The Large *D* limit of General Relativity

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Why large *D*?

$$R_{\mu\nu}=0$$

encode a vast amount of physics

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encode a vast amount of physics

but very hard to solve

(really!)

Why large *D*?

- Theories often
  - have parameters that can be varied
  - simplify at boundaries/origin of parameter space, eg:

    - QED around $e^2 = 0$
    - SU(N) Yang-Mills around N=infty

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- $R_{\mu\nu}=0$ has *D* as natural parameter

- GR simplifies greatly at *D* $\rightarrow\infty$

Why large *D*?

- ➔ Large *D* expansion can be useful
  - – deeper understanding of the theory (reformulation?)
  - – calculations: new perturbative expansion

Large *D* limit of quantum GR?

- Feynman diagrams *Strominger 1981, Bjerrum-Bohr 2003, Canfora 2005*
- Analogue of large *N* of Yang-Mills?
  - $SO(D\text{-}1,1)$ local symmetry of GR

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- Large *N* YM: gluons arrange into worldsheets

Large *D* limit of quantum GR?

- Feynman diagrams *Strominger 1981, Bjerrum-Bohr 2003, Canfora 2005*
- Analogue of large *N* of Yang-Mills?
  - *SO*(*D*-1,1) local symmetry of GR
- Large *N* YM: gluons arrange into worldsheets
- QGR
  - not dominated by planar diagrams
  - *D* not only in graviton polarizations, but also in phase space integrals: **terrible UV behavior**

Large *D* limit of quantum GR?

- Large *D* Kaluza-Klein truncation
  - keeps $\sim D^2$ polarizations of gravitons
  - 4d UV behavior

Not very clear how useful

Classical theory

- Classical GR is well defined at any *D*
- Much of quantum gravity has been learned from semiclassical black hole physics

Similar motivation & techniques: *B. Kol et al*

large-D study of Euclidean Schw zero mode

GR in vacuum

What's this theory?

GR in vacuum

What’s this theory?

A theory of black holes that interact via the grav field between them, and emit and absorb gravitational waves

GR in vacuum

Basic solution

$$ds^2 = -\left(1 - \left(\frac{r_0}{r}\right)^{D-3}\right) dt^2 + \frac{dr^2}{1 - \left(\frac{r_0}{r}\right)^{D-3}} + r^2 d\Omega_{D-2}$$

length scale r_0

GR in vacuum – **large D**

- r_0 not the only scale
- Small *parameter* $1/D$

  $\Rightarrow$ hierarchy of scales

$$\boxed{r_0/D \ll r_0}$$

- This is the **main feature** of large-D GR
- This talk is about its origin & implications

GR in vacuum – **large D**

Different physics at different scales:

- ‘Far’ dynamics at scales $O(r_0\, D^0)$ from horizon
  - flat space, almost trivial
- ‘Near’ dynamics at scales $O(r_0/D)$ from horizon
  - ‘string’ dynamics, non-trivial

GR @ large D

Two main effects:

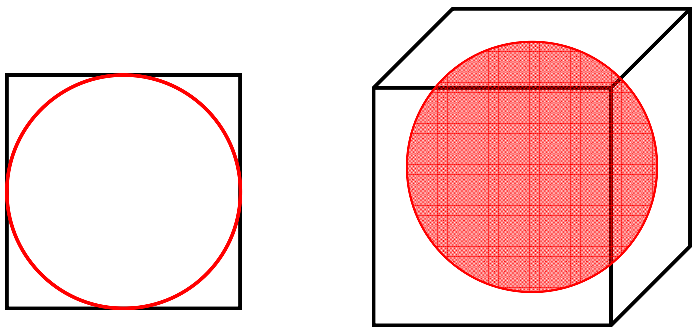
1. Small cross sections
    - elementary geometry effect

2. Interactions localized near horizon
    - gravitational effect

Elementary geometry @ large D

Area of spheres becomes small

compared to hypercubes that enclose them

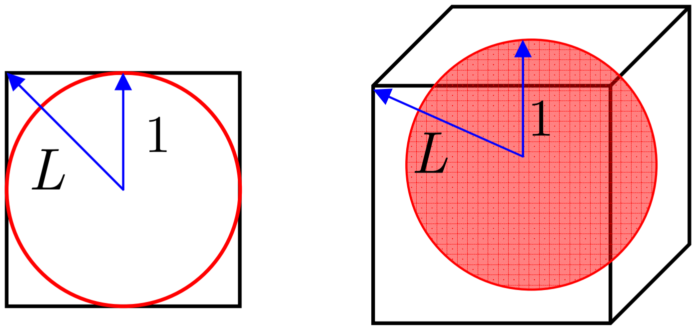


$$\cdots \quad \frac{\mathrm{Area}_{(\mathrm{Sphere})}}{\mathrm{Area}_{(\mathrm{Cube})}} = 2^{2-D}\Omega_{D-2}$$

$$\Omega_{D-2} = \frac{2\pi^{(D-1)/2}}{\Gamma\left(\frac{D-1}{2}\right)} \to \quad \sim D^{-D/2} \to 0$$

