Phase diagram and a sign problem in lattice QCD at strong coupling

A. Ohnishi (YITP)

in collaboration with

T. Ichihara (Kyoto U.), T.Z.Nakano (KKE),

K. Miura (Nagoya U.), N. Kawamoto (Hokkaido U.)

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New Frontiers in QCD 2013

--- *Insight into QCD matter from heavy-ion collisions* ---

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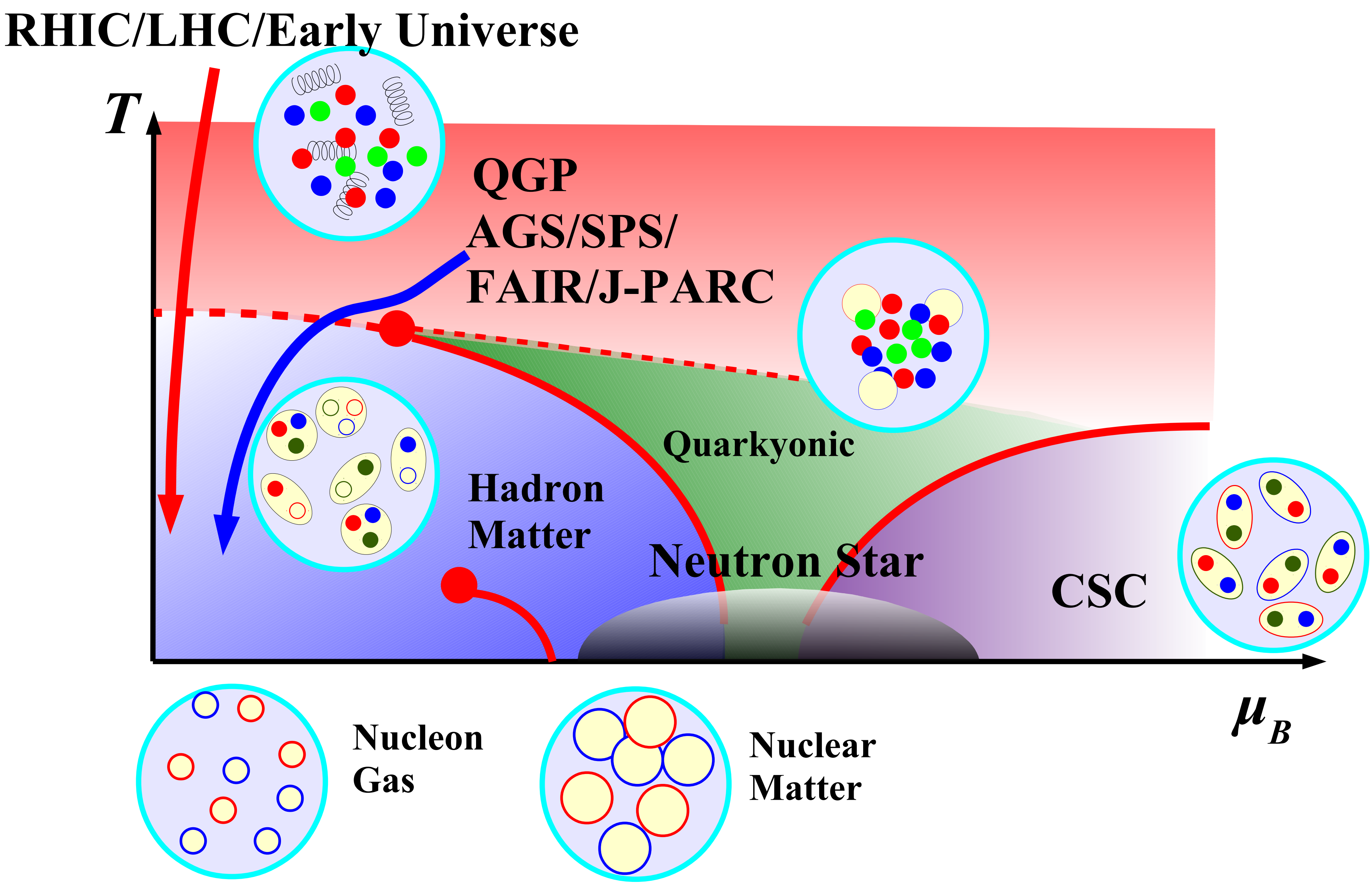
  - Finite coupling effects
    K. Mura, T. Z. Nakano, AO, N. Kawamoto, PRD80(2009), 074034
    T. Z. Nakano, K. Miura, AO, PRD83(2011),016014
  - Fluctuations
    AO, T. Ichihara, T.Z. Nakano, PoS Lattice2012 (2012), 088
    T. Ichihara, T. Z. Nakano, AO, PoS Lattice2013 (2013), to appear
- Summary

QCD Phase Diagram



How can we investigate QCD phase diagram ?

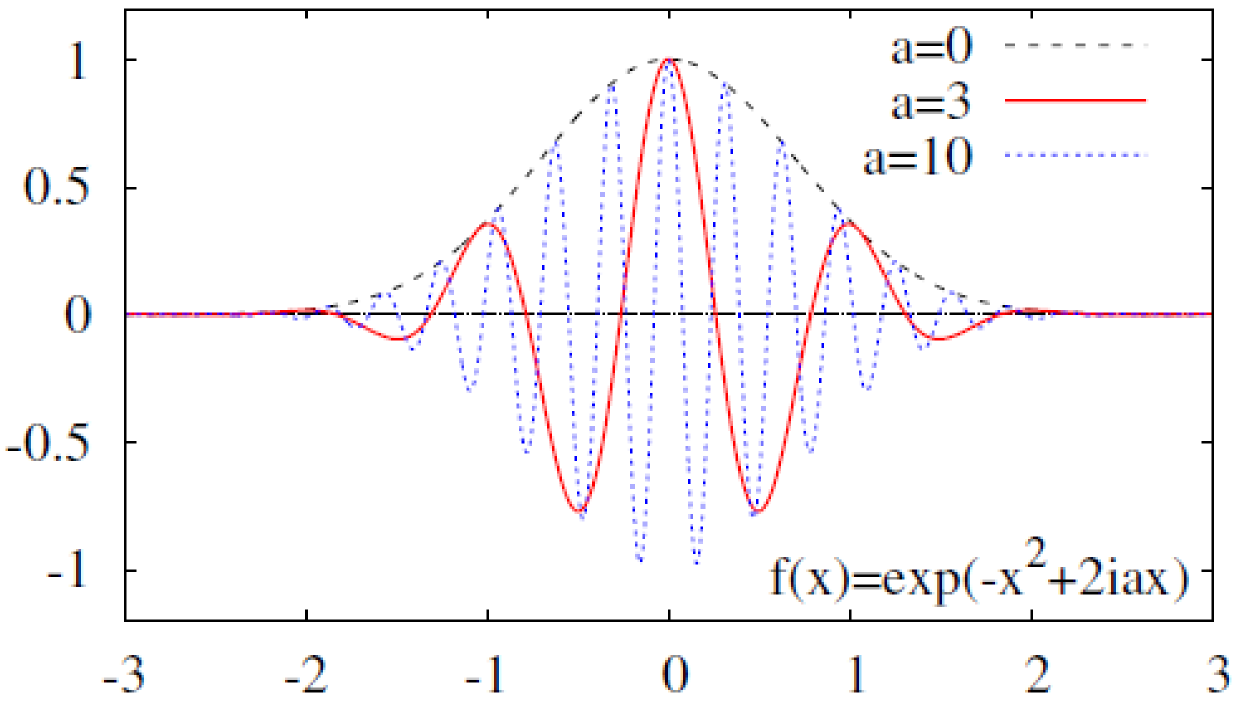
- **Non-pert. & ab initio approach**
  = Monte-Carlo simulation of lattice QCD
  but lattice QCD at finite μ has the sign problem.

Sign Problem

- **Monte-Carlo integral of oscillating function**

$$Z=\int dx\exp(-x^2+2iax)=\sqrt{\pi}\exp(-a^2)$$

$$\langle O\rangle=\frac{1}{Z}\int dx\,O(x)\,e^{-x^2+2iax}$$

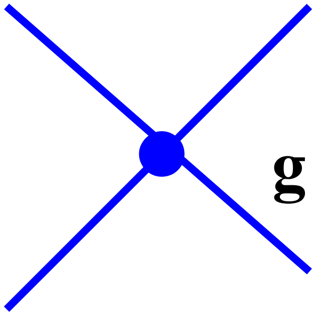


Easy problem for human is not necessarily easy for computers.

Sign Problem (cont.)

- **Generic problem in quantum many-body problems**
  - **Example: Euclid action of interacting Fermions**

$$S=\sum_{x,y}\bar{\psi}_x D_{x,y}\psi_y+g\sum_x(\bar{\psi}\psi)_x(\bar{\psi}\psi)_x$$



  - **Bosonization and MC integral (g>0 → repulsive)**

$$\exp(-g M_x M_x)=\int d\sigma_x \exp(-g\sigma_x^2-2ig\sigma_x M_x)\quad (M_x=(\bar{\psi}\psi)_x)$$

$$Z=\int D[\psi,\bar{\psi},\sigma]\exp\left[-\bar{\psi}(D+2ig\sigma)\psi-g\sum_x\sigma_x^2\right]$$

$$=\int D[\sigma]\ \underline{\mathrm{Det}(D+2ig\sigma)}\exp\left[-g\sum_x\sigma_x^2\right]$$

complex Fermion det.
→ complex stat. weight
→ sign problem

Sign problem in lattice QCD

- **Fermion determinant (= stat. weight of MC integral) becomes complex at finite μ in LQCD.**
  - **γ_5 Hermiticity**

$$Z = \int D[U, q, \bar{q}] \; \exp(-\bar{q} D(\mu, U) q - S_G(U))$$
$$= \int D[U] \; \mathrm{Det}(D(\mu, U)) \; \exp(-S_G(U))$$

$$\gamma_5 D(\mu, U) \gamma_5 = \left[D(-\mu, U^+)\right]^+$$
$$\rightarrow \mathrm{Det}(D(\mu, U)) = \left[\mathrm{Det}(D(-\mu, U^+))\right]^*$$

  - **Fermion det. (Det D) is real for zero μ (and pure imag. μ)**
  - **Fermion det. is complex for finite real μ.**

How can we investigate QCD phase diagram ?

