Introduction to QCD and Jet

Bo-Wen Xiao

Institute of Particle Physics, Central China Normal University

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Overview of the Lectures

- Lecture 1 - Introduction to QCD and Jet
  - QCD basics
  - Sterman-Weinberg Jet in e^+e^- annihilation and Other Jet Observables
  - Collinear Factorization and DGLAP equation
- Lecture 2 - Saturation Physics (Color Glass Condensate)
  - BFKL equation
  - Non-linear small-x evolution equations
  - One loop calculations and Sudakov factors

References:

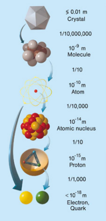
- R.D. Field, Applications of perturbative QCD A lot of detailed examples.
- R. K. Ellis, W. J. Stirling and B. R. Webber, QCD and Collider Physics
- CTEQ, Handbook of Perturbative QCD
- CTEQ website.
- John Collins, The Foundation of Perturbative QCD Includes a lot new development.
- Yu. L. Dokshitzer, V. A. Khoze, A. H. Mueller and S. I. Troyan, Basics of Perturbative QCD More advanced discussion on the small-x physics.
- S. Donnachie, G. Dosch, P. Landshoff and O. Nachtmann, Pomeron Physics and QCD
- V. Barone and E. Predazzi, High-Energy Particle Diffraction
- Y. Kovchegov and E. Levin, Quantum Chromodynamics at High Energy

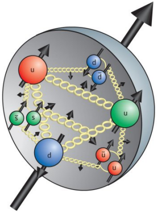
QCD Lagrangian

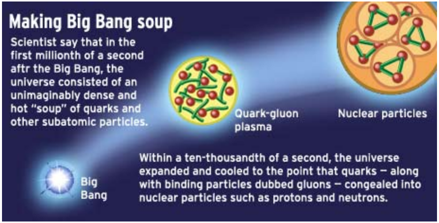
$$L=\bar{\psi}(i\gamma\cdot\partial-m_q)\psi-\frac{1}{4}F^{\mu\nu a}F_{\mu\nu a}-g_s\bar{\psi}\gamma\cdot A\psi$$

with $F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu-gf_{abc}A^b_\mu A^c_\nu$.

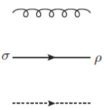
- Non-Abelian gauge field theory. Lagrangian is invariant under SU(3) gauge transformation.
- Basic elements:
  - Quark Ψ^i with 3 colors, 6 flavors and spin $1/2$.
  - Gluon $A^{a\mu}$ with 8 colors and spin 1.







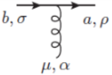
QCD Feynman Rules



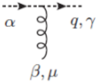
$$\left[-g_{\mu\nu}+(1-\xi)\frac{p_\mu p_\nu}{p^2+i0}\right]\frac{i}{p^2+i0}$$

$$\frac{i(\not p+m_f)_{\rho\sigma}}{p^2-m_f^2+i0}$$

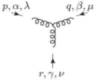
$$\frac{i}{p^2+i0}$$



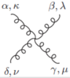
$$-ig\mu^\epsilon(t^\alpha)_{ab}\gamma^\mu_{\rho\sigma}$$



$$-g\mu^\epsilon f_{\alpha\beta\gamma}q^\mu$$



$$-g\mu^\epsilon f_{\alpha\beta\gamma}\left[(p-q)^\nu g^{\lambda\mu}+(q-r)^\lambda g^{\mu\nu}+(r-p)^\mu g^{\nu\lambda}\right]$$



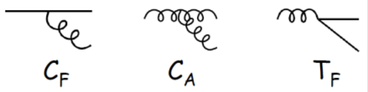
$$\begin{aligned}&-ig^2\mu^{2\epsilon}f_{\epsilon\alpha\beta}f_{\epsilon\gamma\delta}(g^{\kappa\mu}g^{\lambda\nu}-g^{\kappa\nu}g^{\lambda\mu})\\&-ig^2\mu^{2\epsilon}f_{\epsilon\alpha\gamma}f_{\epsilon\beta\delta}(g^{\kappa\lambda}g^{\mu\nu}-g^{\kappa\nu}g^{\lambda\mu})\\&-ig^2\mu^{2\epsilon}f_{\epsilon\alpha\delta}f_{\epsilon\beta\gamma}(g^{\kappa\lambda}g^{\mu\nu}-g^{\kappa\mu}g^{\lambda\nu})\end{aligned}$$

Color Structure

Fundamental representation: T^a_{ij} and Adjoint representation: $t^a_{bc} = -if_{abc}$

The effective color charge:

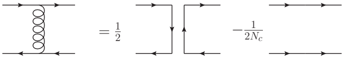
- $\left[T^a, T^b\right] = if^{abc}T^c$
- $\text{Tr}\left(T^aT^b\right) = T_F\delta^{ab}$
- $T^aT^a = C_F \times 1$
- $f^{abc}f^{abd} = C_A\delta^{cd}$



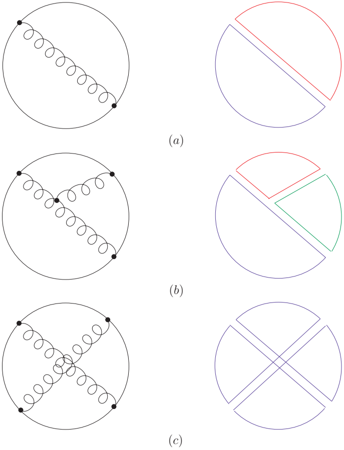
Symbol	SU(n)	SU(3)
T_F	$\frac{1}{2}$	$\frac{1}{2}$
C_F	$\frac{n^2-1}{2n}$	$\frac{4}{3}$
C_A	n	3

Fierz identity and Large N_c limit

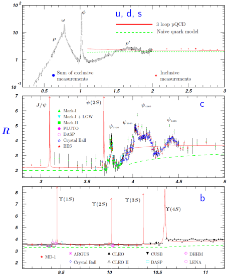
- Fierz identity: $T^a_{ij}T^a_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N_c}\delta_{ij}\delta_{kl}$



- Large N_c limit: $3 \gg 1$



Evidence for colors



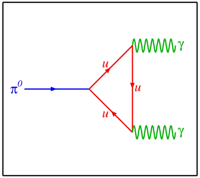
- The ratio between the $e^+e^- \to$ hadrons total cross section and the $e^+e^- \to \mu^+\mu^-$ cross section.

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_{u,d,s,\dots} e_i^2 \left[1 + \frac{\alpha_s(Q^2)}{\pi}\right]$$

  - $N_c \sum_{u,d,s} e_i^2 = 2$
  - $N_c \sum_{u,d,s,c} e_i^2 = \frac{10}{3}$
  - $N_c \sum_{u,d,s,c,b} e_i^2 = \frac{11}{3}$.

Triangle anomaly:

π^0 Decay



- The decay rate is given by the quark triangle loop:

$$\Gamma\left(\pi^0 \to \gamma\gamma\right) = N_c^2 \left(e_u^2 - e_d^2\right)^2 \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2} = 7.7\text{eV}$$

  - $f_\pi = 92.4$MeV is $\pi^- \to \mu^-\nu$ decay constant.
  - The data give $\Gamma\left(\pi^0 \to \gamma\gamma\right) = 7.7 \pm 0.6$eV.
  - Nonrenormalization of the anomaly. [Adler, Bardeen, 69]

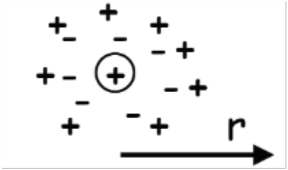
QCD beta function and running coupling

[Gross, Wilczek and Politzer, 73]

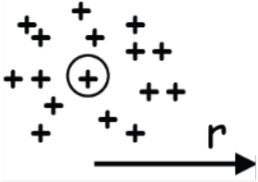
- The QCD running coupling

$$\alpha_s(Q) = \frac{2\pi}{\left(\frac{11}{6}N_c - \frac{2}{3}T_F n_f\right)\ln Q^2/\Lambda^2}$$

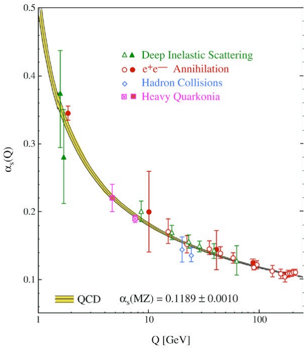
- QED has only fermion loop contributions, thus its coupling runs in opposite direction.



QED like contribution



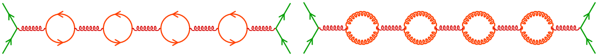
gluon contribution



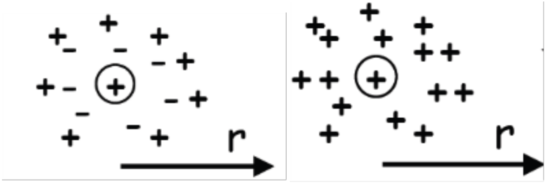
QCD beta function and running coupling

The QCD running coupling

$$\alpha_s(Q) = \frac{2\pi}{\left(\frac{11}{6}N_c - \frac{2}{3}T_F n_f\right)\ln Q^2/\Lambda^2}$$



Screening Anti-Screening

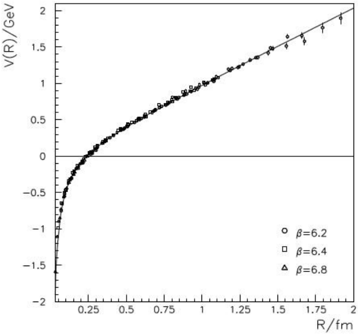


Quark loop QED like contribution Non-Abelian gluon contribution

Brief History of QCD beta function

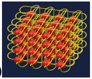
- 1954 Yang and Mills introduced the non-Abelian gauge thoery.
- 1965 Vanyashin and Terentyev calculated the beta function for a massive charged vector field theory.
- 1971 ’t Hooft computed the one-loop beta function for SU(3) gauge theory, but his advisor (Veltman) told him it wasn’t interesting.
- 1972 Gell-Mann proposed that strong interaction is described by SU(3) gauge theory, namely QCD.
- 1973 Gross and Wilczek, and independently Politzer, computed the 1-loop beta-function for QCD.
- 1999 ’t Hooft and Veltman received the 1999 Nobel Prize for proving the renormalizability of QCD.
- 2004 Gross, Wilczek and Politzer received the Nobel Prize.

Confinement

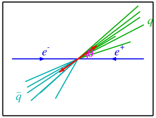


- Non-perturbative QCD. Mass gap between gluon and hadrons. Millennium Prize Problem!
- Linear potential ⇒ constant force.
- Intuitively, confinement is due to the force-carrying gluons having color charge, as compared to photon which does not carry electric charge.
- Color singlet hadrons : no free quarks and gluons in nature

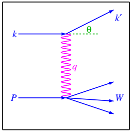
How to test QCD ?

