QCD effective potential with strong magnetic field

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Introduction

- Recently, QCD under strong magnetic fields attracts much attention.

- In particular, it is interesting question how the QCD vacuum and hadron properties are affected when the strength of the fields approach or exceed the QCD scale.
  - Chiral magnetic effect
  - (Inverse) Magnetic catalysis
  - Charged rho meson in strong magnetic field

- Such strong magnetic fields are generated in the relativistic heavy ion collisions.

- Lattice QCD can simulate strongly interacting quark and gluon systems in the presence of the strong magnetic fields.
In QCD under the strong magnetic field...

two kinds of strong dynamics coexist.

- Strongly interacting quark and gluon dynamics
- Non-linear QED dynamics with strong B field
Gluon couples to quark and gluon itself.

Magnetic field (photon) does not couple to gluon directly but interacts with quarks.

The effect of magnetic field must be reflected on QCD through the quark.

Quark propagator non-linearly interacting with photons and gluons
Using quark loop non-linearly interacting with gluon and photon, one can calculate effective Lagrangian for QCD+QED.
Recent Lattice simulation

\[ \langle -S_g \rangle = \langle + \frac{1}{4} F^2 \rangle \text{ increases with } eB \]

Euler-Heisenberg effective action

\[ S_{\text{eff}}^{(2,2)}(\mathcal{F}_{\mu\nu}; B) = -\frac{V_4}{180\pi^2} \frac{(qB)^2}{m^4} \left[ 3 \text{tr} B_\parallel^2 + \text{tr} B_\perp^2 + \text{tr} E_\perp^2 - \frac{5}{2} \text{tr} E_\parallel^2 \right] \]

Caution: Euler-Heisenberg effective action is basically the expansion of e*field/m, and in the current case this expansion obviously breaks down.

\[ \longrightarrow \text{ Full order calculation with respect to the fields is needed!} \]
QCD Lagrangian with electromagnetic fields

\[ \mathcal{L} = -\frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} - \frac{1}{4} f_{\mu \nu} f^{\mu \nu} + \bar{q}(i \gamma_\mu D^\mu - M_q)q \]

**Covariant derivative**

\[ D_\mu = \partial_\mu - ig A_\mu^a T^a - ie Q a_\mu \]

**Field strengths**

\[ F_{\mu \nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c \]

\[ f^{\mu \nu} = \partial^\mu a^\nu - \partial^\nu a^\mu \]

\[ \partial f = 0 : \text{constant fields} \]

**Charge and mass matrices**

\[ Q = \text{diag}(Q_{q_1}, Q_{q_2}, \cdots, Q_{q_f}) , \quad M_q = \text{diag}(m_{q_1}, m_{q_2}, \cdots, m_{q_f}) \]
Background field method

\[ A^a = \hat{A}^a + \mathcal{A}^a \]

\( \hat{A}^a \): Slowly varying classical background field

\( \mathcal{A}^a \): Quantum fluctuation

We apply the Covariantly-constant field as the background field.

\[ \hat{D}^{ab}_{\rho} \hat{F}^{b}_{\mu\nu} = 0 \quad \hat{D}^{ab}_{\rho} = \partial \delta^{ab} + g f^{acb} \hat{A}^c \]

\( \hat{F} \) is varying very slowly (\( \partial \hat{F} = 0 \))

\[ \hat{F}^a_{\mu\nu} = F_{\mu\nu} \hat{n}^a \quad \hat{n}^2 = 1 \]

\[ \hat{A}^a_{\mu} = A_{\mu} \hat{n}^a \]

Gauge fixing (background gauge)

\[ \hat{D}^{ab}_{\mu} \mathcal{A}^{b\mu} = 0 \]
Effective action for $\hat{A}$

\[
\exp\left[i S_{\text{eff}}(\hat{A}_\mu)\right] = \int \mathcal{D}A_\mu \mathcal{D}c \mathcal{D}\bar{c} \mathcal{D}q \mathcal{D}\bar{q} \exp\left\{ i \int d^4 x \left[ -\frac{1}{4} \left( \hat{F}_{\mu\nu}^a + (\hat{D}_\mu^a A_\nu^b - \hat{D}_\nu^b A_\mu^b) + gf^{abc} A_\mu^a A_\nu^b \right)^2 
- \frac{1}{2\xi} (\hat{D}_\mu^a A_\mu^b)^2 - \bar{c}^a (\hat{D}_\mu D_\mu)^{ac} c^c + \bar{q}(i\gamma_\mu \hat{D}^\mu - M_q) q + \bar{q}(ig\gamma_\mu A^{a\mu} \cdot T^a) q - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} \right] \right\}
\]