Elementary geometry @ large D

Lots of space in diagonal directions



...

$$L^2 = x_1^2 + \cdots + x_{D-1}^2 = D - 1$$

$$L \to D^{1/2} \gg 1$$

Sphere of *finite radius* but *zero area*

➔ vanishing cross sections

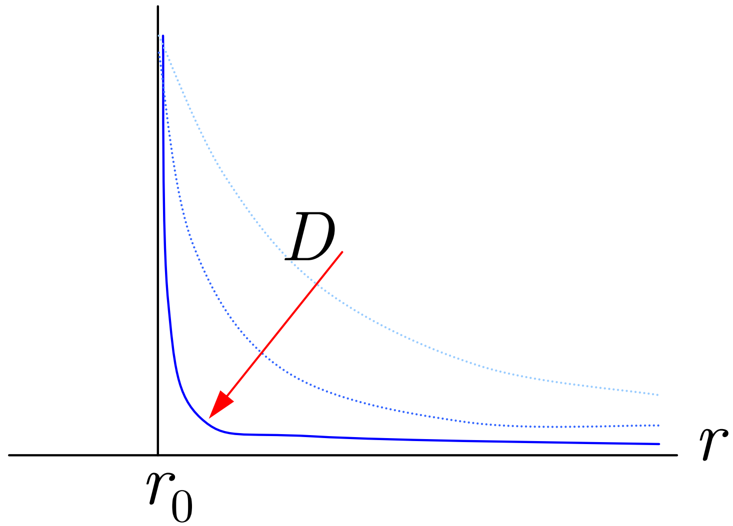
Localization of interactions

- Large potential gradient:

$$\Phi(r) \sim (r_0/r)^{D-3}$$

$$\Rightarrow \nabla\Phi\big|_{r_0} \sim D/r_0$$

$\Phi(r)$



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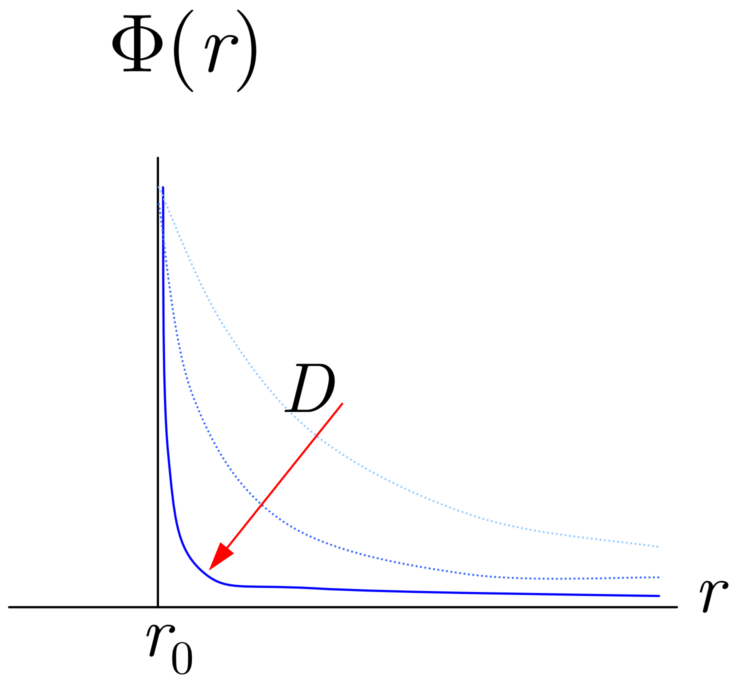
$$r_0/D \ll r_0$$