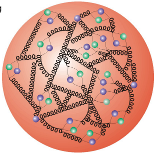
- Non-perturbative part:
  - Hadron mass (Lattice QCD) 
  - Parton distributions (No free partons in the initial state)
  - Fragmentation function (No free quarks and gluons in the final state)
- Perturbative QCD: needs to have Factorization to separate the short distances (perturbative) physics from the long distance (non perturbative) physics.
  - e^+e^- annihilation.
  - Deep inelastic scattering.
  - Hadron-hadron collisions, such as Drell-Yan processes.

$e^+e^- \to$ Hadrons

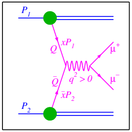


Kinematics of Lepton-Nucleon Scattering





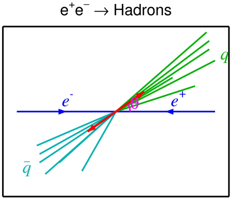
Drell-Yan Process

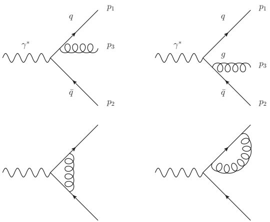


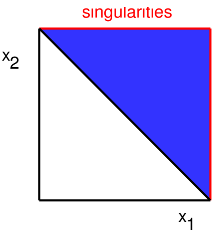
- Collinear factorization demonstrates that collinear parton distribution and fragmentation function are universal.

Outline

e^+e^- annihilation







- Born diagram () gives $\sigma_0 = \alpha_{em}\sqrt{s}N_c \sum_q e_q^2 \left(\frac{4\pi\mu^2}{s}\right)^{\epsilon} \frac{\Gamma[2-\epsilon]}{\Gamma[2-2\epsilon]}$
- NLO: real contribution (3 body final state). $x_i \equiv \frac{2E_i}{Q}$ with $Q=\sqrt{s}$

$$\frac{d\sigma_3}{dx_1 dx_2} = C_F \frac{\alpha_s}{2\pi}\sigma_0 \frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

$$\text{with} \quad \frac{1}{(1-x_1)(1-x_2)} = \frac{1}{x_3}\left[\frac{1}{(1-x_1)}+\frac{1}{(1-x_2)}\right]$$

  - Energy conservation $\Rightarrow x_1+x_2+x_3=2.$
  - $(p_1+p_3)^2 = 2p_1\cdot p_3 = (Q-p_2)^2 = Q^2(1-x_2)$
  - $x_2 \to 1 \Rightarrow \vec{p}_3 \,||\, \vec{p}_1 \Rightarrow$ Collinear Divergence (Similarly $x_1 \to 1$)
  - $x_3 \to 0 \Rightarrow$ Soft Divergence.

Dimensional Regularization

To generate a finite contribution to the total cross section, use the standard procedure dimensional regularization:

- Analytically continue in the number of dimensions from $d = 4$ to $d = 4 - 2\epsilon$.
- Convert the soft and collinear divergence into poles in ϵ.
- To keep g_s dimensionless, substitue $g_s \to g_s \mu^\epsilon$ with renormalization scale μ.

At the end of the day, one finds

$$\sigma_r = \sigma_0 \frac{\alpha_s(\mu)}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma[1-\epsilon]}{\Gamma[1-2\epsilon]} \left[\frac{2}{\epsilon^2} + \frac{3}{\epsilon} + \frac{19}{2} - \frac{2\pi^2}{3}\right]$$

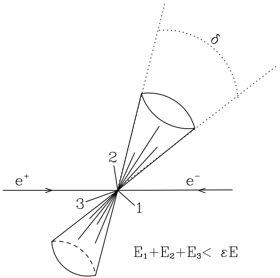
$$\sigma_v = \sigma_0 \frac{\alpha_s(\mu)}{2\pi} C_F \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\epsilon} \frac{\Gamma[1-\epsilon]}{\Gamma[1-2\epsilon]} \left[-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{2\pi^2}{3}\right]$$

and the sum $\lim_{\epsilon \to 0} \sigma = \sigma_0 \left(1 + \frac{3}{4} C_F \frac{\alpha_s(\mu)}{\pi} + \mathcal{O}(\alpha_s^2)\right)$.

- Cancellation between real and virtual for total cross section. Bloch-Nordsieck theorem
- For more exclusive observables, the cancellation is not always complete. One needs to do subtractions of $\frac{1}{\epsilon} + \ln 4\pi - \gamma_E$ ($\overline{\text{MS}}$ scheme).
- Sterman-Weinberg Jets.

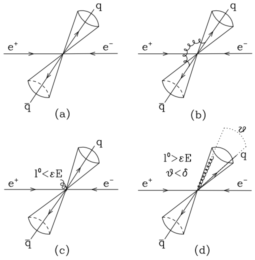
Sterman-Weinberg Jets

Definition: We define, an event contributes if we can find two cones of opening angle δ that contain all of the energy of the event, excluding at most a fraction ϵ of the total, as the production of a pair of Sterman Weinberg jets.



- Jets in experiments are defined as a collimated distribution of hadrons with total energy E within the jet cone size $R \equiv \sqrt{\delta\phi^2 + \delta\eta^2}$.
- Jets in QCD theory are defined as a collimated distribution of partons. Need to assume the parton-hadron duality.
- Jet finding algorithm: (k_t, cone and anti-k_t)See other lecture. [M. Cacciari, G. P. Salam and G. Soyez, 08]

$e^+e^- \to \gamma^* \to$ jets



- a. The Born contribution: σ_0, following the earlier calculation:
- b. The virtual contribution: $-\sigma_0 C_F \frac{\alpha_s}{2\pi} \int_0^E \frac{dl}{l} \int_0^\pi \frac{4d\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$
- c. The soft real contribution: $\sigma_0 C_F \frac{\alpha_s}{2\pi} \int_0^{\epsilon E} \frac{dl}{l} \int_0^\pi \frac{4d\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$
- d. The hard real contribution: $\sigma_0 C_F \frac{\alpha_s}{2\pi} \int_{\epsilon E}^E \frac{dl}{l} \left[\int_0^\delta + \int_{\pi-\delta}^\pi\right] \frac{4d\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$
- sum $= \sigma_0 \left[1 - C_F \frac{\alpha_s}{2\pi} \int_{\epsilon E}^E \frac{dl}{l} \int_\delta^{\pi-\delta} \frac{4d\cos\theta}{1-\cos^2\theta}\right] = \sigma_0 \left[1 - \frac{4C_F\alpha_s}{\pi} \ln\epsilon \ln\delta\right]$
- More complete results including finite ϵ, δ corrections. [B.G. Weeks, 1979]

Infrared Safety

- We have encountered two kinds of divergences: collinear divergence and soft divergence.
- Both of them are of the Infrared divergence type.That is to say, they both involve long distance.
  - According to uncertainty principle, soft $\leftrightarrow$ long distance;
  - Also one needs an infinite time in order to specify accurately the particle momenta, and therefore their directions.
- For a suitable defined inclusive observable (e.g., $\sigma_{e^+e^-\to\text{hadrons}}$), there is a cancellation between the soft and collinear singularities occurring in the real and virtual contributions. Kinoshita-Lee-Nauenberg theorem
- Any new observables must have a definition which does not distinguish between

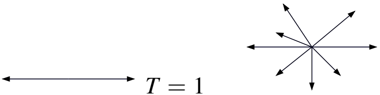
$$\text{parton} \leftrightarrow \text{parton} + \text{soft gluon}$$
$$\text{parton} \leftrightarrow \text{two collinear partons}$$

- Observables that respect the above constraint are called infrared safe observables. Infrared safety is a requirement that the observable is calculable in pQCD.
- Other infrared safe observables, for example, Thrust: $T = \max \frac{\sum_i |p_i \cdot n|}{\sum_i |p_i|}$ ...

Thrust

Global observable reflecting the structure of the hadronic events in e^+e^-:

$$T = \max_{\vec{n}} \frac{\sum_i |p_i \cdot n|}{\sum_i |p_i|}$$



$T = 1$ $\quad$ $T = 1/2$

- For 3-particle event, in terms of x_1 and x_2, the cross section is

$$\frac{d\sigma_3}{\sigma_0 dx_1 dx_2} = \frac{C_F \alpha_s}{2\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

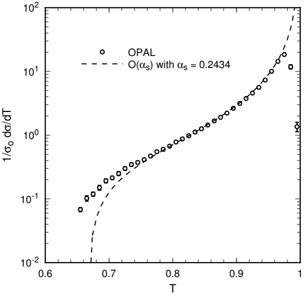
- In this case, $T = \max[x_1, x_2, x_3]$
- By symmetrizing x_i, and requiring $x_1 > x_2 > x_3$, we get $T = x_1 > 2/3$ and

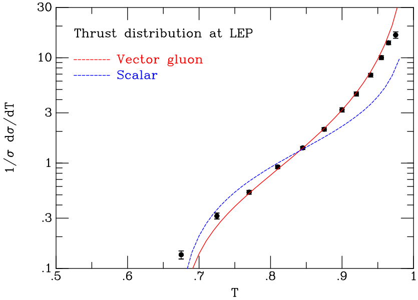
$$\frac{d\sigma_3}{\sigma_0 dT} = \frac{2C_F\alpha_s}{2\pi} \int_{1-2T}^{T} \mathrm{d}x_2 \left[\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} + (x_1 \to x_3) + (x_2 \to x_3) \right]$$

$$= \frac{C_F\alpha_s}{2\pi} \left[\frac{2(3T^2 - 3T + 2)}{T(1-T)} \ln \frac{2T-1}{1-T} - \frac{3(3T-2)(2-T)}{1-T} \right]$$

Thrust

$$\frac{d\sigma_3}{\sigma_0 dT} = \frac{C_F \alpha_s}{2\pi}\left[\frac{2(3T^2-3T+2)}{T(1-T)}\ln\frac{2T-1}{1-T} + \frac{3(2-3T)(2-T)}{1-T}\right]$$





- Deficiency at low T due to kinematics. $T > 2/3$ at this order.
- Miss the data when $T \to 1$ due to divergence. Sudakov factor!

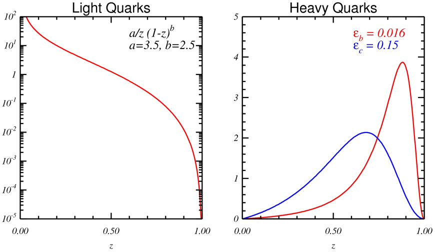
$$\frac{d\sigma}{\sigma_0 dT}\Big|_{T\to 1} \sim \frac{4C_F\alpha_s}{2\pi}\frac{4}{(1-T)}\ln\frac{1}{1-T}\exp\left[-\frac{\alpha_s C_F}{\pi}\ln^2(1-T)\right]$$

- Indication of gluon being a vector boson instead of a scalar.