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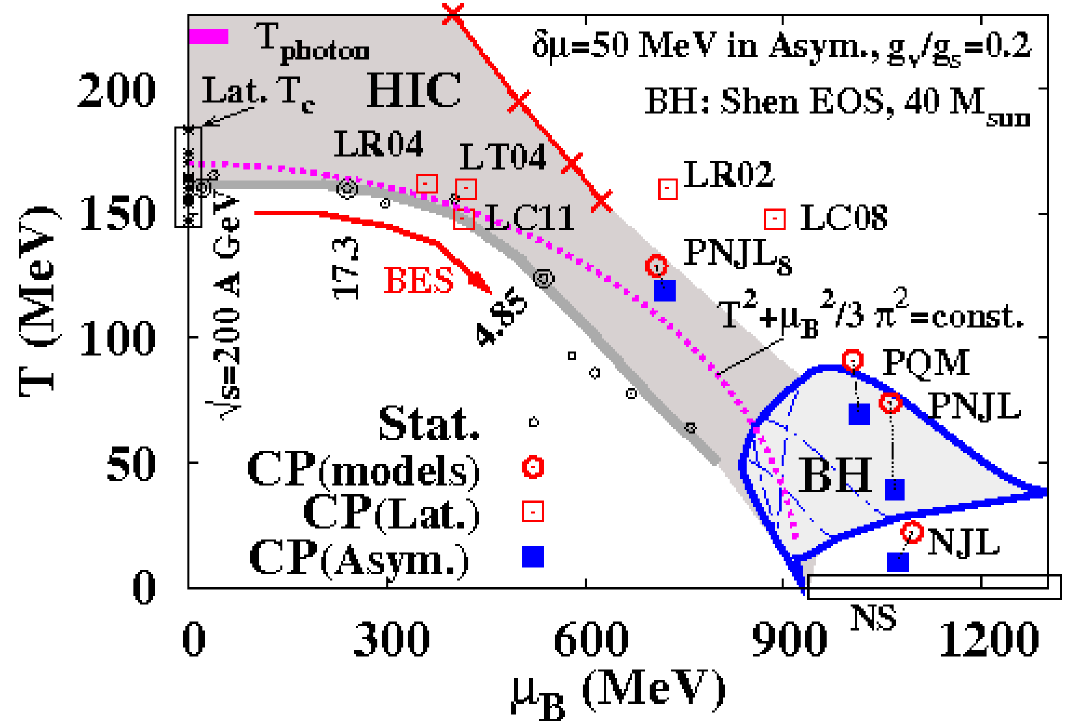
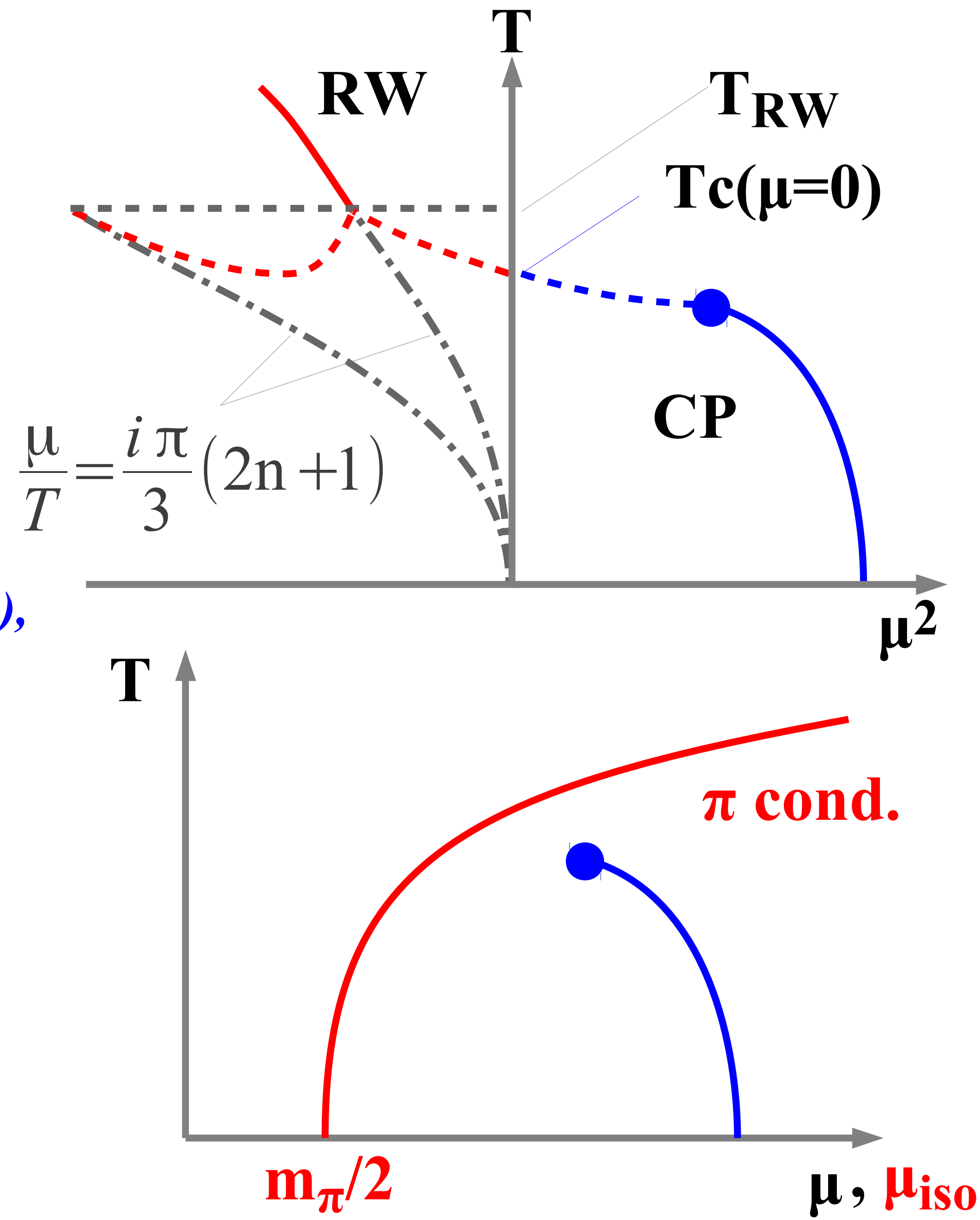
- **Effective model and/or Approximations are necessary.**
  - **Effective models:**
    NJL, PNJL, PQM, ...
    Model dependence is large.
  - **Approximate methods:**
    Taylor expansion,
    Imag. μ , Canonical,
    Re-weighting,
    Fugacity expansion,
    Histogram method,
    Complex Langevin,
    Strong coupling lattice QCD



Lattice QCD at fnite μ

- **Various method work at small μ (μ/T < 1).**
- **Large μ**
  - **Roberge-Weiss transition → Conv. rad. of μ/T < π/3 at T>T_{RW}**
  - **No go theorem**
    Splittorff ('06), Han, Stephanov ('08), Hanada, Yamamoto ('11), Hidaka, Yamamoto ('11)
    - **Phase quenched sim. ~ Isospin chem. pot.**
    - **CP would be hidden in π cond.**



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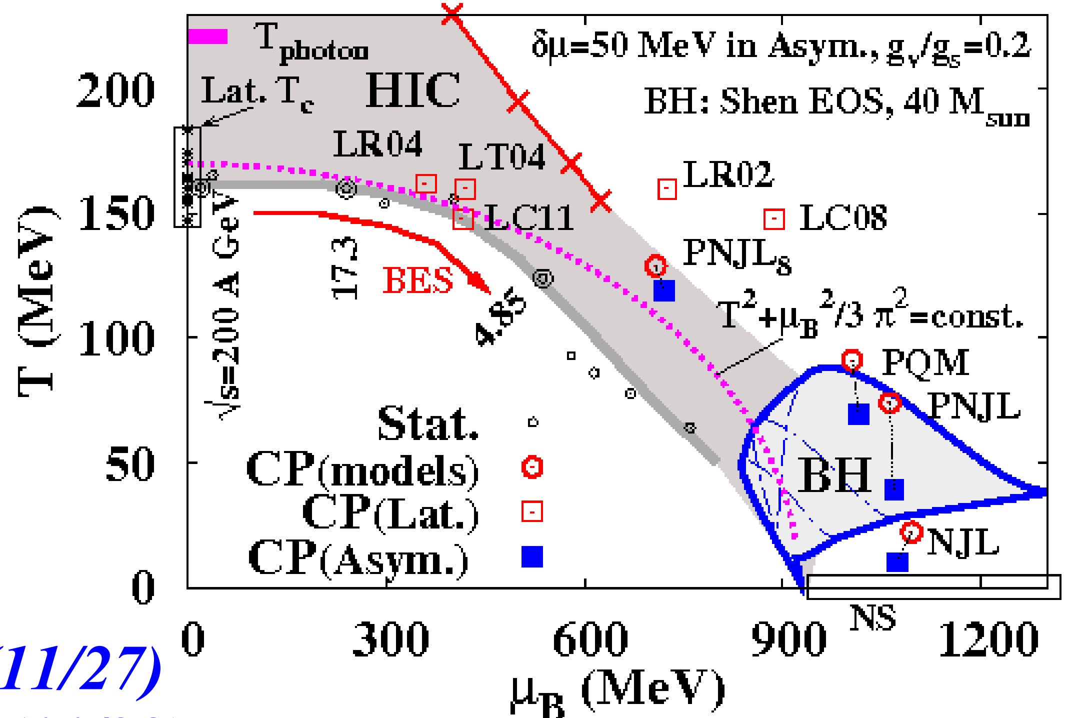
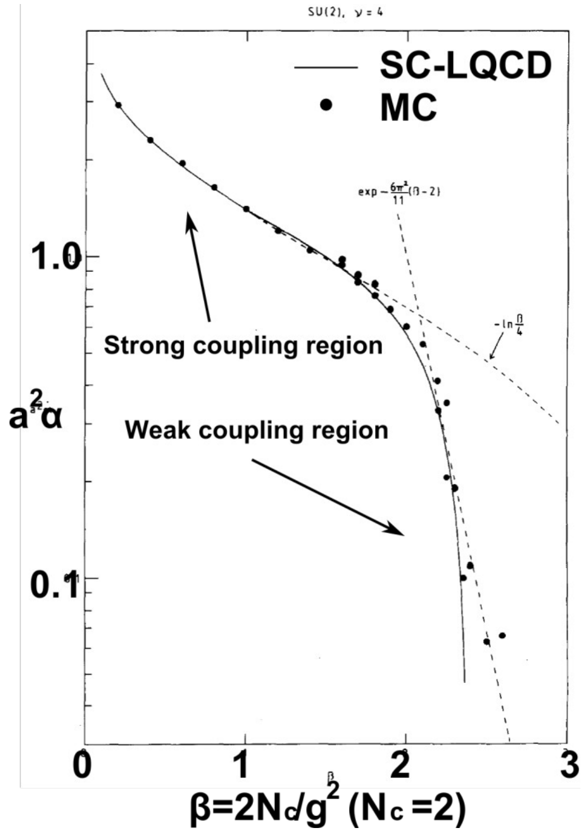
    Complex Langevin, *Aarts (11/28)*
    Strong coupling lattice QCD — **In this talk**



