Mixing shear and dilation in marginal solids

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Mixing shear and dilation...

I. Dilation induced by shear

II. Tuning shear compliance with pre-tension
Mixing shear and dilation...

I. Dilation induced by shear

II. Tuning shear compliance with pre-tension
Counterintuitive dilatancy

O. Reynolds 1885
Weaire & Hutzler, Phil. Mag. 2003
Conti & MacKintosh PRL 2008

Packings expand, networks contract: Why the difference?

packings:  
system expands  
or pressure increases

networks:  
system contracts  
or pressure decreases
Dilatancy enhanced near jamming

Ren, Dijksman & Behringer
PRL 2013
A nonlinear effect

\[ \epsilon = \frac{1}{2} R_p \gamma^2 + \ldots \]

symmetry:

\[ R_p = \left( \frac{\partial^2 \epsilon}{\partial \gamma^2} \right)_\gamma \]

normal stress \( p_0 \)

Reynolds dilatancy coefficient

Ren, Dijksman & Behringer, PRL 2013
Weaire & Hutzler, Phil. Mag. 2003
Reynolds coefficient

assume a hyperelastic solid:

energy \[ dU = -p \, dV - \sigma V \, d\gamma \]

“enthalpy” \[ dH = V \, dp - \sigma V \, d\gamma \]

Maxwell \[ \left( \frac{\partial V}{\partial \gamma} \right)_p = - \left( \frac{\partial \sigma V}{\partial p} \right)_\gamma \]

expression for \( R_p \)

Weaire & Hutzler, Phil. Mag. 2003
BPT, Gran. Matt. 2013
Reynolds coefficient

\[ R_p = \left( \frac{\partial G}{\partial p} \right)_\gamma - \frac{G}{E} \]

shear modulus \( G > 0 \)
Young’s modulus \( E > 0 \)
typically \( E > G \)

Weaire & Hutzler, Phil. Mag. 2003
BPT, Gran. Matt. 2013
Reynolds coefficient

\[ R_p = \left( \frac{\partial G}{\partial p} \right) \gamma - \frac{G}{E} \]

Shear modulus \( G > 0 \)
Young's modulus \( E > 0 \)
Typically \( E > G \)

Magnitude \( \gg 1 \) in marginal solids

Does compression stiffen or soften the shear modulus?

Weaire & Hutzler, Phil. Mag. 2003
BPT, Gran. Matt. 2013
Soft spheres

\[ R_p \approx \left( \frac{\partial G}{\partial p} \right)^\gamma \]

\[ G \sim p^{1/2} \quad \text{(Hookean)} \]

\[ R_p \sim \frac{1}{p^{1/2}} > 0 \]

O’Hern, Silbert, Liu & Nagel, PRE 2003

BPT, Gran. Matt. 2013
Packings expand: verified in model foams

1. Physical intuition?
2. What about networks?

Weaire & Hutzler, Phil. Mag. 2003
Mixing shear and dilation...

I. Dilation induced by shear

II. Tuning shear compliance with pre-tension
Tuning with tension

\[ k_{\text{eff}} \sim p \]

unloaded state = floppy = tunable!
Networks

z < z_c
floppy

z > z_c
rigid

coordination
Networks

- \( z < z_c \) with coordination ON
- \( z > z_c \) with coordination OFF

Tension vs. Probability

- \( p = 0 \)
- \( p = 10^{-3} \)
- \( p = 10^{-1} \)

Color Scale: \( f/f_{\text{max}} \)
Manipulating marginal matter

Brown et al, PNAS 2010

un jammed = OFF

jammed = ON

jamming transition as a switch
Rigidity induced by tension

$z < z_c$

$p = 0$

$p = 10^{-3}$

$p = 10^{-1}$

jamming transition as a **switch**

...or a **knob**

measure $G$
Shear modulus

\( G(p, z) \)  

\[
\frac{G(p, z)}{|z - z_c|^{\mu}} = g \left( \frac{p}{|z - z_c|^{\lambda}} \right)
\]
Critical scaling

\[ G \sim |\Delta z|^\mu \]

\( \mu = 1.1 \quad \lambda = 2.1 \)