Fragmentation function

Factorization of single inclusive hadron production in e^+e^-:

$$\frac{1}{\sigma_0}\frac{\mathrm{d}\sigma(e^+e^-\to h+X)}{\mathrm{d}x}=\sum_i\int_x^1 C_i\left(z,\alpha_s(\mu^2),s/\mu^2\right)D_{h/i}(x/z,\mu^2)+\mathcal{O}(1/s)$$



- $D_{h/i}(x/z,\mu^2)$ encodes the probability that the parton i fragments into a hadron h carrying a fraction z of the parton's momentum.
- Energy conservation $\Rightarrow$

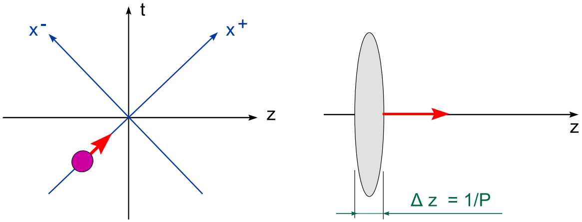
$$\sum_h\int_0^1 dzzD_i^h(z,\mu^2)=1$$

- Heavy quark fragmentation function: Peterson fragmentation function

Outline

Light Cone coordinates and gauge

For a relativistic hadron moving in the $+z$ direction



- In this frame, the momenta are defined

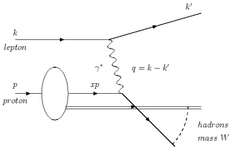
$$P^+ = \frac{1}{\sqrt{2}}(P^0 + P^3) \quad \text{and} \quad P^- = \frac{1}{\sqrt{2}}(P^0 - P^3) \to 0$$

- $P^2 = 2P^+P^- - P_\perp^2$
- Light cone gauge for a gluon with momentum $k^\mu = (k^+, k^-, k_\perp)$, the polarization vector reads

$$k^\mu \epsilon_\mu = 0 \Rightarrow \quad \epsilon = (\epsilon^+ = 0, \epsilon^- = \frac{\epsilon_\perp \cdot k_\perp}{k^+}, \epsilon_\perp^\pm) \quad \text{with} \quad \epsilon_\perp^\pm = \frac{1}{\sqrt{2}}(1, \pm i)$$

Deep inelastic scattering

Summary of DIS:



$$\frac{d\sigma}{dE' d\Omega} = \frac{\alpha_{\mathrm{em}2}}{Q^4}\frac{E'}{E} L_{\mu\nu} W^{\mu\nu}$$

with $L_{\mu\nu}$ the leptonic tensor and $W^{\mu\nu}$ defined as

$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1 + \frac{1}{m_p^2}\left(P^\mu - \frac{P\cdot q}{q^2} q^\mu\right)\left(P^\nu - \frac{P\cdot q}{q^2} q^\nu\right) W_2$$

Introduce the dimensionless structure function:

$$F_1 \equiv W_1 \quad \text{and} \quad F_2 \equiv \frac{Q^2}{2 m_p x} W_2$$

$$\Rightarrow \frac{d\sigma}{dx dy} = \frac{\alpha_{4\pi s \mathrm{em}^2}}{Q^4}\left[(1-y)F_2 + xy^2 F_1\right] \quad \text{with} \quad y = \frac{P\cdot q}{P\cdot k}.$$

Quark Parton Model: Callan-Gross relation

$$F_2(x) = 2xF_1(x) = \sum_q e_q^2 x \left[f_q(x) + f_{\bar{q}}(x)\right].$$

Callan-Gross relation

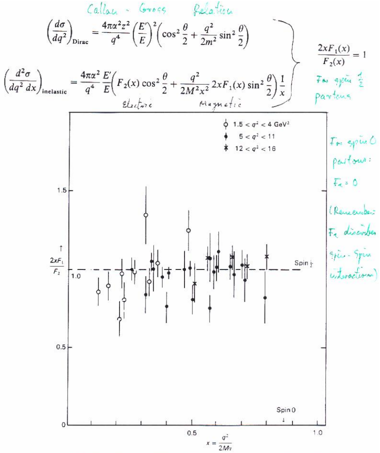
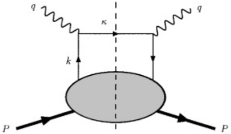


Figure 8.10 The ratio $2xF_1/F_2$ measured in SLAC electron-nucleon scattering experiments. For spin-½ partons, with $g = 2$, a ratio of unity is expected in the limit of large q^2—the Callan-Gross relation. (Data compiled from published SLAC data.)

- The relation ($F_L = F_2 - 2xF_1$) follows from the fact that a spin-$\frac{1}{2}$ quark cannot absorb a longitudinally polarized vector boson.
- In contrast, spin-0 quark cannot absorb transverse bosons and so would give $F_1 = 0$.

Parton Density

The probabilistic interpretation of the parton density.



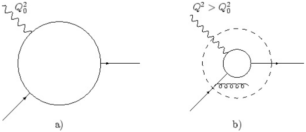
$$\Rightarrow \quad f_q(x) = \int \frac{d\zeta^-}{4\pi} e^{ixP^+\zeta^-} \langle P \left| \bar{\psi}(0)\gamma^+\psi(0,\zeta^-) \right| P \rangle$$

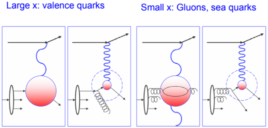
Comments:

- Gauge link $\mathcal{L}$ is necessary to make the parton density gauge invariant.

$$\mathcal{L}(0,\zeta^-) = \mathcal{P}\exp\left(\int_0^{\zeta^-} ds_\mu A^\mu\right)$$

- Choose light cone gauge $A^+ = 0$ and B.C., one can eliminate the gauge link.
- Now we can interpret $f_q(x)$ as parton density in the light cone frame.
- Evolution of parton density: Change of resolution

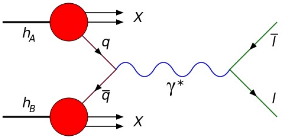




- At low-x, dominant channels are different.

Drell-Yan process

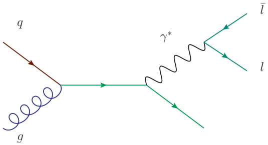
For lepton pair productions in hadron-hadron collisions:



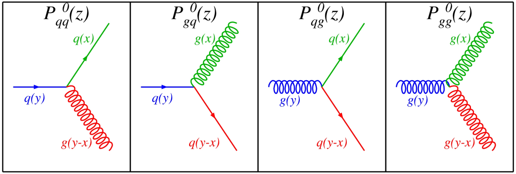
the cross section is

$$\frac{d\sigma}{dM^2 dY} = \sum_q x_1 f_q(x_1) x_2 f_{\bar{q}}(x_2) \frac{1}{3} e_q^2 \frac{4\pi\alpha^2}{3M^4} \quad \text{with} \quad Y = \frac{1}{2} \ln \frac{x_1}{x_2}.$$

- Collinear factorization proof shows that $f_q(x)$ involved in DIS and Drell-Yan process are the same.
- At low-x and high energy, the dominant channel is $qg \to q\gamma^*(l^+ l^-)$.



Splitting function



$$\mathcal{P}_{qq}^{0}(\xi) = \frac{1+\xi^2}{(1-\xi)_+} + \frac{3}{2}\delta(1-\xi),$$

$$\mathcal{P}_{gq}^{0}(\xi) = \frac{1}{\xi}\left[1+(1-\xi)^2\right],$$

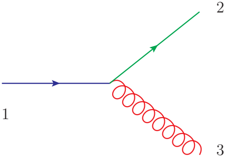
$$\mathcal{P}_{qg}^{0}(\xi) = \left[(1-\xi)^2+\xi^2\right],$$

$$\mathcal{P}_{gg}^{0}(\xi) = 2\left[\frac{\xi}{(1-\xi)_+} + \frac{1-\xi}{\xi} + \xi(1-\xi)\right] + \left(\frac{11}{6} - \frac{2N_f T_R}{3N_c}\right)\delta(1-\xi).$$

- $\xi = z = \frac{x}{y}$.
- $\int_0^1 \frac{d\xi f(\xi)}{(1-\xi)_+} = \int_0^1 \frac{d\xi [f(\xi)-f(1)]}{1-\xi} \Rightarrow \int_0^1 \frac{d\xi}{(1-\xi)_+} = 0$

Derivation of $\mathcal{P}^0_{qq}(\xi)$

The real contribution:

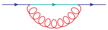


$$k_1 = (P^+, 0, 0_\perp) \quad ; \quad k_2 = (\xi P^+, \frac{k_\perp^2}{\xi P^+}, k_\perp)$$

$$k_3 = ((1-\xi)P^+, \frac{k_\perp^2}{(1-\xi)P^+}, -k_\perp) \quad \epsilon_3 = (0, -\frac{2k_\perp \cdot \epsilon_\perp^{(3)}}{(1-\xi)P^+},$$

$$|V_{q\to qg}|^2 = \frac{1}{2}\mathrm{Tr}\left(\not{k}_2\gamma_\mu\not{k}_1\gamma_\nu\right)\sum \epsilon_3^{*\mu}\epsilon_3^\nu = \frac{2k_\perp^2}{\xi(1-\xi)}\frac{1+\xi^2}{1-\xi}$$

$$\Rightarrow \qquad \mathcal{P}_{qq}(\xi) = \frac{1+\xi^2}{1-\xi} \quad (\xi < 1)$$

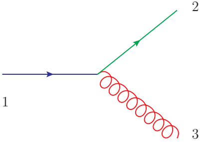
- Including the virtual graph  , use $\int_a^1 \frac{d\xi g(\xi)}{(1-\xi)_+} = \int_a^1 \frac{d\xi g(\xi)}{1-\xi} - g(1)\int_0^1 \frac{d\xi}{1-\xi}$

$$\frac{\alpha_s C_F}{2\pi}\left[\int_x^1 \frac{d\xi}{\xi} q(x/\xi)\frac{1+\xi^2}{1-\xi} - q(x)\int_0^1 d\xi \frac{1+\xi^2}{1-\xi}\right]$$

$$= \frac{\alpha_s C_F}{2\pi}\left[\int_x^1 \frac{d\xi}{\xi} q(x/\xi)\frac{1+\xi^2}{(1-\xi)_+} - q(x)\underbrace{\int_0^1 d\xi \frac{1+\xi^2}{(1-\xi)_+}\Big]}_{=-\frac{3}{2}}.$$

Derivation of $\mathcal{P}^0_{qq}(\xi)$

The real contribution:



$$k_1 = (P^+, 0, 0_\perp) \quad ; \quad k_2 = \left(\xi P^+, \frac{k_\perp^2}{\xi P^+}, k_\perp\right)$$

$$k_3 = \left((1-\xi)P^+, \frac{k_\perp^2}{(1-\xi)P^+}, -k_\perp\right) \quad \epsilon_3 = \left(0, -\frac{2k_\perp \cdot \epsilon_\perp^{(3)}}{(1-\xi)P^+}, \epsilon_\perp^{(3)}\right)$$

$$|V_{q\to qg}|^2 = \frac{1}{2}\mathrm{Tr}\left(\not{k}_2\gamma_\mu\not{k}_1\gamma_\nu\right)\sum \epsilon_3^{*\mu}\epsilon_3^\nu = \frac{2k_\perp^2}{\xi(1-\xi)}\frac{1+\xi^2}{1-\xi}$$

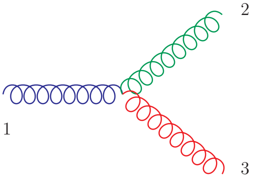
$$\Rightarrow \qquad \mathcal{P}_{qq}(\xi) = \frac{1+\xi^2}{1-\xi} \quad (\xi < 1)$$