Localization of interactions

- r_0 fixed outside horizon

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^{D-3} \longrightarrow 1 \qquad \text{for } r > r_0$$

$$ds^2 \to -dt^2 + dr^2 + r^2 d\Omega_{D-2}$$

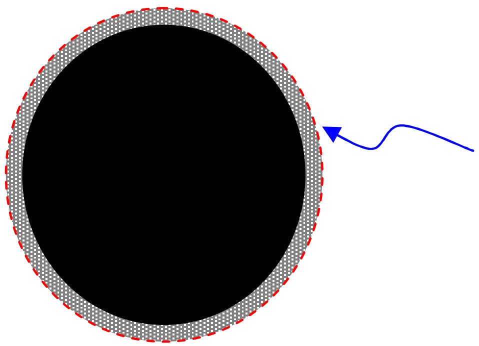
Flat, empty space at $r > r_0$

no gravitational field

Localization of interactions

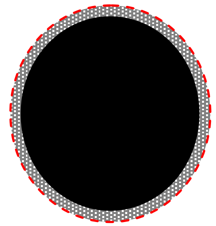
Gravitational potential appreciable only in *thin* near-horizon region

$$(r_0/r)^{D-3} = O(1) \quad \Leftrightarrow \quad r - r_0 < r_0/D$$



$$r - r_0 \sim r_0/D$$

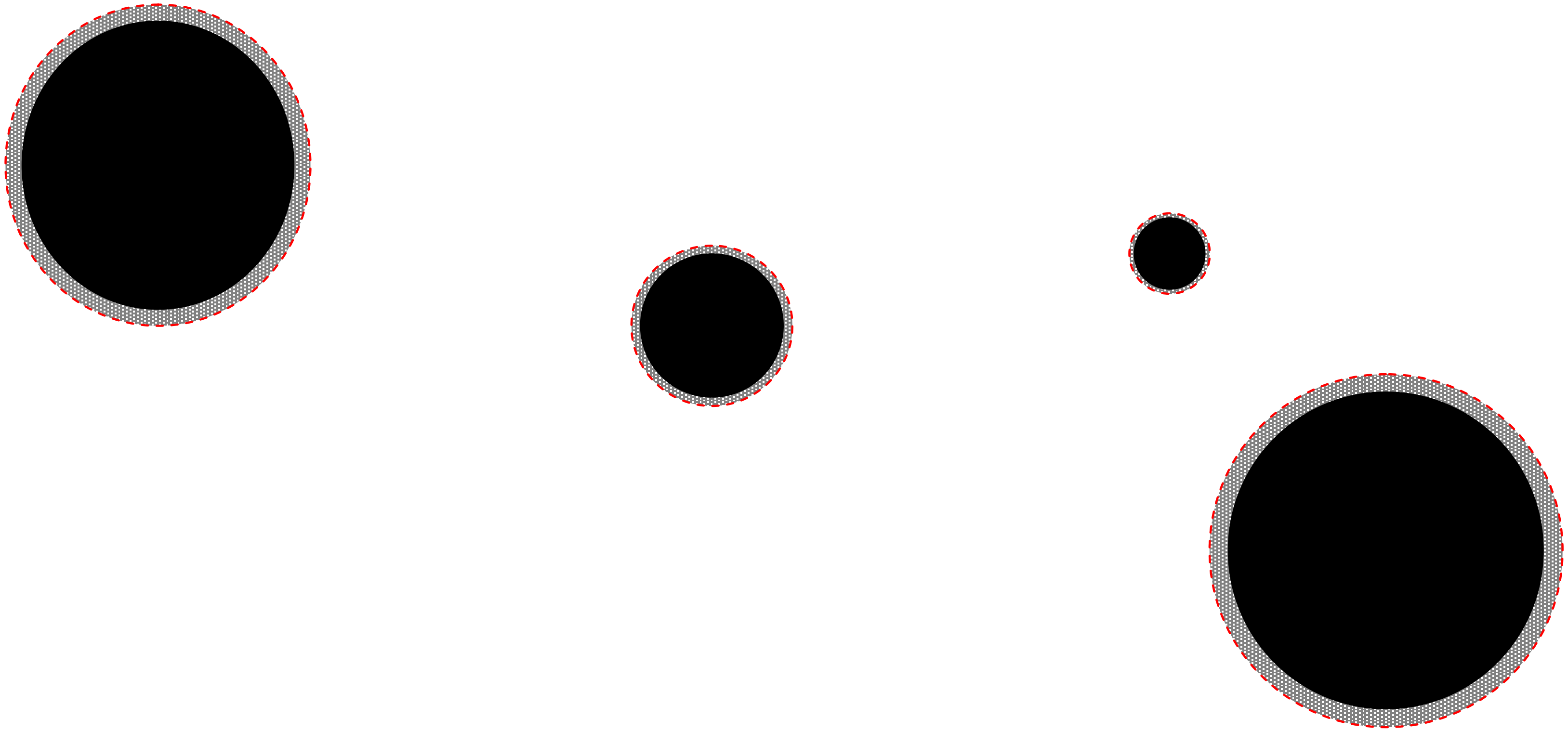
Localization of interactions



- Interactions only within $r - r_0 < r_0/D$
- No capture of particles outside this '*sphere of influence*'
- Probed by frequencies $\omega > D/r_0$