Functional integral for second order fluctuations with $\xi = 1$

**Gluon**

\[
\int \mathcal{D}A e^{i \int d^4 x \frac{1}{2} A_\mu^{a\mu} \{- (\hat{D}^2)^{ac} g_{\mu\nu} - 2gf^{abc} \hat{F}^b_{\mu}\} A^{c\nu}} = \det \left[ - (\hat{D}^2)^{ac} g_{\mu\nu} - 2gf^{abc} \hat{F}^b_{\mu}\right]^{-1/2}
\]

**Ghost**

\[
\int \mathcal{D}c \mathcal{D}\bar{c} e^{i \int d^4 x \bar{c} \left[ - (\hat{D}^2)^{ac}\right] c} = \det \left[ - (\hat{D}^2)^{ac}\right]^{+1}
\]

**Quark**

\[
\int \mathcal{D}q \mathcal{D}\bar{q} e^{i \int d^4 x \bar{q}(i\gamma_\mu \hat{D}^\mu - M_q) q} = \det \left[ i\gamma_\mu \hat{D}^\mu - M_q \right]
\]

The results are well known.

This work
The quark contribution to the effective action

\[ i \Delta S_q = \log \det [i \gamma_\mu \hat{D}^\mu - M_q] \]

\[ \hat{D}^\mu = \partial^\mu - igA^\mu a^T a^T - i e Q a^H \]

### Diagonalization in color space

**SU(2)**

\[
\begin{pmatrix}
  w_1 & 0 \\
  0 & w_2 \\
\end{pmatrix}
\]

**SU(3)**

\[
\begin{pmatrix}
  w_1 & 0 & 0 \\
  0 & w_2 & 0 \\
  0 & 0 & w_3 \\
\end{pmatrix}
\]

\[ w_1 = \frac{1}{\sqrt{3}} \cos \left( \Theta + \frac{\pi}{6} \right), \quad w_2 = -\frac{1}{\sqrt{3}} \cos \left( \Theta - \frac{\pi}{6} \right), \quad w_3 = \frac{1}{\sqrt{3}} \sin (\Theta) \]

\[ \sin^2 3\Theta = 3C_2, \quad C_2 = \left[ d^{abc} \hat{n}^a \hat{n}^b \hat{n}^c \right]^2 \]

\[ \sum_{a=1}^{N_c} w_a^2 = \frac{1}{2}, \quad \sum_{a=1}^{N_c} w_a = 0 \]


We apply the Schwinger’s proper time method to evaluate the effective potential.

Performing the proper time integral, we derive the analytic expression of the effective potential of quark part.

In order to focus on the chromo-magnetic field, we employ the pure chromo-magnetic background for the gluon field.
Effective potential for quark part

\[ V_q = V_{q^{fin}} + V_{q^{div}} \]

\[ H_c = \sqrt{\vec{H}_c^2}, \quad B = \sqrt{\vec{B}^2} \]

\[ V_{q^{fin}} = \sum_{a=1}^{N_c} \sum_{i=1}^{N_f} \left\{ -\frac{a_{a,i}^2}{24\pi^2} \left( \log(2a_{a,i}) + 12\zeta'(-1, \frac{m_{q_i}^2}{2a_{a,i}}) - 1 \right) 
+ \frac{m_{q_i}^2 a_{a,i}}{8\pi^2} \log\left(\frac{2a_{a,i}}{m_{q_i}^2}\right) - \frac{m_{q_i}^4}{16\pi^4} \left( \log\left(\frac{2a_{a,i}}{m_{q_i}^2}\right) + \frac{1}{2} \right) \right\} \]

\[ V_{q^{div}} = \frac{N_f}{48\pi^2} (gH_c)^2 \log\Lambda^2 + \frac{N_c}{24\pi^2} \left( \sum_{i=1}^{N_f} Q_{q_i}^2 \right)^2 (eB)^2 \log\Lambda^2 \]

\[ a_{a,i} = \sqrt{\left( gw_a \vec{H}_c + eQ_{q_i} \vec{B} \right)^2} = \sqrt{g^2w_a^2 H_c^2 + e^2 Q_{q_i}^2 B^2 + 2gw_a eQ_{q_i} H_c B \cos\theta_{HB}} \]
\[ \tilde{B} \rightarrow \tilde{B} \]

Color $SU(2)$ case with $Q_q = 1$

\[ \tilde{H}_c \]

\[ \tilde{B} \rightarrow \begin{array}{c} q_{w_1}(e) \\ q_{w_2}(e) \end{array} \]
Color $SU(2)$ case with $Q_q = 1$
Gluon + ghost part effective potential