- Regularize $\frac{1}{1-\xi}$ to $\frac{1}{(1-\xi)_+}$ by including the divergence from the virtual graph.
- Probability conservation:

$$P_{qq} + \mathrm{d}P_{qq} = \delta(1-\xi) + \frac{\alpha_s C_F}{2\pi}\mathcal{P}^0_{qq}(\xi)dt \quad \text{and} \quad \int_0^1 \mathrm{d}\xi \mathcal{P}_{qq}(\xi) = 0,$$

$$\Rightarrow \mathcal{P}_{qq}(\xi) = \frac{1+\xi^2}{(1-\xi)_+} + \frac{3}{2}\delta(1-\xi) = \left(\frac{1+\xi^2}{1-\xi}\right)_+ .$$

Derivation of $\mathcal{P}^0_{gg}(\xi)$



$$k_1 = (P^+, 0, 0_\perp) \quad \epsilon_1 = (0, 0, \epsilon_\perp^{(1)}) \quad \text{with} \quad \epsilon_\perp^\pm = \frac{1}{\sqrt{2}}(1, \pm i)$$

$$k_2 = (\xi P^+, \frac{k_\perp^2}{\xi P^+}, k_\perp) \quad \epsilon_2 = (0, \frac{2k_\perp \cdot \epsilon_\perp^{(2)}}{\xi P^+}, \epsilon_\perp^{(2)})$$

$$k_3 = ((1-\xi)P^+, \frac{k_\perp^2}{(1-\xi)P^+}, -k_\perp) \quad \epsilon_3 = (0, -\frac{2k_\perp \cdot \epsilon_\perp^{(3)}}{(1-\xi)P^+}, \epsilon_\perp^{(3)})$$

$$V_{g\to gg} = (k_1 + k_3)\cdot \epsilon_2 \epsilon_1 \cdot \epsilon_3 + (k_2 - k_3)\cdot \epsilon_1 \epsilon_2 \cdot \epsilon_3 - (k_1 + k_2)\cdot \epsilon_3 \epsilon_1 \cdot \epsilon_2$$

$$\Rightarrow \qquad |V_{g\to gg}|^2 = |V_{+++}|^2 + |V_{+-+}|^2 + |V_{++-}|^2 = 4k_\perp^2 \frac{[1-\xi(1-\xi)]^2}{\xi^2(1-\xi)^2}$$

$$\Rightarrow \qquad \mathcal{P}_{gg}(\xi) = 2\left[\frac{1-\xi}{\xi} + \frac{\xi}{1-\xi} + \xi(1-\xi)\right] \quad (\xi < 1)$$

- Regularize $\frac{1}{1-\xi}$ to $\frac{1}{(1-\xi)_+}$
- Momentum conservation:

$$\int_0^1 d\xi\, \xi \left[\mathcal{P}_{qq}(\xi) + \mathcal{P}_{gq}(\xi)\right] = 0 \qquad \int_0^1 d\xi\, \xi \left[2\mathcal{P}_{qg}(\xi) + \mathcal{P}_{gg}(\xi)\right] = 0,$$

$\Rightarrow$ the terms which is proportional to $\delta(1-\xi)$.

DGLAP equation

In the leading logarithmic approximation with $t = \ln \mu^2$, the parton distribution and fragmentation functions follow the DGLAP[Dokshitzer, Gribov, Lipatov, Altarelli, Parisi, 1972-1977] evolution equation as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{array}{c} q(x,\mu) \\ g(x,\mu) \end{array}\right] = \frac{\alpha(\mu)}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[\begin{array}{cc} C_F P_{qq}(\xi) & T_R P_{qg}(\xi) \\ C_F P_{gq}(\xi) & N_c P_{gg}(\xi) \end{array}\right]\left[\begin{array}{c} q(x/\xi,\mu) \\ g(x/\xi,\mu) \end{array}\right],$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\begin{array}{c} D_{h/q}(z,\mu) \\ D_{h/g}(z,\mu) \end{array}\right] = \frac{\alpha(\mu)}{2\pi}\int_z^1 \frac{d\xi}{\xi}\left[\begin{array}{cc} C_F P_{qq}(\xi) & C_F P_{gq}(\xi) \\ T_R P_{qg}(\xi) & N_c P_{gg}(\xi) \end{array}\right]\left[\begin{array}{c} D_{h/q}(z/\xi,\mu) \\ D_{h/g}(z/\xi,\mu) \end{array}\right],$$

Comments:

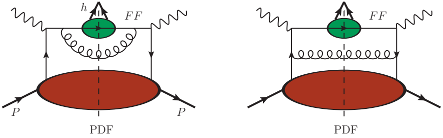
- In the double asymptotic limit, $Q^2 \to \infty$ and $x \to 0$, the gluon distribution can be solved analytically and cast into

$$xg(x,\mu^2) \simeq exp\left(2\sqrt{\frac{\alpha_s N_c}{\pi}\ln\frac{1}{x}\ln\frac{\mu^2}{\mu_0^2}}\right) \quad \text{Fixed coupling}$$

$$xg(x,\mu^2) \simeq exp\left(2\sqrt{\frac{N_c}{\pi b}\ln\frac{1}{x}\ln\frac{\ln\mu^2/\Lambda^2}{\ln\mu_0^2/\Lambda^2}}\right) \quad \text{Running coupling}$$

- The full DGLAP equation can be solved numerically.

Collinear Factorization at NLO



Use $\overline{\text{MS}}$ scheme ($\frac{1}{\hat{\epsilon}} = \frac{1}{\epsilon} + \ln 4\pi - \gamma_E$) and dimensional regularization, DGLAP equation reads

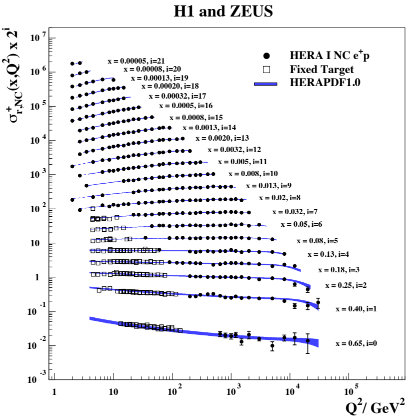
$$\begin{bmatrix} q(x,\mu) \\ g(x,\mu) \end{bmatrix} = \begin{bmatrix} q^{(0)}(x) \\ g^{(0)}(x) \end{bmatrix} - \frac{1}{\hat{\epsilon}} \frac{\alpha(\mu)}{2\pi} \int_x^1 \frac{d\xi}{\xi} \begin{bmatrix} C_F P_{qq}(\xi) & T_R P_{qg}(\xi) \\ C_F P_{gq}(\xi) & N_c P_{gg}(\xi) \end{bmatrix} \begin{bmatrix} q(x/\xi) \\ g(x/\xi) \end{bmatrix},$$

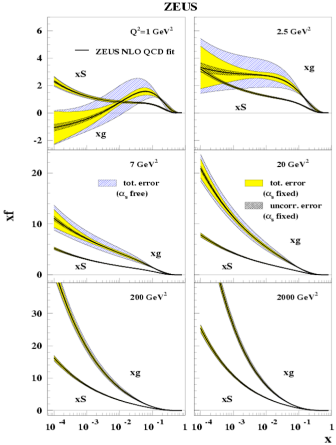
and

$$\begin{bmatrix} D_{h/q}(z,\mu) \\ D_{h/g}(z,\mu) \end{bmatrix} = \begin{bmatrix} D^{(0)}_{h/q}(z) \\ D^{(0)}_{h/g}(z) \end{bmatrix} - \frac{1}{\hat{\epsilon}} \frac{\alpha(\mu)}{2\pi} \int_z^1 \frac{d\xi}{\xi} \begin{bmatrix} C_F P_{qq}(\xi) & C_F P_{gq}(\xi) \\ T_R P_{qg}(\xi) & N_c P_{gg}(\xi) \end{bmatrix} \begin{bmatrix} D_{h/q}(z/\xi) \\ D_{h/g}(z/\xi) \end{bmatrix}$$

- Soft divergence cancels between real and virtual diagrams;
- Gluon collinear to the initial state quark $\Rightarrow$ parton distribution function; Gluon collinear to the final state quark $\Rightarrow$ fragmentation function. KLN theorem does not apply.
- Other kinematical region of the radiated gluon contributes to the NLO ($\mathcal{O}(\alpha_s)$ correction) hard factor.

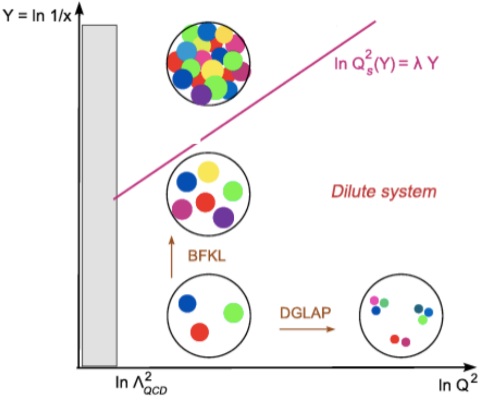
DGLAP evolution





- NLO DGLAP fit yields negative gluon distribution at low Q^2 and low x.
- Does this mean there is no gluons in that region? No

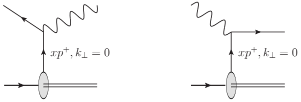
Phase diagram in QCD



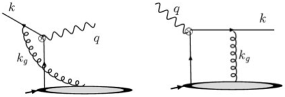
- Low Q^2 and low x region $\Rightarrow$ saturation region.
- Use BFKL equation and BK equation instead of DGLAP equation.
- BK equation is the non-linear small-x evolution equation which describes the saturation physics.

Collinear Factorization vs $k_\perp$ Factorization

Collinear Factorization



$k_\perp$ Factorization(Spin physics and saturation physics)



- The incoming partons carry no $k_\perp$ in the Collinear Factorization.
- In general, there is intrinsic $k_\perp$. It can be negligible for partons in protons, but should be taken into account for the case of nucleus target with large number of nucleons ($A \to \infty$).
- $k_\perp$ Factorization: High energy evolution with $k_\perp$ fixed.
- Initial and final state interactions yield different gauge links. (Process dependent)
- In collinear factorization, gauge links all disappear in the light cone gauge, and PDFs are universal.

Outline

k_t dependent parton distributions

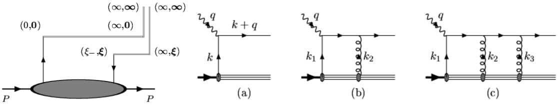
The unintegrated quark distribution

$$f_q(x,k_\perp)=\int\frac{d\xi^-d^2\xi_\perp}{4\pi(2\pi)^2}e^{ixP^+\xi^-+i\xi_\perp\cdot k_\perp}\langle P\left|\bar{\psi}(0)\mathcal{L}^\dagger(0)\gamma^+\mathcal{L}(\xi^-,\xi_\perp)\psi(\xi_\perp,\xi^-)\right|P\rangle$$

as compared to the integrated quark distribution

$$f_q(x)=\int\frac{d\xi^-}{4\pi}e^{ixP^+\xi^-}\langle P\left|\bar{\psi}(0)\gamma^+\mathcal{L}(\xi^-)\psi(0,\xi^-)\right|P\rangle$$

- The dependence of $\xi_\perp$ in the definition.
- Gauge invariant definition.
- Light-cone gauge together with proper boundary condition $\Rightarrow$ parton density interpretation.
- The gauge links come from the resummation of multiple gluon interactions.
- Gauge links may vary among different processes.