Wyart, Liang, Kabla & Mahadevan, PRL 2008
Ellenbroek, Zeravcic & Van Hecke, EPL 2009
Zaccone & Scossa-Romano, PRB 2011
Broedersz, Mao, Lubensky & MacKintosh Nat. Mat. 2011
Tighe, PRL 2012
Critical scaling

\[ G \sim p^{\mu/\lambda} \]

\[ \mu = 1.1 \quad \lambda = 2.1 \]
Critical scaling

\[ \frac{G}{|\Delta z|^{\mu}} \sim \frac{p}{|\Delta z|^{\lambda-\mu}} \]

\[ \mu = 1.1 \quad \lambda = 2.1 \]
Critical scaling

$G / |\Delta z|^\mu$

$\mu = 1.1 \quad \lambda = 2.1$

$p / |\Delta z|^\lambda$

$0.01 \quad 0.1 \quad 1 \quad 10 \quad 100$
reversible tuning with tension
diverging susceptibility

only rigid under tension
\( G \sim p/\Delta z \)

critical regime
\( G \sim p^{1/2} \)

“classical” jammed solid
\( G \sim \Delta z \)

excess coordination \( \Delta z = z - z_c \)
Spectra

tension

\[ z < z_c \]

Density of States (DOS)

frequency \( \omega \)

Spectra

Density of States (DOS)

zero tension

z < z_c

area $\sim |\Delta z|$

finite frequency modes

density gap: Düring et al, Soft Matter 2012
Density of States (DOS)

$\text{area} \sim |\Delta z|$

finite frequency modes

$\sqrt{k_{\text{eff}}} \sim \sqrt{p}$

$\omega$

$z < z_c$

tension

Spectra

Spectra

\[ p = 10^{-3} \]

\[ z \rightarrow z_c \]

DOS vs. frequency \( \omega \)
Spectra

\[ z = 3.5 \]

\begin{align*}
\text{DOS} & \quad \text{frequency } \omega \\
\text{pressure} & \quad \text{pressure}
\end{align*}
Simple scaling argument

\[
\frac{1}{G} \sim \frac{1}{N} \sum_n \frac{1}{\omega_n^2}
\]

modulus $\leftrightarrow$ modes

\[ G \sim \frac{p}{|\Delta z|} \text{ w/ high susceptibility} \quad \frac{\partial G}{\partial p} \sim \frac{1}{|\Delta z|} \]
Critical scaling

upper branch:

\[ G = G_0 \Delta z \sqrt{1 + c p / \Delta z^2} \]
Critical scaling

\[
\frac{(G - G_{p \to 0})}{\Delta z^\mu} = \frac{p}{|\Delta z|^\lambda}
\]
Network dilatancy

\[ R_p \simeq \left( \frac{\partial G}{\partial p} \right) \gamma \]

\[ G \propto \Delta z \sqrt{1 - \text{const} \cdot \frac{p}{\Delta z^2}} \]

\[ R_p \sim -\frac{1}{G} < 0 \]
Network dilatancy

constant normal stress: \( R_p \sim 1/G \)

constant volume: \( R_V \sim \text{const} \)

finite size effect: Goodrich, Dagois-Bohy et al. (in prep)
Intuiting dilatancy near jamming

$R_p \sim \frac{\partial G}{\partial p} < 0$

networks contract

$R_p \sim \frac{\partial G}{\partial p} > 0$

packings expand
Relating networks and packings

![Graph showing the relationship between connectivity $z$ and effective volume fraction $\phi$.]

Katgert & Van Hecke, EPL 2010

$$G \propto \Delta z \sqrt{1 - \text{const} \cdot \frac{p}{\Delta z^2}}$$

**Stability:** $G > 0$

$$\Delta z \geq \text{const} \cdot p^{1/2}$$

cf. Wyart et al, PRE 2005

**Higher pressure = more contacts**

$$G \sim p^{1/2}$$
Dilatancy and strain stiffening

\[ G(z, p, \gamma) = G_0(z, p) f(\gamma / \gamma^*) \]

**Ansatz:**

- \( \epsilon = R \gamma^2 / 2 \)
- \( \gamma^* = 2 / R \)
Dilatancy and strain stiffening

\[ \frac{G}{G_0(\Delta z, p)} \]

\[ p = 10^{-3} \ldots 10^{-1} \]
\[ \Delta z = 0.01 \ldots 0.5 \]
\[ \nu = 1.1 \]

linear response

strain stiffening

cf Wyart et al., PRL 2008
Nonlinear bulk modulus

\[ \mu = 1.1 \quad \lambda = 2.1 \]

\[ K_{\text{app}} \equiv \frac{p}{\epsilon} \]

\[ K_{\text{app}} / |\Delta z|^{\mu} \]

\[ p / |\Delta z|^{\lambda} \]

c.f. Sheinman, Broedersz, MacKintosh, PRE 2012

Wyart, Liang, Kabla, Mahadevan, PRL 2008
packings expand / networks contract

tunable shear modulus

enhanced near jamming

PhD and Postdoc positions available