Real part

\[ ReV_g = V_{g}^{fin} + V_{g}^{div} \]

\[ V_{g}^{fin} = \frac{11N_c}{96\pi^2} (gH_c)^2 \left\{ \log(gH_c) - c_g + \frac{1}{N_c} \sum_{a=1}^{N_c} \lambda_a^2 \log \lambda_a^2 \right\} \]

\[ V_{g}^{div} = -\frac{11N_c}{96\pi^2} (gH_c)^2 \log \Lambda^2 \]

Color charges

\[ SU(2) \]
\[ \lambda_1 = +1, \lambda_2 = -1 \]

\[ SU(3) \]
\[ \lambda_1^2 = \frac{1}{2} \left[ 1 - \cos \left( 2\Theta - \frac{\pi}{3} \right) \right], \lambda_2^2 = \frac{1}{2} \left[ 1 - \cos \left( 2\Theta + \frac{\pi}{3} \right) \right], \lambda_3^2 = \frac{1}{2} \left[ 1 + \cos \left( 2\Theta \right) \right] \]

Imaginary part

\[ ImV_g = -\frac{N_c}{16\pi^2} (gH_c)^2 \]: Nilesen-Olesen instability

N. Nielsen and Olesen, Nucl. Phys. B144 (1978)
Logarithmic divergences and renormalization

\[ V^{\text{div}} = \frac{1}{2} \left\{ -\frac{1}{(4\pi)^2} \left[ \frac{11}{3} N_c - \frac{2}{3} N_f \right] \right\} (gH_c)^2 \log \Lambda^2 + \frac{1}{2} \frac{N_c}{12\pi^2} \left( \sum_{i=1}^{N_f} Q_{qi}^2 \right) (eB)^2 \log \Lambda^2 \]

replacing the couplings and field in the potential by bare couplings \( g_0, e_0 \) and fields \( H_0, B_0 \)

\[ V_{\text{eff}} = \frac{H_0^2}{2} + \frac{B_0^2}{2} + V_0^{\text{div}} + V_0^{\text{fin}} \]

and rescale the couplings and fields as

\[ H_0 = Z_H^{1/2} H_C, \quad B_0 = Z_B^{1/2} B \]

\[ g = Z_H^{1/2} g_0, \quad e = Z_B^{1/2} e_0 \]

The rescale factors are given by

\[ Z_H = 1 + \delta_H, \quad Z_B = 1 + \delta_B \]
Using renormalized couplings $g, e$ and fields $H_c, B$ we can write the effective potential as

$$V_{eff} = \frac{1}{2} Z_H H_c^2 + \frac{1}{2} Z_B B^2 + V^{div} + V^{fin}$$

Introducing the counterterms so that log divergences cancel

$$\delta_H = \frac{g^2}{(4\pi)^2} \left[ \frac{11}{3} N_c - \frac{2}{3} N_f \right] \log \left( \frac{\Lambda^2}{\mu^2} \right)$$

$$\delta_B = -\frac{N_e e^2}{12\pi^2} \left( \sum_{i=1}^{N_f} Q_{q_i}^2 \right) \log \left( \frac{\Lambda^2}{\mu^2} \right)$$

we finally get renormalized effective potential

$$V_{eff} = \frac{H_c^2}{2} + \frac{B^2}{2} + V^{fin}$$

Furthermore, we can calculate the beta functions of QCD and QED

$$\beta_{QCD} = \mu \frac{\partial g}{\partial \mu} = \frac{1}{2} g \mu \frac{\partial Z_H}{\partial \mu} = -\frac{g^3}{(4\pi)^2} \left[ \frac{11}{3} N_c - \frac{2}{3} N_f \right]$$

$$\beta_{QED} = \mu \frac{\partial e}{\partial \mu} = \frac{1}{2} e \mu \frac{\partial Z_B}{\partial \mu} = +\frac{N_e e^3}{12\pi^2} \left( \sum_{i=1}^{N_f} Q_{q_i}^2 \right)$$

\[\text{correct one-loop } \beta \text{ functions}\]
We investigate the magnetic field dependence of the QCD effective potential.

In this study, we consider the color SU(3) case with the three flavor (u,d,s).