Two Different Gluon Distributions

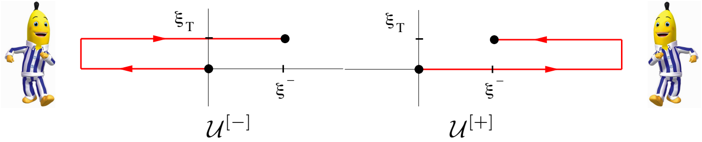
[F.Dominguez, BX and F. Yuan, PRL, 11]

I. Weizsäcker Williams gluon distribution: Gauge Invariant definitions

$$xG^{(1)} = 2\int \frac{d\xi^- d\xi_\perp}{(2\pi)^3 P^+} e^{ixP^+\xi^- - ik_\perp\cdot\xi_\perp} \mathrm{Tr}\langle P|F^{+i}(\xi^-,\xi_\perp)\mathcal{U}^{[+]\dagger}F^{+i}(0)\mathcal{U}^{[+]}|P\rangle.$$

II. Color Dipole gluon distributions: Gauge Invariant definitions

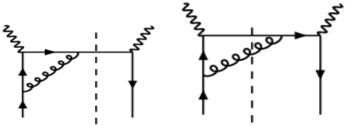
$$xG^{(2)} = 2\int \frac{d\xi^- d\xi_\perp}{(2\pi)^3 P^+} e^{ixP^+\xi^- - ik_\perp\cdot\xi_\perp} \mathrm{Tr}\langle P|F^{+i}(\xi^-,\xi_\perp)\mathcal{U}^{[-]\dagger}F^{+i}(0)\mathcal{U}^{[+]}|P\rangle.$$



- The WW gluon distribution is the conventional gluon distributions.
  Quadrupole $\Rightarrow$ Direct measurement: DIS dijet, etc.
- The dipole gluon distribution has no such interpretation.
  Dipole $\Rightarrow$ γ-jet correlation in pA.

TMD factorization

One-loop factorization:

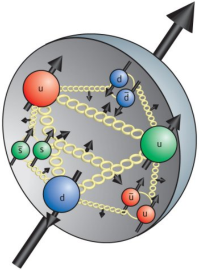


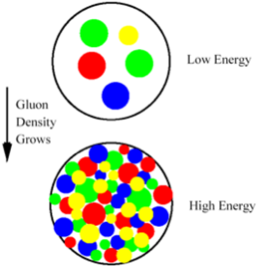
For gluon with momentum k

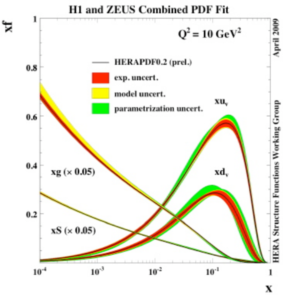
- k is collinear to initial quark $\Rightarrow$ parton distribution function;
- k is collinear to the final state quark $\Rightarrow$ fragmentation function.
- k is soft divergence (sometimes called rapidity divergence) $\Rightarrow$ Wilson lines (Soft factor) or small-x evolution for gluon distribution.
- Other kinematical region of the radiated gluon contributes to the NLO ($\mathcal{O}(\alpha_s)$ correction) hard factor.
- See new development in Collins' book.

Outline

Deep into low-x region of Protons



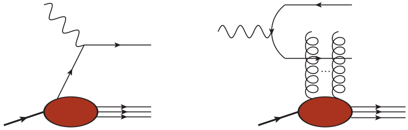




- Gluon splitting functions ($\mathcal{P}^0_{qg}(\xi)$ and $\mathcal{P}^0_{gg}(\xi)$) have $1/(1-\xi)$ singularities.
- Partons in the low-x region is dominated by gluons.
- Resummation of the $\alpha_s \ln \frac{1}{x}$.

Dual Descriptions of Deep Inelastic Scattering

[A. Mueller, 01; Parton Saturation-An Overview]



Bjorken frame   Dipole frame

Bjorken frame

$$F_2(x,Q^2) = \sum_q e_q^2 x \left[f_q(x,Q^2) + f_{\bar{q}}(x,Q^2) \right].$$

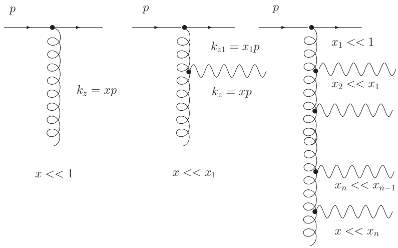
Dipole frame

$$F_2(x,Q^2) = \sum_f e_f^2 \frac{Q^2}{4\pi^2\alpha_{\rm em}} \int_0^1 \mathrm{d}z \int \mathrm{d}^2 x_\perp \mathrm{d}^2 y_\perp \left[|\psi_T(z,r_\perp,Q)|^2 + |\psi_L(z,r_\perp,Q)|^2 \right] \times [1 - S(r_\perp)], \quad \text{with} \quad r_\perp = x_\perp - y_\perp.$$

- Bjorken: partonic picture of a hadron is manifest. Saturation shows up as a limit on the occupation number of quarks and gluons.
- Dipole: partonic picture is no longer manifest. Saturation appears as the unitarity limit for scattering. Easy to resum the multiple gluon interactions.

BFKL evolution

[Balitsky, Fadin, Kuraev, Lipatov;74] The infrared sensitivity of Bremsstrahlung favors the emission of small-x gluons:



Probability of emission:

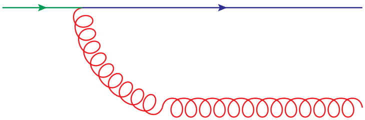
$$dp \sim \alpha_s N_c \frac{dk_z}{k_z} = \alpha_s N_c \frac{dx}{x}$$

In small-x limit and Leading log approximation:

$$p \sim \sum_{n=0}^{\infty} \alpha_s^n N_c^n \int_x^1 \frac{dx_n}{x_n} \cdots \int_{x_2}^1 \frac{dx_1}{x_1} \sim \exp\left(\alpha_s N_c \ln \frac{1}{x}\right)$$

- Exponential growth of the amplitude as function of rapidity;
- As compared to DGLAP which resums $\alpha_s C \ln \frac{Q^2}{\mu_0^2}$.

Dipole model. [Mueller, 94]
Consider a Bremsstrahlung emission of soft gluon $z_g \ll 1$,



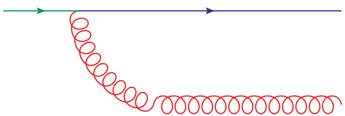
and use LC gauge $\epsilon = (\epsilon^+ = 0, \epsilon^- = \frac{\epsilon_\perp \cdot k_\perp}{k^+}, \epsilon_\perp^\pm)$

$$\mathcal{M}(k_\perp) = -2igT^a \frac{\epsilon_\perp \cdot k_\perp}{k_\perp^2}$$

- $q \to qg$ vertex and Energy denominator.
- Take the limit $k_g^+ \to 0$.
- Similar to the derivation of $\mathcal{P}_{qq}(\xi)$.

The dipole splitting kernal

The Bremsstrahlung amplitude in the coordinate space



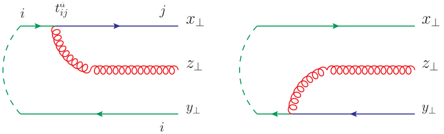
$$\mathcal{M}(x_\perp - z_\perp) = \int d^2k_\perp e^{ik_\perp\cdot(x_\perp - z_\perp)}\mathcal{M}(k_\perp)$$

$$\text{Use} \quad \int d^2k_\perp \frac{\epsilon_\perp \cdot k_\perp}{k_\perp^2} e^{ik_\perp \cdot b_\perp} = 2\pi i \frac{\epsilon_\perp \cdot b_\perp}{b_\perp^2},$$

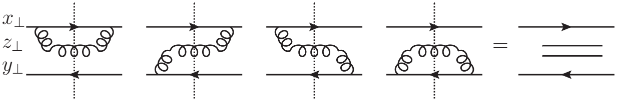
$$\Rightarrow \qquad \mathcal{M}(x_\perp - z_\perp) = 4\pi g T^a \frac{\epsilon_\perp \cdot (x_\perp - z_\perp)}{(x_\perp - z_\perp)^2}$$

The dipole splitting kernal

Consider soft gluon emission from a color dipole in the coordinate space $(x_\perp, y_\perp)$



$$\mathcal{M}(x_\perp, z_\perp, y_\perp) = 4\pi g T^a \left[\frac{\epsilon_\perp \cdot (x_\perp - z_\perp)}{(x_\perp - z_\perp)^2} - \frac{\epsilon_\perp \cdot (y_\perp - z_\perp)}{(y_\perp - z_\perp)^2}\right] \Rightarrow$$



$$= \frac{\alpha_s N_c}{2\pi^2} \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (y_\perp - z_\perp)^2}$$

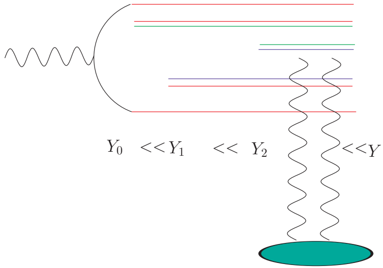
- The probability of dipole splitting at large N_c limit

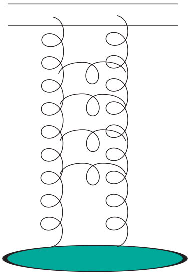
$$dP_{\text{splitting}} = \frac{\alpha_s N_c}{2\pi^2} \frac{(x_\perp - y_\perp)^2}{(x_\perp - z_\perp)^2 (x_\perp - z_\perp)^2} \mathrm{d}^2 z_\perp \mathrm{d}Y \quad \text{with} \quad \mathrm{d}Y = \frac{\mathrm{d}k_g^+}{k_g^+}$$

- Gluon splitting ⇔ Dipole splitting.

BFKL evolution in Mueller's dipole model

[Mueller; 94] In large N_c limit, BFKL evolution can be viewed as dipole branching in a fast moving $q\bar{q}$ dipole in coordinate space:





$n(r, Y)$ dipoles of size r.

BFKL Pomeron

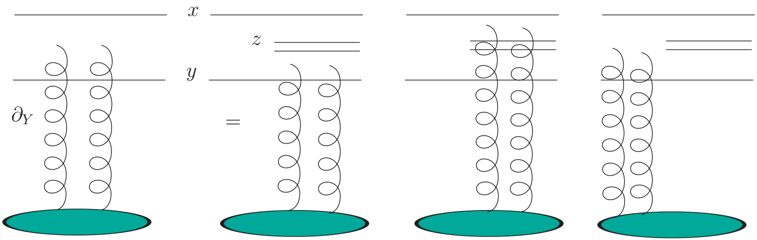
The T matrix ($T \equiv 1 - S$ with S being the scattering matrix) basically just counts the number of dipoles of a given size,

$$T(r, Y) \sim \alpha_s^2 n(r, Y)$$

- The probability of emission is $\bar{\alpha}_s \frac{(x-y)^2}{(x-z)^2(z-y)^2}$;
- Assume independent emissions with large separation in rapidity.