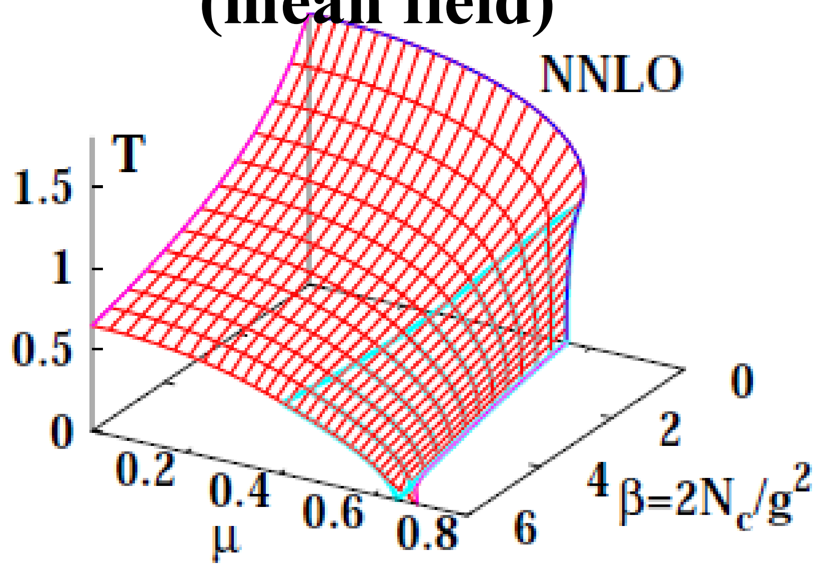
Strong Coupling Lattice QCD

Pure YM



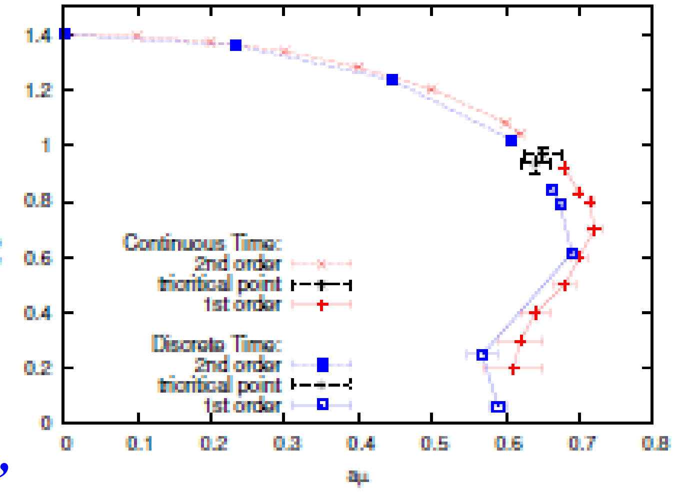
Wilson ('74), Creutz ('80), Munster ('80, '81), Lottini, Philipsen, Langelage's ('11)

Phase diagram (mean field)



Kawamoto ('80), Kawamoto, Smit ('81), Damagaard, Hochberg, Kawamoto ('85), Bilic, Karsch, Redlich ('92), Fukushima ('03); Nishida ('03), Kawamoto, Miura, AO, Ohnuma ('07). Miura, Nakano, AO, Kawamoto ('09) Nakano, Miura, AO ('10)

Fluctuations



Mutter, Karsch ('89), de Forcrand, Fromm ('10), de Forcrand, Unger ('11), AO, Ichihara, Nakano ('12), Ichihara, Nakano, AO ('13)

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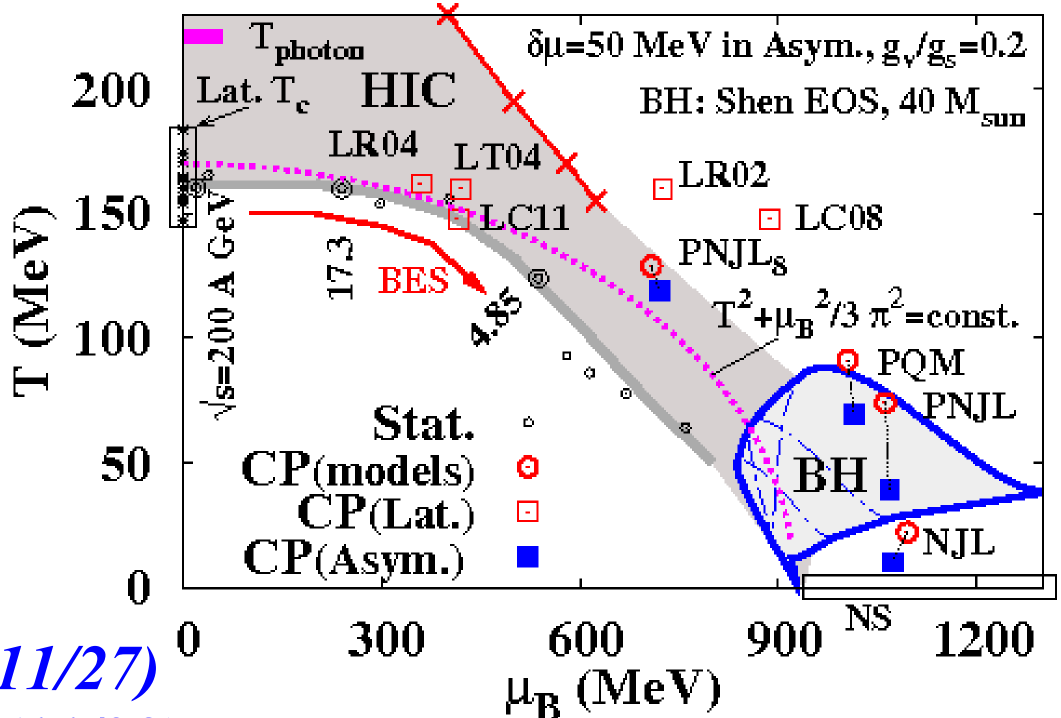


We discuss the phase diagram and a sign problem in SC-LQCD

Strong coupling lattice QCD

SC-LQCD: Setups & Disclaimer

- **We investigate the phase diagram and try to understand nuclear matter based on the strong-coupling lattice QCD (SC-LQCD).**
  - **Effective potential (free E. density) → phase boundary & EOS**
- **Setups & Disclaimer**
  - **Effective action in SCL ($1/g^0$), NLO ($1/g^2$), NNLO ($1/g^4$) terms and Polyakov loop.**

    NLO: Faldt-Petersson ('86), Bilic-Karsch-Redlich ('92)
    Conversion radius > 6 in pure YM ? Osterwalder-Seiler ('78)
  - **One species of unrooted staggered fermion (N_f=4 @ cont.)**

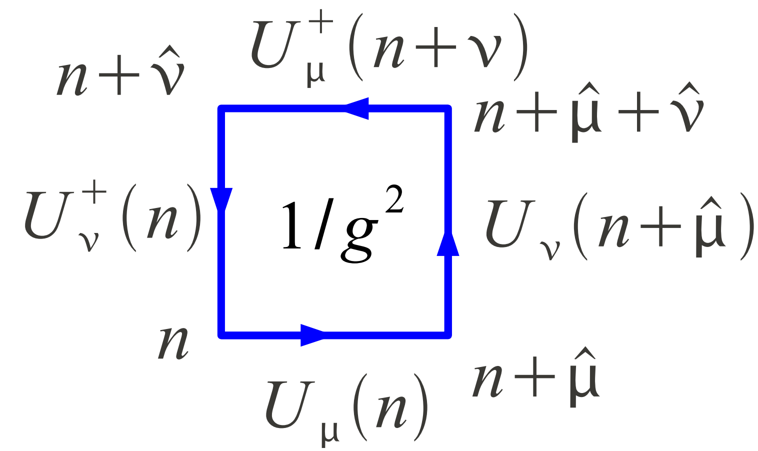
    Moderate N_f deps. of phase boundary: BKR92, Nishida('04), D'Elia-Lombardo ('03)
  - **Leading order in 1/d expansion (d=3=space dim.)**
    → Min. # of quarks for a given plaquette configurations, no spatial B prop.
  - **Different from "strong couling" in "large N_c"**

Still far from "Realistic", but SC-LQCD would tell us useful qualitative features of the phase diagram and EOS.

Lattice QCD action

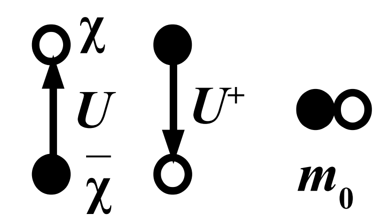
- **Gluon field → Link variables** $U_\mu(x) \simeq \exp(i g A_\mu)$
- **Gluon action → Plaquette action**

$$S_G = \frac{2N_c}{g^2} \sum_{plaq.} \left[1 - \frac{1}{N_c} \mathrm{Re\,tr}\, U_{\mu\nu}(n) \right]$$



  - **Loop → surface integral of "rotation" $F_{\mu\nu}$ in the U(1) case.**

- **Quark kinetic term (staggered fermion)**



$$S_F = \frac{1}{2} \sum_x \left[\bar{\chi}_x e^{\mu} U_{0,x} \chi_{x+\hat{0}} - \bar{\chi}_{x+\hat{0}} e^{-\mu} U_{0,x}^+ \chi_x \right]$$

$$+ \frac{1}{2} \sum_{x,j} \eta_j(x) \left[\bar{\chi}_x U_{x,j} \chi_{x+\hat{j}} - \bar{\chi}_{x+\hat{j}} U_{x,j}^+ \chi_x \right] + \sum_x m_0 M_x$$

$$= S_F^{(t)} + S_F^{(s)} + \sum_x m_0 M_x$$

Link integral → Area Law

- **One-link integral**

$$\int dU\, U_{ab} U^{+}_{cd} = \frac{1}{N_c} \delta_{ad} \delta_{bc}$$

- **Wilson loop in pure Yang-Mills theory**

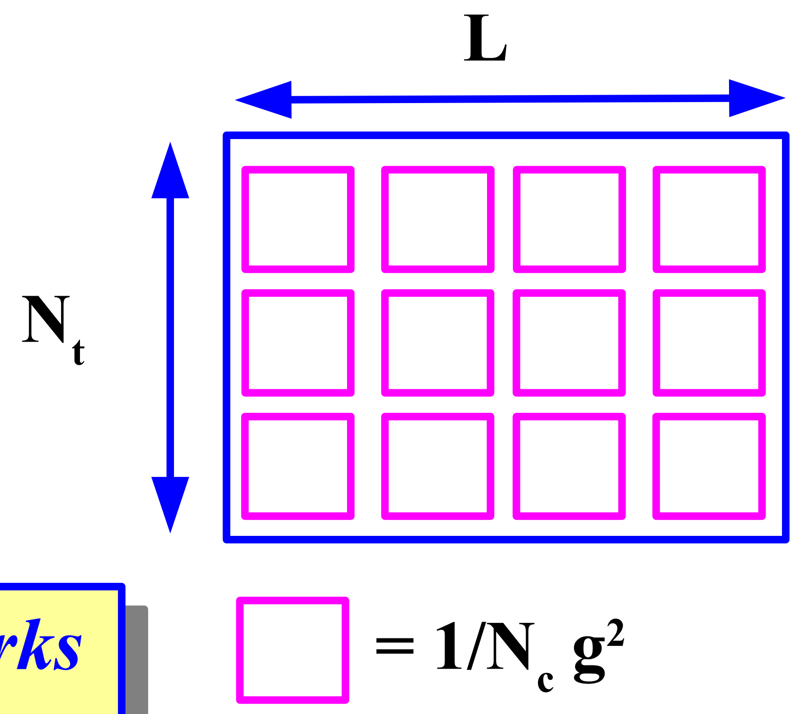
$$\langle W(C = L \times N_\tau) \rangle = \frac{1}{Z} \int DU\, W(C) \exp\left[\frac{1}{g^2} \sum_P \mathrm{tr}\left(U_P + U_P^{+}\right) \right]$$