We use the following parameters

- $Q_u = +\frac{2}{3}$, $Q_d = Q_s = -\frac{1}{3}$
- $m_u = m_d = 5 \text{ MeV}$, $m_s = 140 \text{ MeV}$
- $\alpha_s = 1$, $\alpha_{EM} = \frac{1}{137}$, $\mu = 1 \text{ GeV}$
Chromo-magnetic fields prefer to be parallel (or anti-parallell) to the external magnetic field, which is consistent with recent lattice results.

\[ H_{c\parallel} > H_{c\perp} \]
Chromo-magnetic fields prefer to be parallel (or anti-parallel) to the external magnetic field, which is consistent with recent lattice results.\[ H_c^\parallel > H_c^\perp \]
The one-loop YM effective potential $H_c^2/2 + V_{YM}$ has a minimum away from the origin, which corresponds to the dynamical generation of the chromomagnetic condensate.

This result is qualitatively in agreement with LQCD and FRG analyses.

Quark loop contributions attenuate the gluonic contributions.

How the condensate behaves in the presence of the magnetic field?
QCD effective potential with finite magnetic fields

We defined the normalized potential:

\[ \bar{V}(H_c, B) = V(H_c, B) - V(0, B) \]

As the magnetic field increases, the minimum shift to the right hand side.

The chromo-magnetic condensate increases with an increasing magnetic field.

This behavior is quite similar to the recent observed gluonic magnetic catalysis in lattice QCD.
In the massless limit of the quark $m_q \to 0$, one can obtain the analytic expression of $(gH_c)^2_{min}$ with $eB = 0$:

$$(gH_c)^2_{\text{min}, 0} = \mu^4 \exp \left\{ -\frac{8\pi}{b_0 \alpha_s} - 1 + \frac{2}{b_0} \left( 11N_c c_g - \frac{2N_f}{3} c_q \right) \right\}, \quad b_0 = \frac{11N_c}{3} - \frac{2N_f}{3}$$

where $c_g$ and $c_q$ are some constants.

In the small $eB$ region, $(gH_c)_{\text{min}, 0} \gg eB$, we find

$$(gH_c)^2_{\text{min}} = (gH_c)^2_{\text{min}, 0} + \frac{(4\pi)^2}{b_0} \frac{N_c}{12\pi^2} \left( \sum_{i=1}^{N_f} Q_{q_i}^2 \right) (eB)^2$$

Note that the coefficient of the second term is the ratio of the coefficients of $\beta_{\text{QCD}}$ and $\beta_{\text{QED}}$.

In the large $eB$ region, $eB > (gH_c)_{\text{min}}$, $(gH_c)^2_{\text{min}}$ still monotonically increases as the magnetic field increases.
In our results, quark loop contributions should be important, since only $V_q$ has $B$-dependence.

To see the importance, we define the following quantity

$$\Delta \bar{V}(H_c, B) = \bar{V}(H_c, B) - \bar{V}(H_c, 0) = V_q(H_c, B) - V_q(0, B) - V_q(H_c, 0) = \frac{i}{\int d^4x} \log \left[ \frac{\det(i\hat{\mathcal{D}}(H_c, B) - M_q)}{\det(i\hat{\mathcal{D}}(H_c, 0) - M_q) \det(i\hat{\mathcal{D}}(0, B) - M_q)} \right].$$

- $\Delta \bar{V}$ is negative in the whole region of $gH_c$-eB plane.
- $\Delta \bar{V}$ is monotonically decreasing as either $gH_c$ or eB increases.
Using the definition of $\Delta \bar{V}$, we can rewrite the normalized effective potential

$$\bar{V}(H_c, B) = \bar{V}(H_c, 0) + \Delta \bar{V}(H_c, B)$$

$$= \frac{H_c^2}{2} + V_{YM} + [V_q(H_c, 0) + \Delta \bar{V}(H_c, B)]$$

$V_q(H_c, 0)$: B-independent part of quark loop which attenuates the gluonic contributions.

$\Delta \bar{V}(H_c, B)$: B-dependent part of quark loop which enhances the gluonic contributions.

Thanks to the property of the B-dep. part of the quark loop, $(gH_c)^2_{min}$ monotonically increases with an increasing magnetic field.

This property of the quark loop supports the gluonic magnetic catalysis at zero temperature, observed in current lattice data.
Summary

- We derive the analytic expression of the one-lop QCD effective potential with the magnetic field.

- After the renormalization of couplings and fields, we obtain the correct $\beta$-functions of both QCD and QED.

- Our result shows that the chromo-magnetic field prefers to be parallel (or anti-parallel) to the external magnetic field, which is consistent with recent lattice results.

- Quark loop contributions with magnetic fields enhance the gluonic contributions and thus the chromo-magnetic condensate monotonically increase with an increasing magnetic field.

  This result supports the gluonic magnetic catalysis at zero temperature, observed in current lattice data.