BFKL equation

Consider a slight change in rapidity and the Bremsstrahlung emission of soft gluon (dipole splitting)

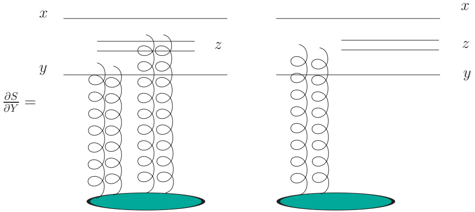


$$\partial_Y T(x, y; Y) = \frac{\bar{\alpha}_s}{2\pi} \int d^2 z \frac{(x-y)^2}{(x-z)^2 (z-y)^2} \left[T(x, z; Y) + T(z, y; Y) - T(x, y; Y) \right]$$

Outline

Kovchegov equation

[Kovchegov; 99] [Mueller; 01] Including non-linear effects: ($T \equiv 1 - S$)

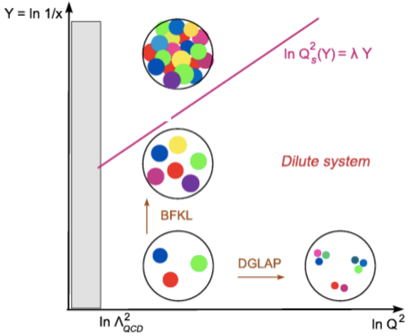


$$\partial_Y S(x-y;Y) = \frac{\alpha N_c}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(z-y)^2}\left[S(x-z;Y)S(z-y;Y) - S(x-y;Y)\right]$$

$$\partial_Y T(x-y;Y) = \frac{\alpha N_c}{2\pi^2}\int d^2z \frac{(x-y)^2}{(x-z)^2(z-y)^2} \times \left[T(x-z;Y)+T(z-y;Y)-T(x-y;Y) - \underbrace{T(x-z;Y)T(z-y;Y)}_{saturation}\right]$$

- Linear BFKL evolution results in fast energy evolution.
- Non-linear term $\Rightarrow$ fixed point ($T = 1$) and unitarization, and thus saturation.

Phase diagram in QCD



- Low Q^2 and low x region $\Rightarrow$ saturation region.
- Balitsky-Kovchegov equation is the non-linear small-x evolution equation which describes the saturation physics.

Balitsky-Kovchegov equation vs F-KPP equation

[Munier, Peschanski, 03] Consider the case with fixed impact parameter, namely, T_{xy} is only function of $r = x - y$. Then, transforming the B-K equation into momentum space:

BK equation: $\partial_Y T = \bar{\alpha}\chi_{\text{BFKL}}(-\partial_\rho)T - \bar{\alpha}T^2$ with $\bar{\alpha} = \frac{\alpha N_c}{\pi}$

Diffusion approximation $\Rightarrow$

F-KPP equation: $\partial_t u(x,t) = \partial_x^2 u(x,t) + u(x,t) - u^2(x,t)$

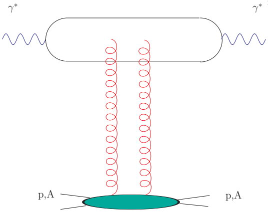
- $u \Rightarrow T$, $\bar{\alpha}Y \Rightarrow t$, $\varrho = \log(k^2/k_0^2) \Rightarrow x$, with k_0 being the reference scale;
- B-K equation lies in the same universality class as the F-KPP [Fisher-Kolmogrov-Petrovsky-Piscounov; 1937] equation.
- F-KPP equation admits traveling wave solution $u = u\,(x - vt)$ with minimum velocity;
- the non-linear term saturates the solution in the infrared.

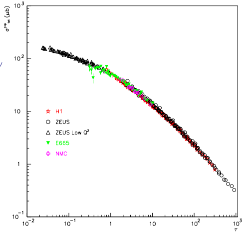
Geometrical scaling

Geometrical scaling in DIS:

$$T(r,Y) = T\left[r^2 Q_s^2(Y)\right] = \left[r^2 Q_s^2(Y)\right]^{\gamma_c} \underbrace{\exp\left[-\frac{\log^2\left(r^2Q_s^2(Y)\right)}{2\chi''(\gamma_c)\bar{\alpha}Y}\right]}_{\text{Scaling window}}$$

- All data of $\sigma_{tot}^{\gamma^* p}$ when $x \leq 0.01$ and $\frac{1}{r^2} = Q^2 \leq 450 GeV^2$ plotting as function of $\tau = Q^2/Q_s^2$ falls on a curve, where $Q_s^2 = \left(\frac{x_0}{x}\right)^{0.29} GeV^2$ with $x_0 = 3\times 10^{-4}$;
- scaling window: $|\log\left(r^2Q_s^2(Y)\right)| \ll \sqrt{2\chi''(\gamma_c)\bar{\alpha}Y}$.





[Golec-Biernat, Stasto, Kwiecinski; 01]

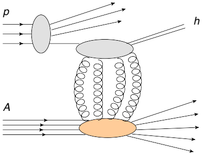
Outline

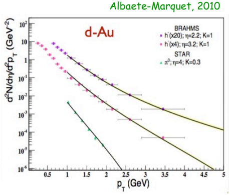
Forward hadron production in pA collisions

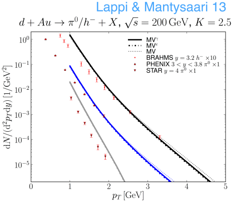
[Dumitru, Jalilian-Marian, 02] Inclusive forward hadron production in pA collisions

$$\frac{d\sigma_{\rm LO}^{pA\to hX}}{d^2p_\perp dy_h} = \int_\tau^1 \frac{dz}{z^2}\left[\sum_f x_p q_f(x_p,\mu)\mathcal{F}(k_\perp)D_{h/q}(z,\mu) + x_p g(x_p,\mu)\tilde{\mathcal{F}}(k_\perp)D_{h/g}(z,\mu)\right].$$

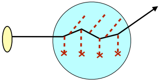
$$p + A \to h(y, p_\perp) + X$$







- Caveats: arbitrary choice of the renormalization scale μ and K factor.
- NLO correction? [Dumitru, Hayashigaki, Jalilian-Marian, 06; Altinoluk, Kovner 11] [Chirilli, Xiao and Yuan, 12]

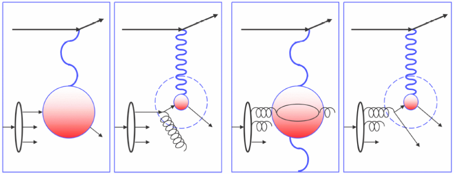


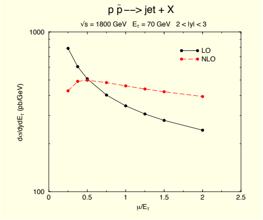
$$\mathcal{F}(k_\perp) = \int \frac{d^2x_\perp d^2y_\perp}{(2\pi)^2} e^{-ik_\perp\cdot(x_\perp - y_\perp)} S_Y^{(2)}(x_\perp, y_\perp)$$

Why do we need NLO calculations?

Large x: valence quarks

Small x: Gluons, sea quarks





- Due to quantum evolution, PDF and FF changes with scale. This introduces large theoretical uncertainties in $xf(x)$ and $D(z)$. Choice of the scale at LO requires information at NLO.
- LO cross section is always a monotonic function of μ, thus it is just order of magnitude estimate.
- NLO calculation significantly reduces the scale dependence. More reliable.
- $K = \frac{\sigma_{LO}+\sigma_{NLO}}{\sigma_{LO}}$ is not a good approximation.
- NLO is vital in establishing the QCD factorization in saturation physics.

NLO Calculation and Factorization

- Factorization is about separation of short distant physics (perturbatively calculable hard factor) from large distant physics (Non perturbative).

$$\sigma \sim xf(x) \otimes \mathcal{H} \otimes D_h(z) \otimes \mathcal{F}(k_\perp)$$

- NLO (1-loop) calculation always contains various kinds of divergences.
  - Some divergences can be absorbed into the corresponding evolution equations.
  - The rest of divergences should be cancelled.
- Hard factor

$$\mathcal{H} = \mathcal{H}^{(0)}_{\text{LO}} + \frac{\alpha_s}{2\pi}\mathcal{H}^{(1)}_{\text{NLO}} + \cdots$$

should always be finite and free of divergence of any kind.

- NLO vs NLL   Naive α_s expansion sometimes is not sufficient!

	LO	NLO	NNLO	$\cdots$
LL	1	$\alpha_s L$	$(\alpha_s L)^2$	$\cdots$
NLL		α_s	$\alpha_s(\alpha_s L)$	$\cdots$
$\cdots$			$\cdots$	$\cdots$

- Evolution $\rightarrow$ Resummation of large logs.
  LO evolution resums LL; NLO $\Rightarrow$ NLL.

Systematic factorization for the $p + A \to H + X$ process
[G. Chirilli, BX and F. Yuan, Phys. Rev. Lett. 108, 122301 (2012)]

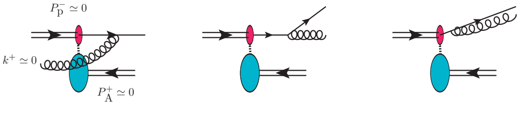
$$\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp} = \sum_a \int \frac{dz}{z^2}\frac{dx}{x}\xi x f_a(x,\mu) D_{h/c}(z,\mu) \int [dx_\perp] S^Y_{a,c}([x_\perp]) \mathcal{H}_{a\to c}(\alpha_s, \xi, [x_\perp]\mu)$$

Collinear divergence: pdfs

Collinear divergence: fragmentation functs

Rapidity divergence: BK evolution

Finite hard factor



Rapidity Divergence   Collinear Divergence (P)   Collinear Divergence (F)

- Typical integrals in real contributions:

$$\underbrace{\int^1 d\xi \frac{1+\xi^2}{(1-\xi)}}_{\text{Rapidity D.}} \underbrace{\int \frac{d^2 q_\perp}{q_\perp^2} e^{-iq_\perp \cdot r_\perp}}_{\text{Collinear D.}}$$

The subtraction of the rapidity divergence

We remove the rapidity divergence from the real and virtual diagrams by the following subtraction:

$$\mathcal{F}(k_\perp) = \mathcal{F}^{(0)}(k_\perp) - \frac{\alpha_s N_c}{2\pi^2}\int_0^1 \frac{d\xi}{1-\xi}\int \frac{d^2x_\perp d^2y_\perp d^2b_\perp}{(2\pi)^2} e^{-ik_\perp\cdot(x_\perp - y_\perp)} \times \frac{(x_\perp - y_\perp)^2}{(x_\perp - b_\perp)^2(y_\perp - b_\perp)^2}\left[S^{(2)}(x_\perp, y_\perp) - S^{(4)}(x_\perp, b_\perp, y_\perp)\right].$$

Decomposing the dipole splitting kernel as

$$\frac{(x_\perp - y_\perp)^2}{(x_\perp - b_\perp)^2(y_\perp - b_\perp)^2} = \frac{1}{(x_\perp - b_\perp)^2} + \frac{1}{(y_\perp - b_\perp)^2} - \frac{2(x_\perp - b_\perp)\cdot(y_\perp - b_\perp)}{(x_\perp - b_\perp)^2(y_\perp - b_\perp)^2}.$$

with the first two terms removed from the virtual diagrams while the last term removed from the real diagrams. Comments:

- This divergence removing procedure is similar to the renormalization of parton distribution and fragmentation function in collinear factorization.
- Splitting functions becomes $\frac{1+\xi^2}{(1-\xi)_+}$ after the subtraction.
- Rapidity divergence disappears when the $k_\perp$ is integrated. Unique feature of unintegrated gluon distributions.