$$= \exp\left(-V(L) N_\tau\right)$$

in the strong coupling limit

$$\langle W(C) \rangle = N \left(\frac{1}{g^2 N} \right)^{L N_\tau}$$

$$\to V(L) = L \log\left(g^2 N\right)$$



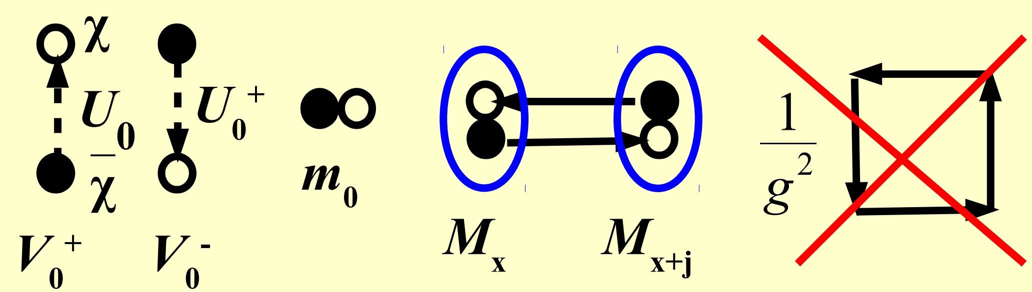
Linear potential between heavy-quarks → Confinement (Wilson, 1974)

Link integral → Effective action

- **Effective action in the strong coupling limit (SCL)**
  - **Ignore plaquette action ($1/g^2$)**
    → We can integrate each link independently !
  - **Integrate out *spatial* link variables of min. quark number diagrams (1/d expansion)**

$$S_{\mathrm{eff}} = S_F^{(t)} - \frac{1}{4N_c}\sum_{x,j} M_x M_{x+\hat{j}} + m_0 \sum_x M_x \quad \left(M_x = \bar{\chi}_x \chi_x\right)$$

Damgaard, Kawamoto, Shigemoto ('84)

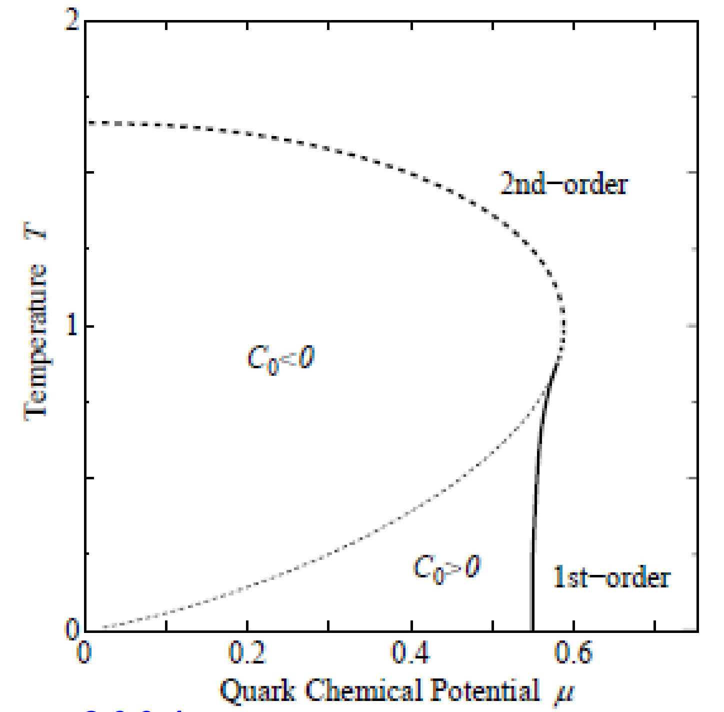


$$\int dU\, U_{ab} U^{+}_{cd} = \delta_{ad}\delta_{bc}/N_c$$

Lattice QCD in SCL
→ Fermion action
with nearest neighbor
four Fermi interaction

Phase diagram in SC-LQCD (mean field)

- **"Standard" simple procedure in Fermion many-body problem**
  - **Bosonize interaction term (Hubbard-Stratonovich transformation)**
  - **Mean field approximation (constant auxiliary field)**
  - **Fermion & temporal link integral**
    Damgaard, Kawamoto, Shigemoto ('84); Faldt, Petersson ('86); Bilic, Karsch, Redlich ('92); Fukushima ('04); Nishida ('04)



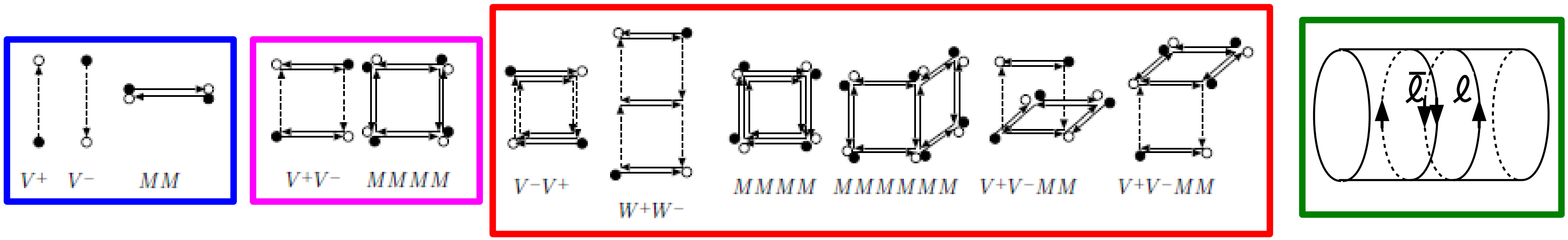
Fukushima, 2004

Finite Coupling Effects

- **Effective Action with finite coupling corrections**
  Integral of exp(-S_G) over spatial links with exp(-S_F) weight → $S_{\rm eff}$

$$S_{\rm eff} = S_{\rm SCL} - \log\langle \exp(-S_G)\rangle = S_{\rm SCL} - \sum_{n=1} \frac{(-1)^n}{n!}\langle S_G^n\rangle_c$$

$\langle S_G^n\rangle_c$=Cumulant (connected diagram contr.) *c.f. R.Kubo('62)*



$$S_{\rm eff} = \frac{1}{2}\sum_x (V_x^+ - V_x^-) - \frac{b_\sigma}{2d}\sum_{x,j>0}[MM]_{j,x}$$ *SCL (Kawamoto-Smit, '81)*

$$+\frac{1}{2}\frac{\beta_\tau}{2d}\sum_{x,j>0}[V^+V^- + V^-V^+]_{j,x} - \frac{1}{2}\frac{\beta_s}{d(d-1)}\sum_{x,j>0,k>0,k\neq j}[MMMM]_{jk,x}$$ *NLO (Faldt-Petersson, '86)*

$$-\frac{\beta_{\tau\tau}}{2d}\sum_{x,j>0}[W^+W^- + W^-W^+]_{j,x} - \frac{\beta_{ss}}{4d(d-1)(d-2)}\sum_{\substack{x,j>0,|k|>0,|l|>0\\ |k|\neq j,|l|\neq j,|l|\neq |k|}}[MMMM]_{jk,x}[MM]_{j,x+\hat{l}}$$

$$+\frac{\beta_{\tau s}}{8d(d-1)}\sum_{x,j>0,|k|\neq j}[V^+V^- + V^-V^+]_{j,x}\left([MM]_{j,x+\hat{k}} + [MM]_{j,x+\hat{k}+\hat{0}}\right)$$ *NNLO (Nakano, Miura, AO, '09)*

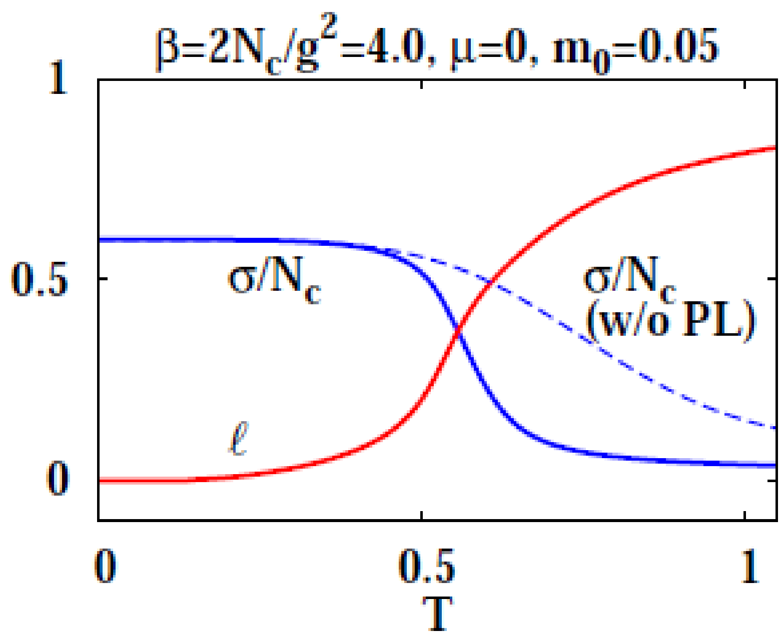
$$-\left(\frac{1}{g^2N_c}\right)^{N_\tau} N_c^2 \sum_{\mathbf{x},j>0}\left(\bar{P}_{\mathbf{x}}P_{\mathbf{x}+\hat{j}} + h.c.\right)$$ *Polyakov loop (Gocksch, Ogilvie ('85), Fukushima ('04) Nakano, Miua, AO ('11))*

