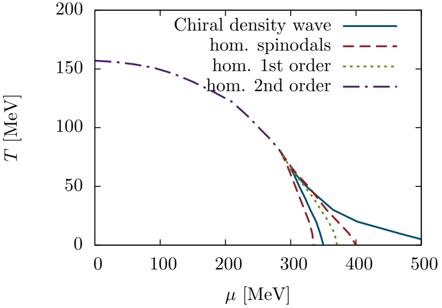
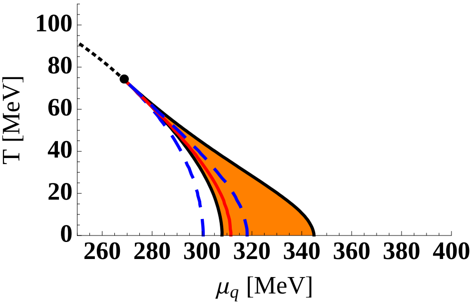


NJL [D. Nickel, PRD (2009)]



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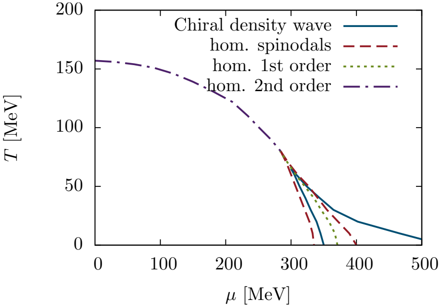

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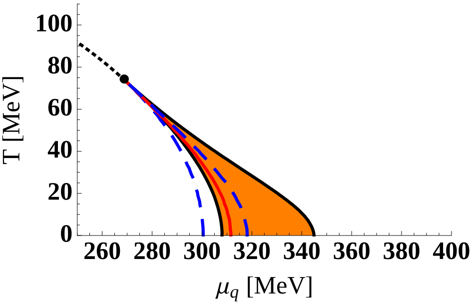

- phase-transition region qualitatively similar

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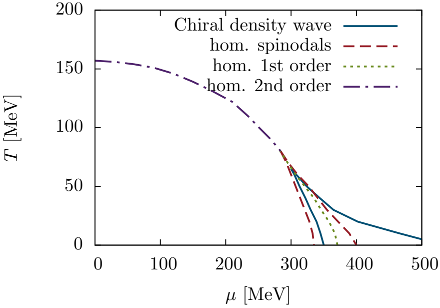

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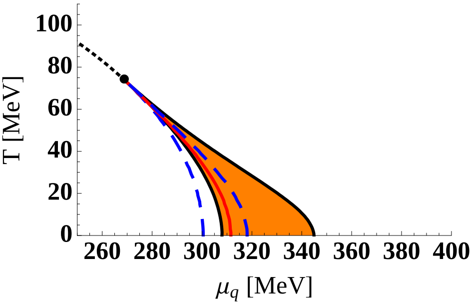

- phase-transition region qualitatively similar
- different behavior at high μ (but CSC not included)

Conclusions

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- Inhomogeneous phases should be considered!

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- NJL model with one- and two-dimensional modulations of $\langle \bar{q}q \rangle$:
  - 1st-order line and critical point covered by an inhomogeneous region
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- fluctuations? ...

Collaborators

Stefano Carignano

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Bundesministerium für Bildung und Forschung

EMMI

HIC for FAIR Helmholtz International Center

HGS-HIRe for FAIR Helmholtz Graduate School for Hadron and Ion Research