The subtraction of the collinear divergence

Remove the collinear singularities by redefining the quark distribution and the quark fragmentation function as follows

$$q(x,\mu) = q^{(0)}(x) - \frac{1}{\hat{\epsilon}}\frac{\alpha_s(\mu)}{2\pi}\int_x^1 \frac{d\xi}{\xi} C_F \mathcal{P}_{qq}(\xi) q\left(\frac{x}{\xi}\right),$$

$$D_{h/q}(z,\mu) = D_{h/q}^{(0)}(z) - \frac{1}{\hat{\epsilon}}\frac{\alpha_s(\mu)}{2\pi}\int_z^1 \frac{d\xi}{\xi} C_F \mathcal{P}_{qq}(\xi) D_{h/q}\left(\frac{z}{\xi}\right),$$

with

$$\mathcal{P}_{qq}(\xi) = \underbrace{\frac{1+\xi^2}{(1-\xi)_+}}_{\text{Real}} + \underbrace{\frac{3}{2}\delta(1-\xi)}_{\text{Virtual}}.$$

Comments:

- Reproducing the DGLAP equation for the quark channel. Other channels will complete the full equation.
- The emitted gluon is collinear to the initial state quark $\Rightarrow$ Renormalization of the parton distribution.
- The emitted gluon is collinear to the final state quark $\Rightarrow$ Renormalization of the fragmentation function.

For the $q \to q$ channel, the factorization formula can be written as

$$\frac{d^3\sigma^{p+A\to h+X}}{dyd^2p_\perp} = \int \frac{dz}{z^2}\frac{dx}{x}\xi xq(x,\mu)D_{h/q}(z,\mu)\int\frac{d^2x_\perp d^2y_\perp}{(2\pi)^2}\left\{S_Y^{(2)}(x_\perp,y_\perp)\left[\mathcal{H}_{2qq}^{(0)}+\frac{\alpha_s}{2\pi}\mathcal{H}_{2qq}^{(1)}\right]\right.$$
$$\left.+\int\frac{d^2b_\perp}{(2\pi)^2}S_Y^{(4)}(x_\perp,b_\perp,y_\perp)\frac{\alpha_s}{2\pi}\mathcal{H}_{4qq}^{(1)}\right\}$$

with $\mathcal{H}_{2qq}^{(0)} = e^{-ik_\perp\cdot r_\perp}\delta(1-\xi)$ and

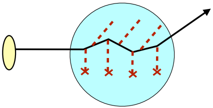
$$\mathcal{H}_{2qq}^{(1)} = C_F\mathcal{P}_{qq}(\xi)\ln\frac{c_0^2}{r_\perp^2\mu^2}\left(e^{-ik_\perp\cdot r_\perp}+\frac{1}{\xi^2}e^{-i\frac{k_\perp}{\xi}\cdot r_\perp}\right)-3C_F\delta(1-\xi)e^{-ik_\perp\cdot r_\perp}\ln\frac{c_0^2}{r_\perp^2 k_\perp^2}$$
$$-(2C_F-N_c)e^{-ik_\perp\cdot r_\perp}\left[\frac{1+\xi^2}{(1-\xi)_+}\widetilde{I}_{21}-\left(\frac{\left(1+\xi^2\right)\ln(1-\xi)^2}{1-\xi}\right)_+\right]$$

$$\mathcal{H}_{4qq}^{(1)} = -4\pi N_c e^{-ik_\perp\cdot r_\perp}\left\{e^{-i\frac{1-\xi}{\xi}k_\perp\cdot(x_\perp-b_\perp)}\frac{1+\xi^2}{(1-\xi)_+}\frac{1}{\xi}\frac{x_\perp-b_\perp}{(x_\perp-b_\perp)^2}\cdot\frac{y_\perp-b_\perp}{(y_\perp-b_\perp)^2}\right.$$
$$\left.-\delta(1-\xi)\int_0^1 d\xi'\frac{1+\xi'^2}{(1-\xi')_+}\left[\frac{e^{-i(1-\xi')k_\perp\cdot(y_\perp-b_\perp)}}{(b_\perp-y_\perp)^2}-\delta^{(2)}(b_\perp-y_\perp)\int d^2r'_\perp\frac{e^{ik_\perp\cdot r'_\perp}}{r'^2_\perp}\right]\right\},$$

where
$$\widetilde{I}_{21} = \int\frac{d^2b_\perp}{\pi}\left\{e^{-i(1-\xi)k_\perp\cdot b_\perp}\left[\frac{b_\perp\cdot(\xi b_\perp-r_\perp)}{b_\perp^2(\xi b_\perp-r_\perp)^2}-\frac{1}{b_\perp^2}\right]+e^{-ik_\perp\cdot b_\perp}\frac{1}{b_\perp^2}\right\}.$$

Single inclusive hadron production up to NLO

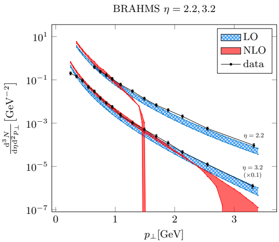
$$
\begin{aligned}
d\sigma \quad &= \int xf_a(x) \otimes D_a(z) \otimes \mathcal{F}_a^{x_g}(k_\perp) \otimes \mathcal{H}^{(0)} \\
&+ \frac{\alpha_s}{2\pi} \int xf_a(x) \otimes D_b(z) \otimes \mathcal{F}_{(N)ab}^{x_g} \otimes \mathcal{H}_{ab}^{(1)}.
\end{aligned}
$$

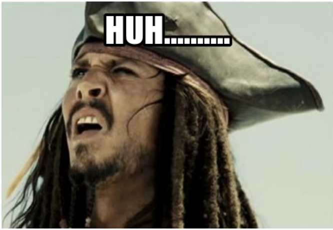


Consistent implementation should include all the NLO α_s corrections.

- NLO parton distributions. (MSTW or CTEQ)
- NLO fragmentation function. (DSS or others.)
- Use NLO hard factors. 4 channels $q \to q$, $q \to g$, $g \to q(\bar{q})$ and $g \to g$
- Use the one-loop approximation for the running coupling
- rcBK evolution equation for the dipole gluon distribution [Balitsky, Chirilli, 08; Kovchegov, Weigert, 07]. Full NLO BK evolution not available.
- Saturation physics at One Loop Order (SOLO). [Stasto, Xiao, Zaslavsky, 13]

Surprise

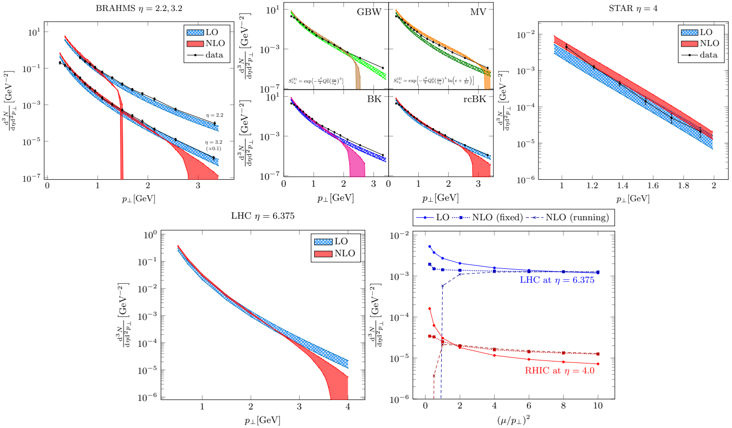




- The abrupt drop of the NLO correction when $p_\perp > Q_s$ was really a surprise!
- What is going wrong?
  - Saturation formalism? Dilute-dense factorization? Not necessarily positive definite! Does this indicate that we need NNLO correction? $\cdots$
  - Some hidden large correction in $\frac{\alpha_s}{2\pi}\mathcal{H}^{(1)}_{\text{NLO}}$?

Numerical implementation of the NLO result

[Stasto, Xiao, Zaslavsky, 13, accepted for publication in PRL]



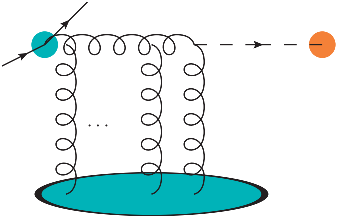
- Agree with data for $p_\perp < Q_s(y)$, and reduced scale dependence, no K factor.
- For more forward rapidity, the agreement gets better and better.
- Additional *plus*-function (threshold) resummation ? Similar to thrust $T \to 1$.

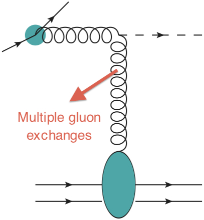
Outline

LO calculation for Higgs production in pA collisions

[A. Mueller, BX and F. Yuan, 12, 13] The effective Lagrangian:

$$\mathcal{L}_{eff} = -\frac{1}{4} g_\phi \Phi F^a_{\mu\nu} F^{a\mu\nu}$$



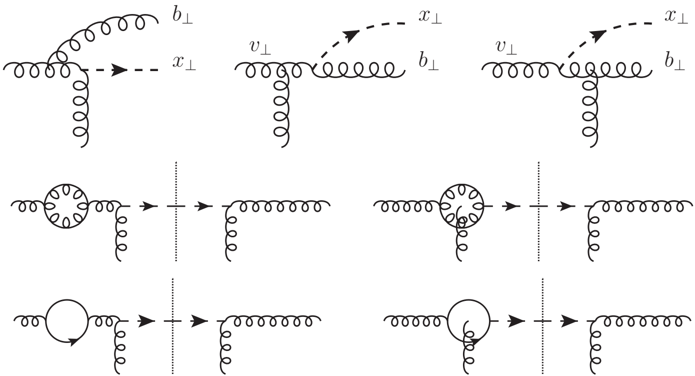


$$\frac{d\sigma^{(\mathrm{LO})}}{dyd^2k_\perp} = \sigma_0 \int \frac{d^2x_\perp d^2x'_\perp}{(2\pi)^2} e^{ik_\perp\cdot(x_\perp - x'_\perp)} x g_p(x) S_Y^{WW}(x_\perp, x'_\perp) \ ,$$

- $\sigma_0 = g_\phi^2/g^2 32(1-\epsilon)$ with $\epsilon = -(D-4)/2$;
- $S_Y^{WW}(x_\perp, y_\perp) = -\left\langle \mathrm{Tr}\left[\partial_\perp^\beta U(x_\perp) U^\dagger(y_\perp) \partial_\perp^\beta U(y_\perp) U^\dagger(x_\perp)\right]\right\rangle_Y$
- Only initial state interaction is present. $\Rightarrow$ WW gluon distribution.
- For *AA* collisions, there exists the true k_t factorization.