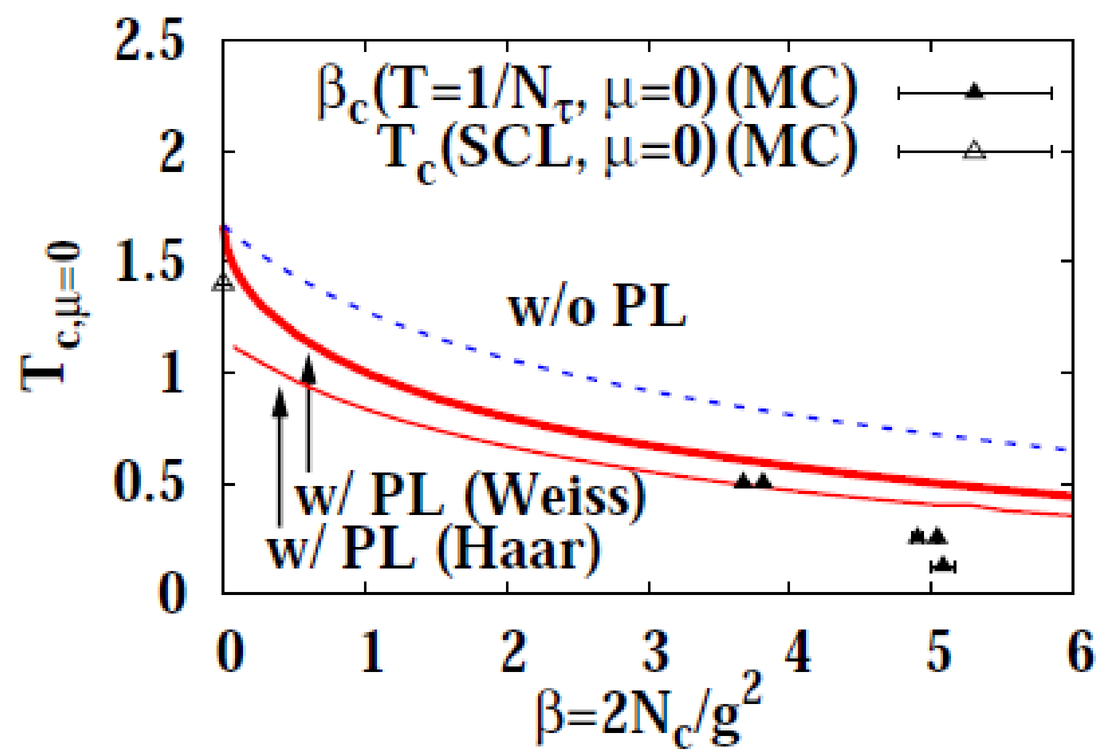
SC-LQCD with Polyakov Loop Effects at μ=0

T. Z. Nakano, K. Miura, AO, PRD 83 (2011), 016014 [arXiv:1009.1518 [hep-lat]]

- **P-SC-LQCD reproduces MC results of $T_c(\mu=0)$ ($\beta= 2N_c/g^2 \le 4$)**

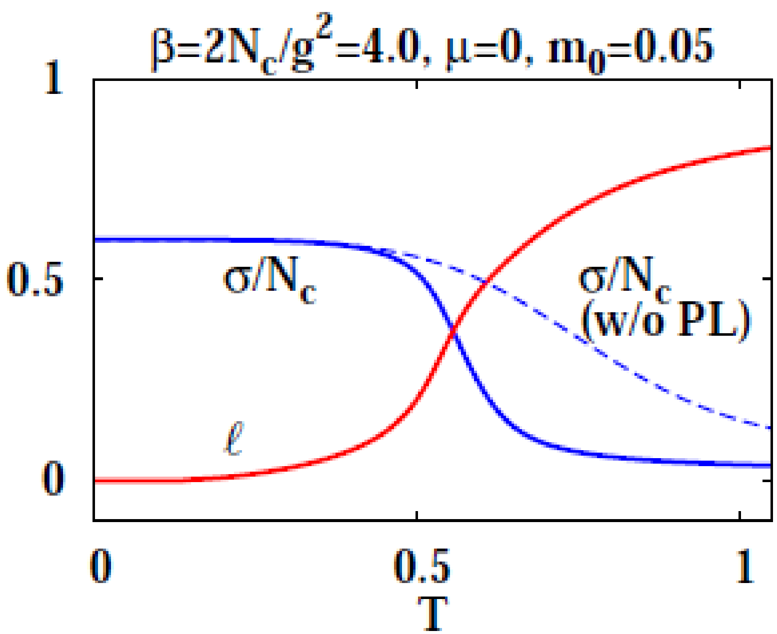
  MC data: SCL (Karsch et al. (MDP), de Forcrand, Fromm (MDP)), $N_\tau=2$ (de Forcrand, private), $N_\tau=4$ (Gottlieb et al.('87), Fodor-Katz ('02)), $N_\tau=8$ (Gavai et al.('90))

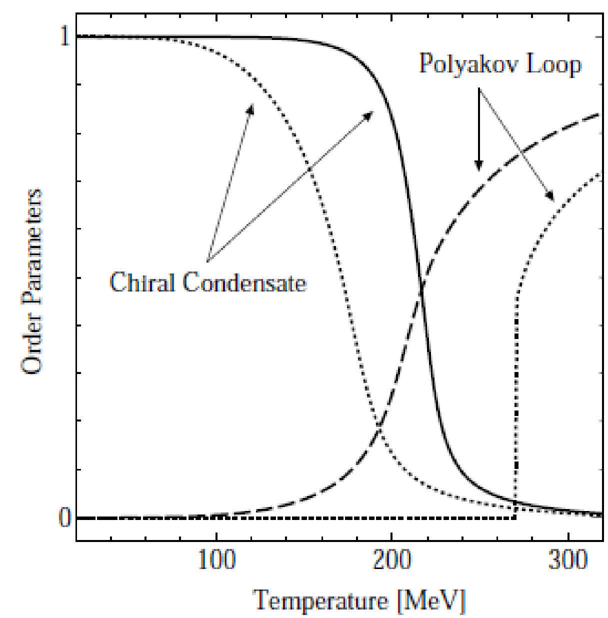




Lattice Unit

- **Comparison of Polyakov loop in SC-LQCD and PNJL**
  - **SC-LQCD: Tc decreases with Polyakov loop**
    (Polyakov loop deconfines hadrons)
  - **PNJL: Tc increases with Polyakov loop**
    (Polyakov loop confine quarks)



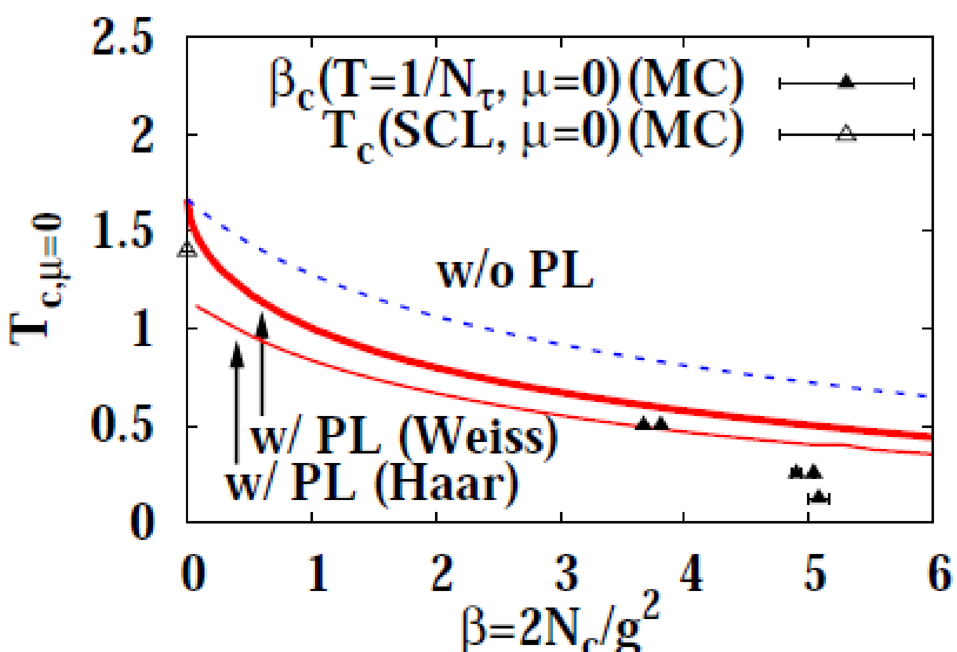


Fukushima ('04)

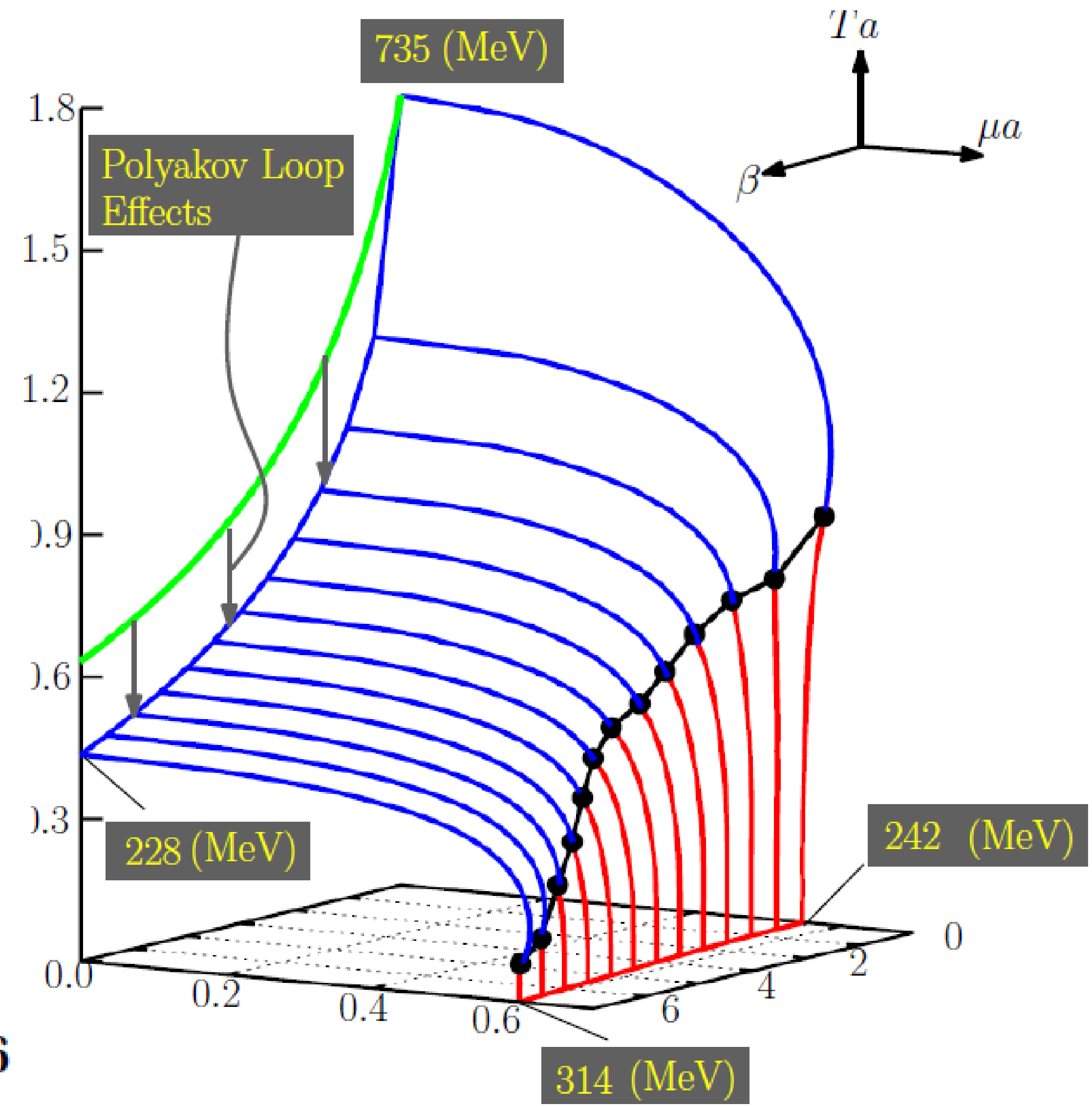
Finite Coupling and Polyakov Loop Effects

Miura, Nakano, AO ('09), Miura, Nakano, AO, Kawamoto ('09), Nakano, Miura, AO ('10), Nakano, Miura, AO, Kawamoto ('11)

- **Finite coupling & Pol. loop reduces T_c while μ_c is stable.**
  - **MC results of T_c at $\mu=0$ are explained at $\beta_g=2N_c/g^2 < 4$.**
  - **Compatible with empiricals.**



Nakano et al. ('11)



Miura et al., in prep.

Beyond the mean field approximation

- **Constant auxiliary field → Fluctuating auxiliary field**

$$S_{\text{eff}} = S_F^{(t)} + \sum_x m_x M_x + \frac{L^3}{4N_c} \sum_{\boldsymbol{k},\tau} f(\boldsymbol{k}) \left[|\sigma_{\boldsymbol{k},\tau}|^2 + |\pi_{\boldsymbol{k},\tau}|^2 \right]$$

$$m_x = m_0 + \frac{1}{4N_c} \sum_j \left((\sigma + i\varepsilon\pi)_{x+\hat{j}} + (\sigma + i\varepsilon\pi)_{x-\hat{j}} \right)$$

$$f(\boldsymbol{k}) = \sum_j \cos k_j \;\; <, \quad \varepsilon = (-1)^{x_0+x_1+x_2+x_3}$$

  - **Auxiliary fields can be integrated out using MC technique (Auxiliary Field Monte-Carlo (AFMC) method)**
    - **Another method: Monomer-Dimer-Polymer simulation**
      Mutter, Karsch ('89), de Forcrand, Fromm ('10), de Forcrand, Unger ('11)
  - **Bosonization of "repulsive" mode: Extended HS transf.**
    → Introducing " *i* " leads to the complex Fermion determinant.
    Miura, Nakano, AO (09), Miura, Nakano, AO, Kawamoto (09)
  - **Weight cancellation from low momentum modes is small, due to the ε factor.**

Origin of the sign problem in AFMC

- **Extended Hubbard-Stratonovich transformation**

Miura, Nakano, AO ('09), Miura, Nakano, AO, Kawamoto ('09)

$$e^{\alpha AB} = \int d\phi\, d\varphi\, e^{-\alpha\left[(\phi + (A+B)/2)^2 + (\varphi + i(A-B)/2)^2 - AB\right]}$$

$$= \int d\phi\, d\varphi\, e^{-\alpha\left[\phi^2 + \varphi^2 + \phi(A+B) + i\varphi(A-B)\right]}$$

Complex

We need "i" to bosonize product of different kind.
→ Fermion determinant becomes complex.

- **Bosonization in AFMC in the strong coupling limit**

$$\exp\left\{\alpha f(\mathrm{k})\left[M_{-\mathbf{k},\tau} M_{\mathbf{k},\tau} - M_{-\bar{\mathbf{k}},\tau} M_{\bar{\mathbf{k}},\tau}\right]\right\}$$

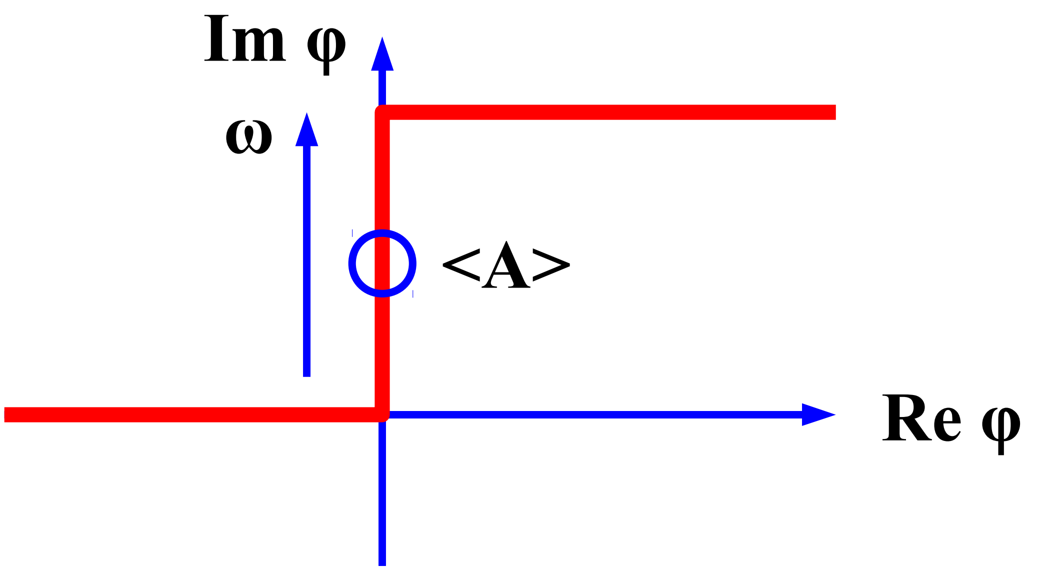
$$= \int d\sigma_{\mathbf{k},\tau}\, d\sigma^*_{\mathbf{k},\tau}\, d\pi_{\mathbf{k},\tau}\, d\pi^*_{\mathbf{k},\tau} \exp\left\{-\alpha f(\mathrm{k})\left[|\sigma_{\mathbf{k},\tau}|^2 + |\pi_{\mathbf{k},\tau}|^2 \right.\right.$$

$$\left.\left. + \sigma^*_{\mathbf{k},\tau} M_{\mathbf{k},\tau} + M_{-\mathbf{k},\tau}\sigma_{\mathbf{k},\tau} - i\pi^*_{\mathbf{k},\tau} M_{\bar{\mathbf{k}},\tau} - iM_{-\bar{\mathbf{k}},\tau}\pi_{\mathbf{k},\tau}\right]\right\}$$

Repulsive interaction in Mean Field Approximation

- **Mean field treatment of repulsive interaction**

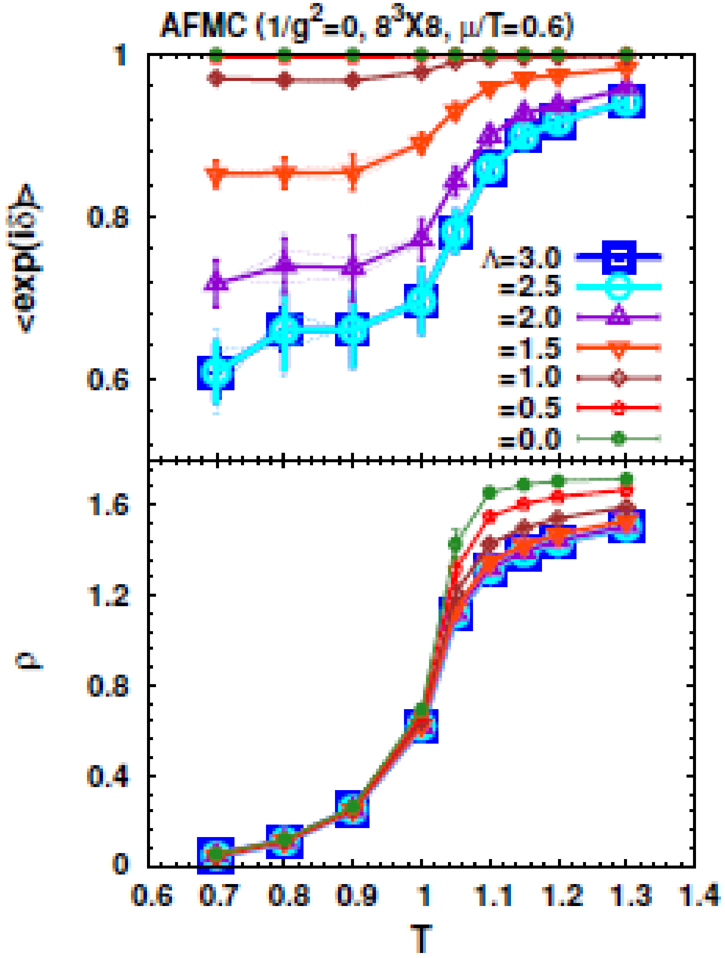
$$e^{-\alpha A^2} = \int d\varphi \exp\left(-\alpha\left[\varphi^2 + 2i\varphi A\right]\right)$$
$$\simeq \exp\left(\alpha\left[\omega^2 - 2\omega A\right]\right) \quad \left(\varphi = i\omega, \quad \omega = \langle A \rangle\right)$$