Some Technical Details

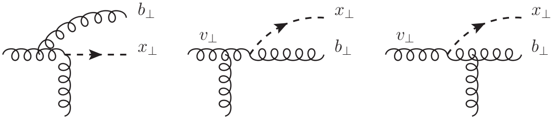
[A. Mueller, BX and F. Yuan, 12, 13] Typical diagrams:



- High energy limit $s \to \infty$ and $M^2 \gg k_\perp^2$. Use dimensional regularization.
- Power counting analysis: take the leading power contribution in terms of $\frac{k_\perp^2}{M^2}$.

Some Technical Details

[A. Mueller, BX and F. Yuan, 12, 13]



The phase space $(l^+, l^-, l_\perp)$ of the radiated gluon can be divided into three regions:

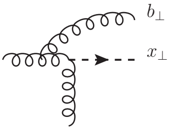
- (a) gluon is collinear to the incoming proton. $\Rightarrow$ DGLAP evolution.
  Subtraction of the collinear divergence and choose $\mu^2 = \frac{c_0^2}{R_\perp^2}$:

$$-\frac{1}{\epsilon} S^{WW}(x_\perp, y_\perp) \mathcal{P}_{gg}(\xi) \otimes xg\left(x, \frac{c_0^2}{R_\perp^2}\right) \quad \text{with} \quad \xi = \frac{l^+}{p^+}$$

- (b) gluon is collinear to the incoming nucleus. $\Rightarrow$ Small-x evolution.
  Subtraction of the rapidity divergence: $\Rightarrow$ non-linear small-x evolution equation

$$xg_p(x) \int_0^1 \frac{d\xi}{\xi} \int \mathbf{K}_{\text{DMMX}} \otimes S^{WW}(x_\perp, y_\perp)$$

- (c) gluon is soft. $\Rightarrow$ Sudakov logarithms.



(a)

- Consider the kinematic constraint for real emission before taking $s \to \infty$, and note that $x_g x_p s = M^2$

$$\int_{\frac{l_\perp^2}{x_p s}}^{1} \frac{d\xi}{\xi} = \ln\left(\frac{x_p s}{l_\perp^2}\right) = \ln\frac{1}{x_g} + \ln\frac{M^2}{l_\perp^2}.$$

- Now we can take $s \to \infty$ and $x_g \to 0$, but keep $x_g x_p s = M^2$.
- The Sudakov contribution gives

$$\mu^{2\epsilon}\int \frac{d^{2-2\epsilon} l_\perp}{(2\pi)^{2-2\epsilon}} e^{-il_\perp \cdot R_\perp} \frac{1}{l_\perp^2} \ln\frac{M^2}{l_\perp^2}$$
$$= \frac{1}{4\pi}\left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{M^2}{\mu^2} + \frac{1}{2}\left(\ln\frac{M^2}{\mu^2}\right)^2 - \frac{1}{2}\left(\ln\frac{M^2 R_\perp^2}{c_0^2}\right)^2 - \frac{\pi^2}{12}\right]$$

Some Technical Details

- Subtraction of the UV-divergence: (Color charge renormalization)

$$\frac{\alpha_s}{\pi}N_c\beta_0\left(-\frac{1}{\epsilon_{UV}}+\ln\frac{M^2}{\mu^2}\right) \quad \text{with} \quad \beta_0=\frac{11}{12}-\frac{N_f}{6N_c}.$$

- Real diagrams ⇒

$$+\frac{\alpha_s}{\pi}N_c\left[\frac{1}{\epsilon^2}-\frac{1}{\epsilon}\ln\frac{M^2}{\mu^2}+\frac{1}{2}\left(\ln\frac{M^2}{\mu^2}\right)^2-\frac{1}{2}\left(\ln\frac{M^2R_\perp^2}{c_0^2}\right)^2-\frac{\pi^2}{12}\right].$$

- Virtual graphs ⇒

$$\frac{\alpha_s}{\pi}N_c\left[-\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\ln\frac{M^2}{\mu^2}-\frac{1}{2}\ln^2\frac{M^2}{\mu^2}+\frac{\pi^2}{2}+\frac{\pi^2}{12}+\beta_0\ln\frac{M^2}{\mu^2}\right]$$

- All divergences now cancel and the remaining contribution is

$$\frac{\alpha_s}{\pi}N_c\left(-\frac{1}{2}\ln^2\frac{M^2R_\perp^2}{c_0^2}+\beta_0\ln\frac{M^2R_\perp^2}{c_0^2}+\frac{\pi^2}{2}\right).$$

Sudakov factor

Final results:

- At one-loop order: $R_\perp \equiv x_\perp - x'_\perp$

$$\frac{d\sigma^{(\text{LO+NLO})}}{\sigma_0 dy d^2k_\perp} = \int \frac{d^2x_\perp d^2x'_\perp}{(2\pi)^2} e^{ik_\perp \cdot R_\perp} xg_p(x, \mu^2 = c_0^2/R_\perp^2) S^{WW}_{Y=\ln 1/x_g}(x_\perp, x'_\perp)$$
$$\times \left[1 + \frac{\alpha_s}{\pi} N_c \left(-\frac{1}{2}\ln^2 \frac{M^2 R_\perp^2}{c_0^2} + \beta_0 \ln \frac{M^2 R_\perp^2}{c_0^2} + \frac{\pi^2}{2}\right)\right].$$

- Collins-Soper-Sterman resummation:

$$\frac{d\sigma^{(\text{resum})}}{dy d^2k_\perp}|_{k_\perp \ll M} = \sigma_0 \int \frac{d^2x_\perp d^2x'_\perp}{(2\pi)^2} e^{ik_\perp \cdot R_\perp} e^{-\mathcal{S}_{\text{sud}}(M^2, R_\perp^2)} S^{WW}_{Y=\ln 1/x_g}(x_\perp, x'_\perp)$$
$$\times x_p g_p(x_p, \mu^2 = \frac{c_0^2}{R_\perp^2}) \left[1 + \frac{\alpha_s}{\pi}\frac{\pi^2}{2} N_c\right],$$

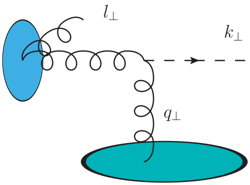
where the Sudakov form factor contains all order resummation

$$\mathcal{S}_{\text{sud}}(M^2, R_\perp^2) = \int_{c_0^2/R_\perp^2}^{M^2} \frac{d\mu^2}{\mu^2}\left[A \ln \frac{M^2}{\mu^2} + B\right].$$

- $A = \sum_{i=1}^{\infty} A^{(i)} \left(\frac{\alpha_s}{\pi}\right)^i$, we find $A^{(1)} = N_c$ and $B^{(1)} = -\beta_0 N_c$.

Probabilistic interpretation of the Sudakov double logarithms

The gluon wave function:



- Now in momentum space, the Higgs transverse momentum is fixed to be $k_\perp \sim q_\perp \ll M$.
- The energetic gluon is annihilated to create the color neutral and heavy Higgs particle, thus, other gluons in its wave function are forced to be produced.
- The Sudakov factor S is just the probability of those gluons having transverse momentum much less than $k_\perp$.

$$\frac{d\sigma^{(\text{resum})}}{dyd^2k_\perp}|_{k_\perp \ll M} = \sigma_0 \int \frac{d^2x_\perp d^2x'_\perp}{(2\pi)^2} e^{ik_\perp \cdot R_\perp} e^{-\mathcal{S}_{\text{sud}}(M^2, R_\perp^2)} S^{WW}_{Y=\ln 1/x_g}(x_\perp, x'_\perp) \times x_p g_p(x_p, \mu^2 = \frac{c_0^2}{R_\perp^2}) \left[1 + \frac{\alpha_s}{\pi}\frac{\pi^2}{2}N_c\right],$$

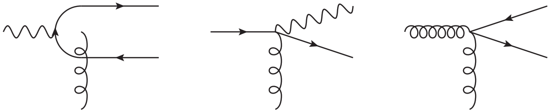
- Four independent (at LL level) renormalization equation.
- Unified description of the *CSS* and small-x evolution? TMD and UGD share the same operator definition.

Define $\mathcal{S}^{WW}(R_\perp, M; x_g) = \mathcal{C} e^{-\mathcal{S}_{\text{sud}}(M^2, R_\perp^2)} S^{WW}_{Y=\ln 1/x_g}(x_\perp, x'_\perp) \Rightarrow$

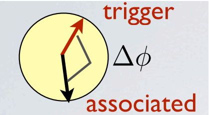
$$\frac{d\sigma^{(\text{resum})}}{dyd^2k_\perp}|_{k_\perp \ll M} = \sigma_0 \int \frac{d^2x_\perp d^2x'_\perp}{(2\pi)^2} e^{ik_\perp \cdot R_\perp} x_p g_p(x_p, \mu^2 = \frac{c_0^2}{R_\perp^2}) \mathcal{S}^{WW}(R_\perp, M; x_g)$$

Sudakov factor for dijet productions in pA collisions and DIS

Consider the dijet productions in pA collisions:



$\mathcal{C}_{q\bar{q}} = \frac{N_c}{2}$ $\qquad$ $\mathcal{C}_{q\gamma} = \frac{N_c}{2} + \frac{C_F}{2}$ $\qquad$ $\mathcal{C}_{g\to q\bar{q}} = N_c$



$$\frac{d\sigma}{dy_1 dy_2 dP_\perp^2 d^2k_\perp} \propto H(P_\perp^2) \int d^2x_\perp d^2y_\perp e^{ik_\perp \cdot R_\perp} e^{-\mathcal{S}_{sud}(P_\perp, R_\perp)} \widetilde{W}_{x_A}(x_\perp, y_\perp) \,.$$

Comments:

- Problems get harder due to presence of both initial and final state emissions.
- For back-to-back dijet processes, $M_J^2 \sim P_\perp^2 \gg k_\perp^2$

$$\mathcal{S}_{\text{sud}} = \frac{\alpha_s \mathcal{C}}{2\pi} \ln^2 \frac{P_\perp^2 R_\perp^2}{c_0^2} \quad \text{with} \quad R_\perp \sim \frac{1}{k_\perp}.$$

- Empirical formula for $\mathcal{C}$: $\mathcal{C} = \sum_i \frac{C_i}{2}$, where C_i is the Casimir color factor of the incoming particles. $C_i = C_F$ for incoming quarks, $C_i = N_c$ for gluons.
- For heavy quarkonium production, there is also a Sudakov factor. [Qiu, Sun, BX, Yuan, 13]