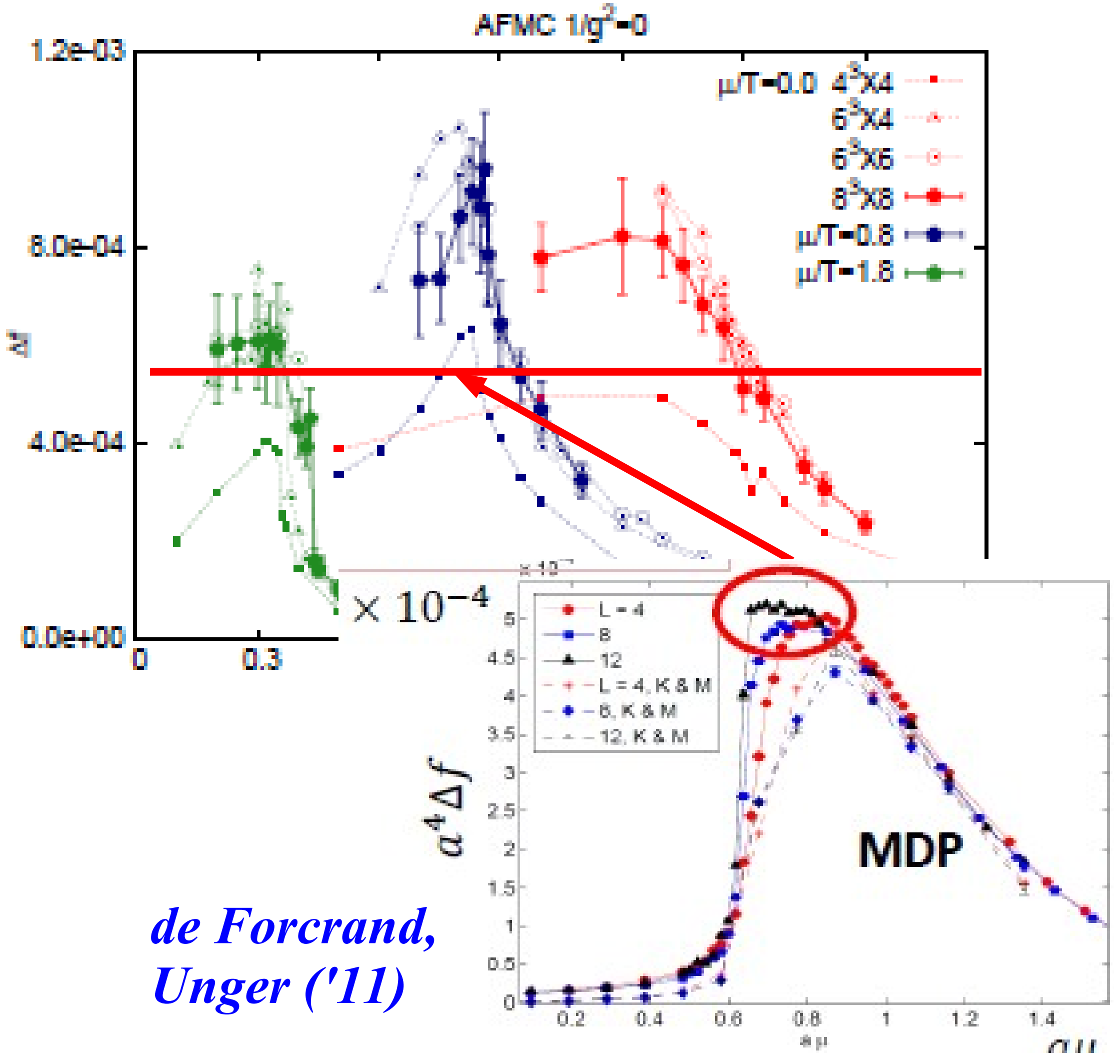


How serious is the weight cancellation ?

- **Statistical weight cancellation in AFMC**

$$\langle \exp(i\delta) \rangle \equiv \exp(-\Omega \Delta f) \ , \quad \Omega = \text{space-time volume}$$





de Forcrand, Unger ('11)

$$\text{cut-off} \quad \sum_j \sin^2 k_j > \Lambda$$

Ichihara, Nakano, AO, PoS Lattice2013 (2013), to appear

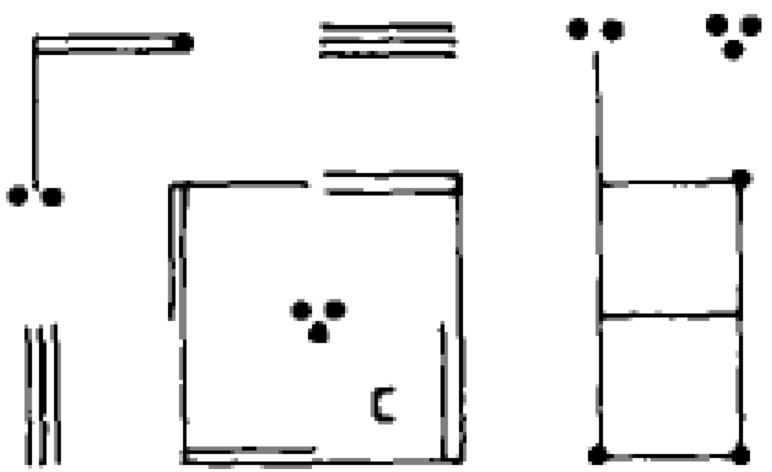
Monomer-Dimer-Polymer simulation

- **The partition function of LQCD can be given as the sum of monomer-dimer-polymer (MDP) configuration weight. The sign problem is mild.** *Karsch, Mutter ('89)*

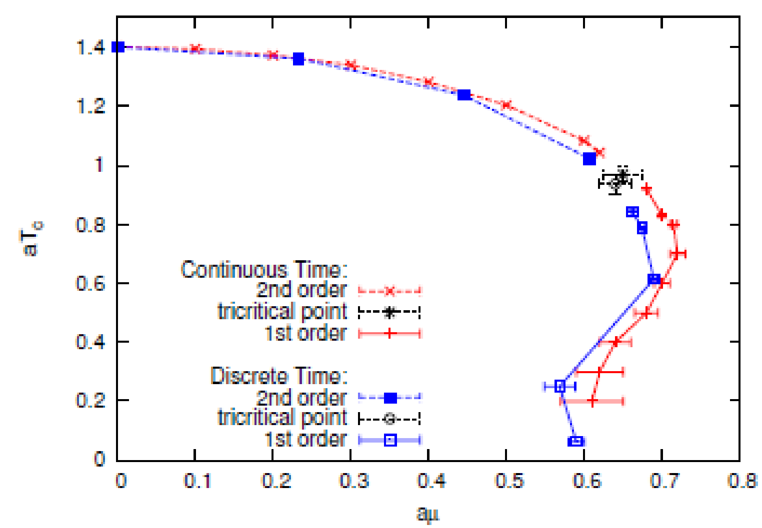
$$Z(2ma, \mu, r) = \sum_K w_K$$

$$w_K = (2ma)^{N_M} r^{2N_t} (1/3)^{N_1 N_2} \prod_x w(x) \prod_C w(C)$$

- **MDP with worm algorithm is applied to study the phase diagram**
de Forcrand, Fromm ('10), de Forcrand, Unger ('11)



Karsch, Mutter ('89)

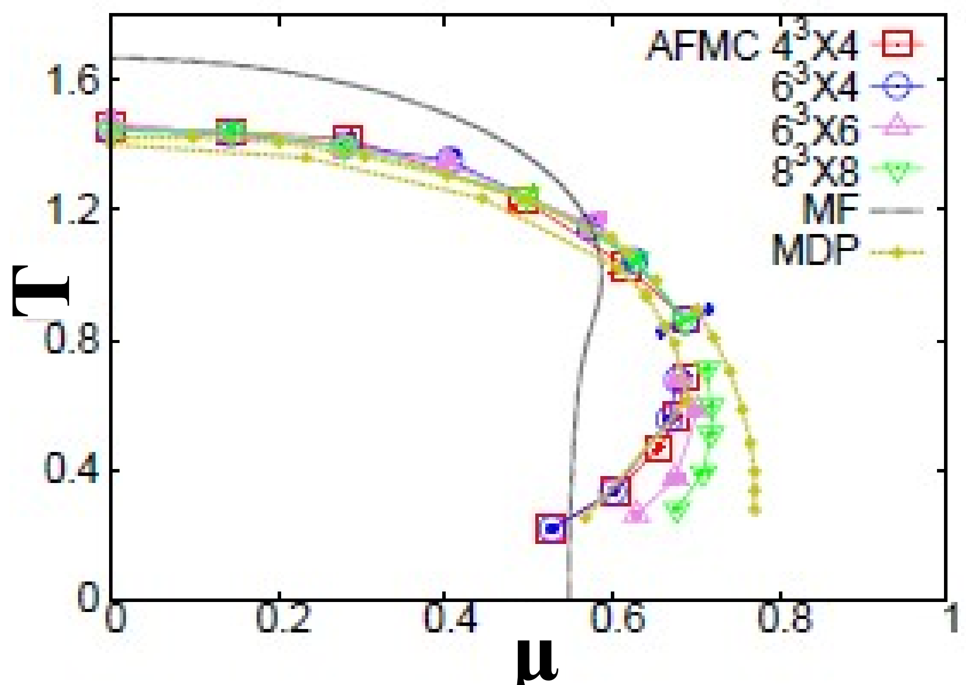


de Forcrand, Unger ('11)

Phase diagram

- **AFMC phase diagram**
  - **Reduction of T_c at μ=0 and enlarged hadron phase at medium T compared with the mean field results.**
  - **Quantitatively consistent with MDP simulation, if extrapolated to $N\tau \to \infty$**
    de Forcrand, Fromm ('09); de Forcrand, Unger ('11)
  - **Spatial size dependence is small.**

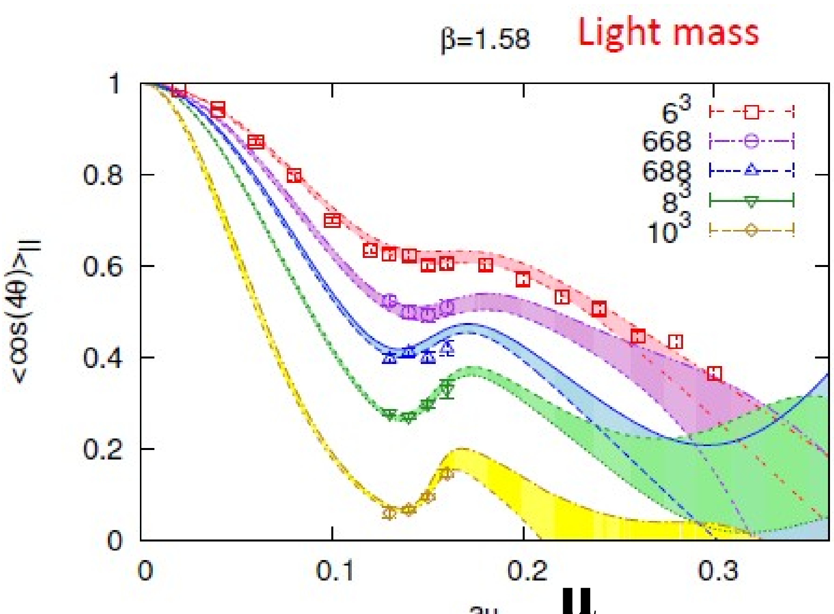
→ Close to the final answer to the phase boundary in the strong coupling limit !

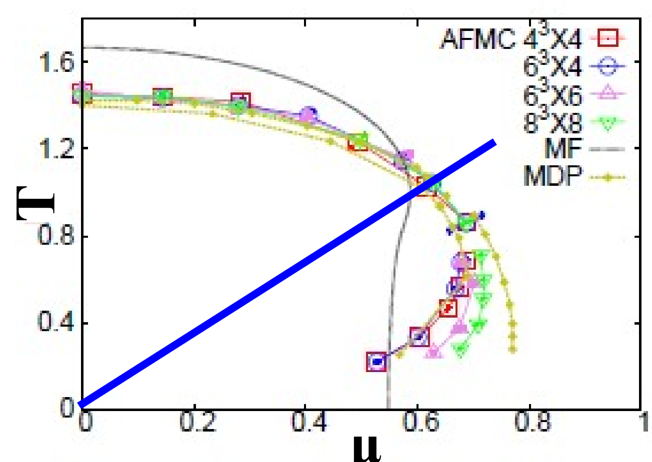


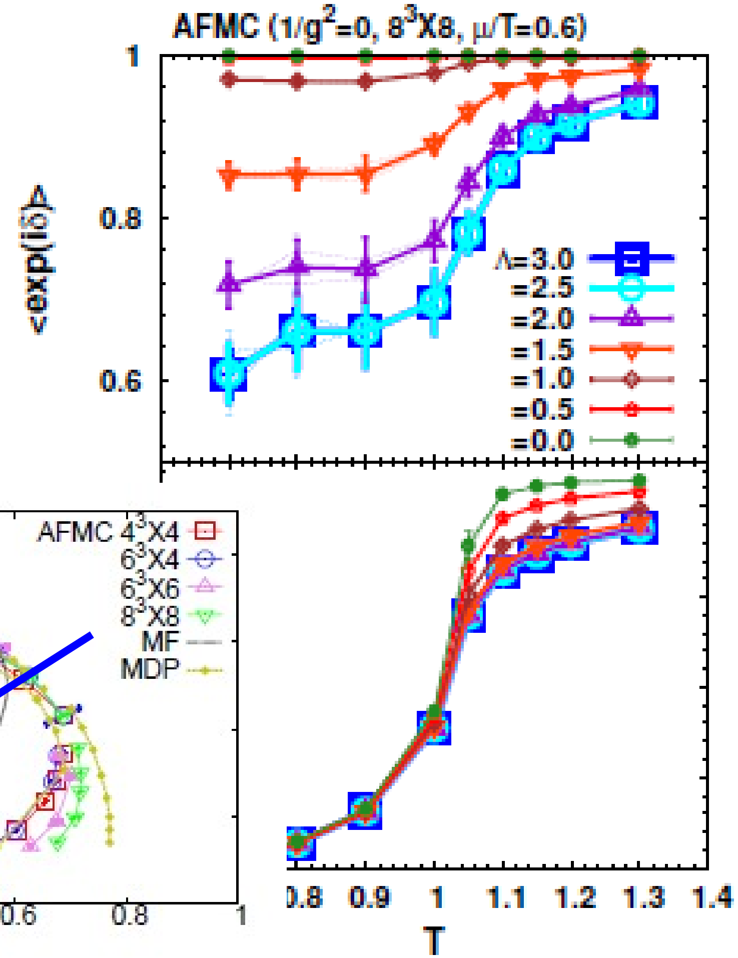
Ichihara, Nakano, AO, PoS Lattice2013 (2013), to appear

Discussion: Comparison with Brute Force Simulation

- **Lattice MC simulation at finite μ and finite β with Nf=4**
  Takeda et al. ('13)
  - **Ave. Phase Factor ~ 0.3 at $a\mu \sim 0.15$ (8^3 x 4, $a\mu_c = am_\pi/2 \sim 0.7$)**
- **AFMC**
  - **Ave. Phase Factor ~ 0.6 around the transition (8^4, SCL)**







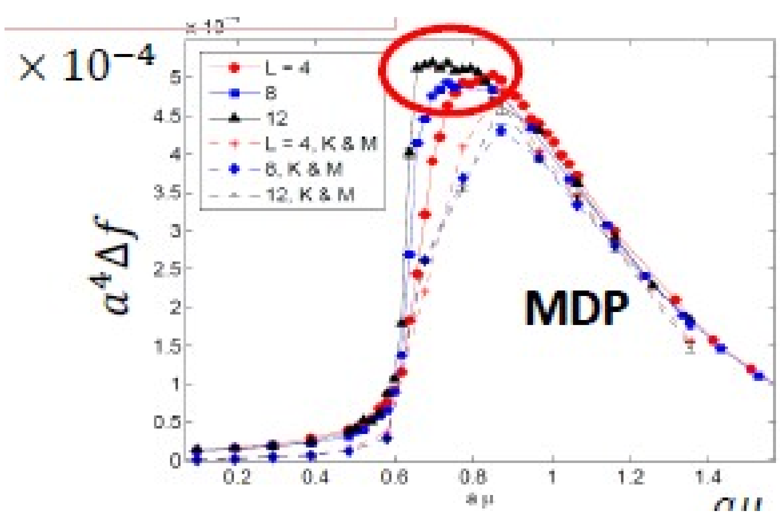
Takeda, Jin, Kuramashi, Y.Nakamura, Ukawa, Lattice 2013

$a\mu_c = am_\pi/2 \sim 0.7$

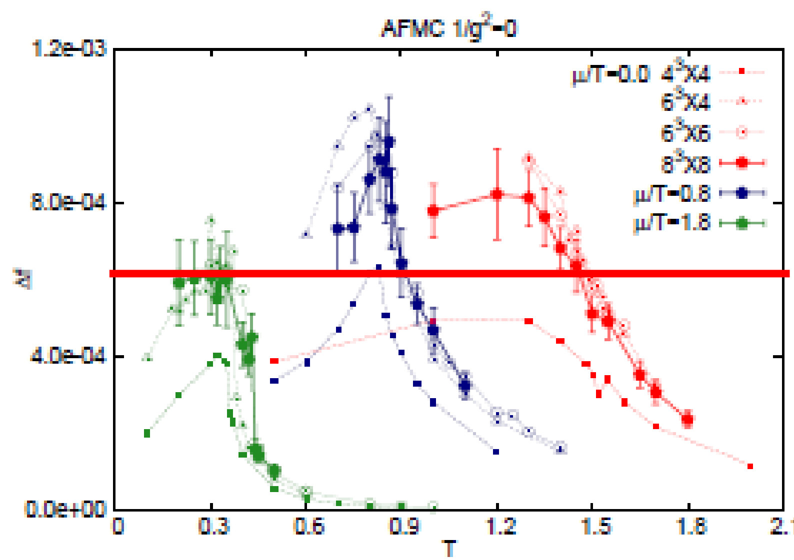
Ichihara, Nakano, AO, Lattice 2013

Discussion: Comparison with MDP

- **MDP simulation on anisotropic lattice at finite T and μ**
  de Forcrand, Fromm ('10), de Forcrand, Unger ('11)
  - **Strong coupling limit.**
    Including finite coupling effects is not straightforward.
  - **Includes higher-order terms in 1/d expansion**
    (spatial baryon hopping, meson-meson interaction)
  - **No sign problem in the continuous time limit ($N\tau \to \infty$).**



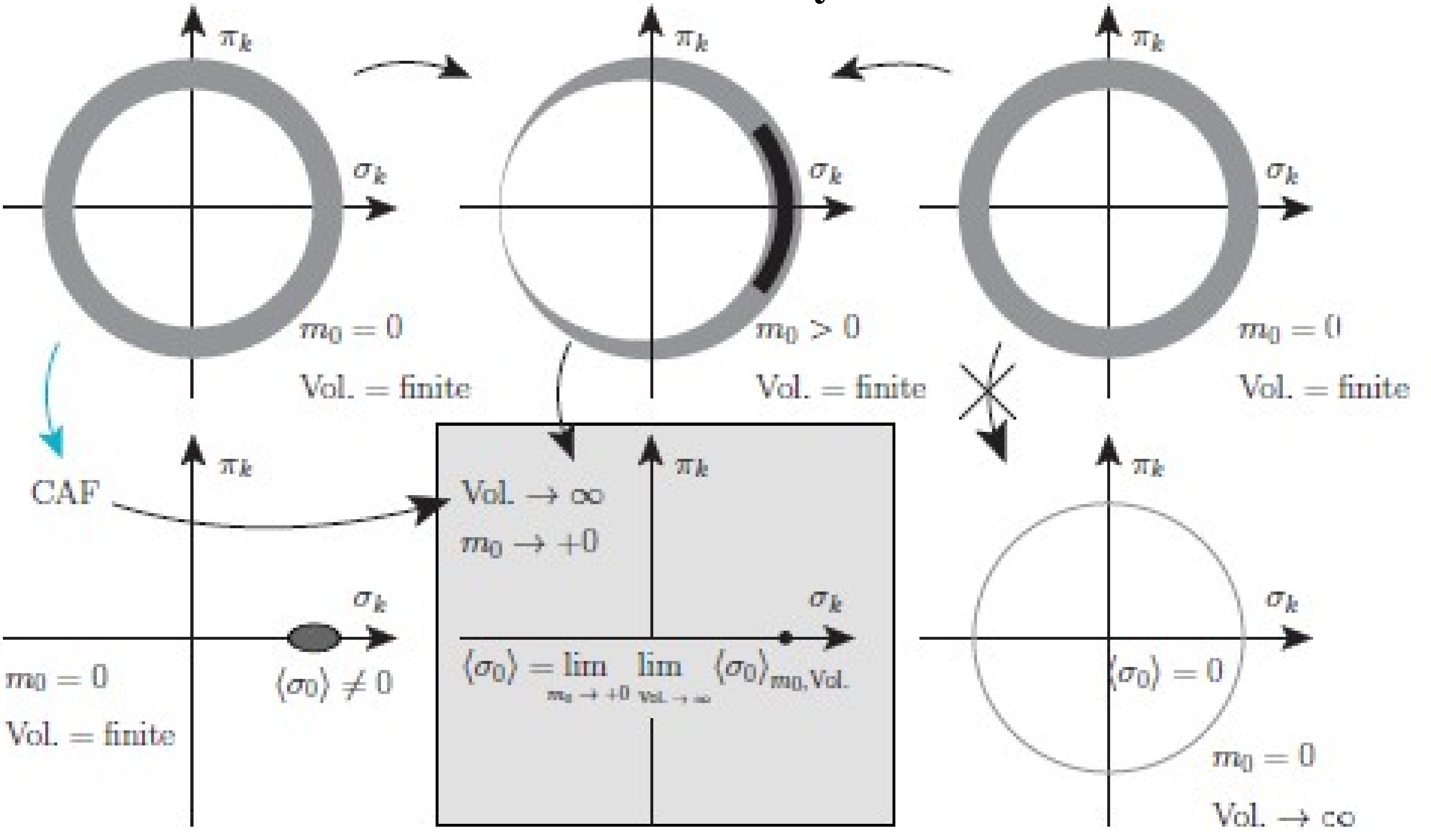
de Forcrand, Unger ('11)



Ichihara, Nakano, AO ('13)

Chiral Angle Fixing

How can we simulate correct thermodynamic chiral limit ?



Summary

- **Strong coupling lattice QCD is a promising tool in finite density lattice QCD.**
  - **Strong coupling limit + finite coupling correction + Polyakov loop → MC results of Tc is roughly reproduced.**
  - **Sign problem could be solved in the strong coupling limit. Two independent methods show the same phase boundary, and the spatial size dependence is small. (Monomer-dimer-polymer simulation, Auxiliary field MC)**
- **Challenge**
  - **Finite coupling + Fluctuations** *Unger et al. ('13)*
    Different type of Fermion
    Minimally doubled fermion, Misumi, Kimura, AO ('12)
    Higher order terms in 1/d expansion,
    ….