GR @ D $\rightarrow\infty$: *far* view   scale $O(r_0\, D^0)$

Holes cut out in Minkowski space

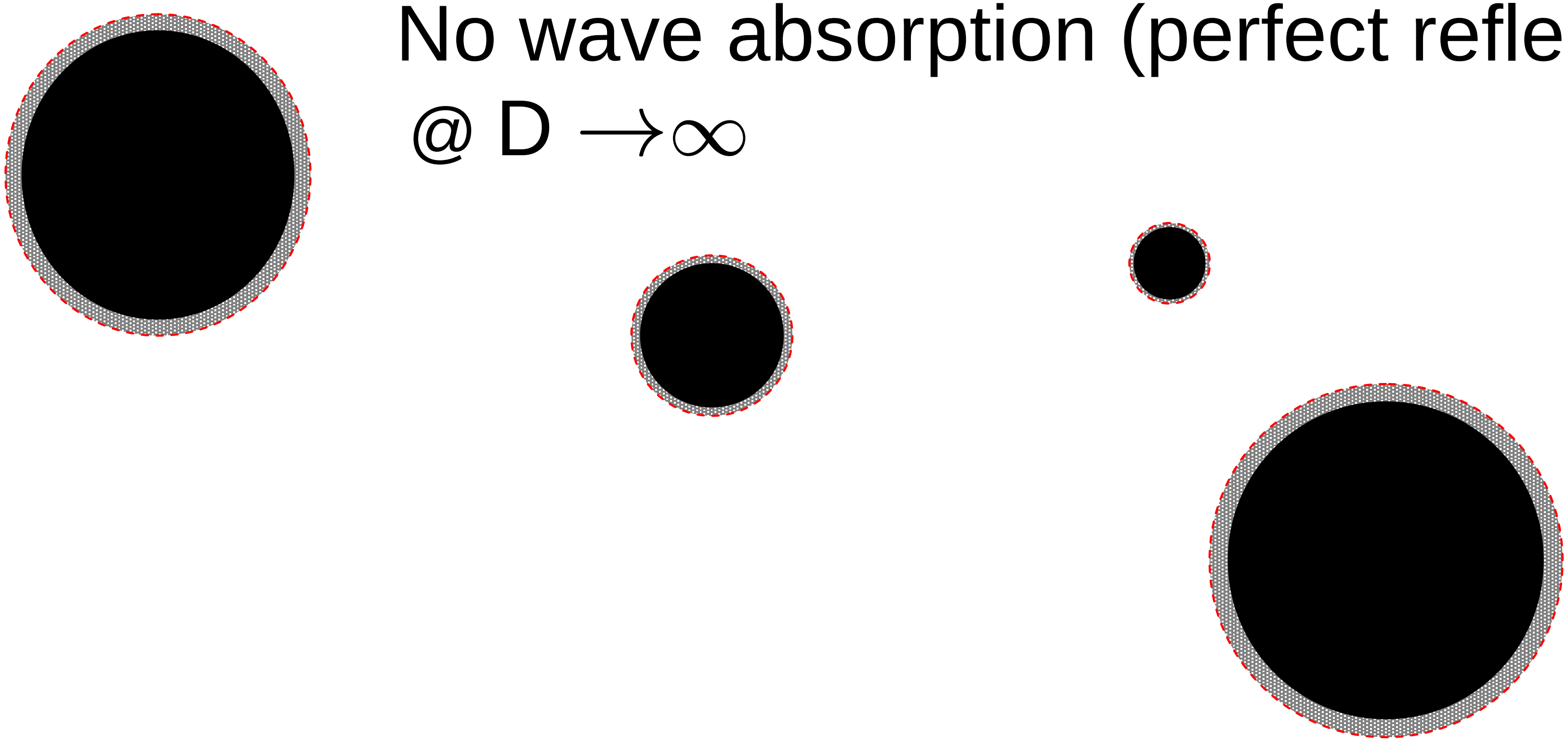


GR @ D $\rightarrow\infty$: *far* view

scale $O(r_0\, D^0)$

Holes cut out in Minkowski space

No wave absorption (perfect reflection)
@ D $\rightarrow\infty$



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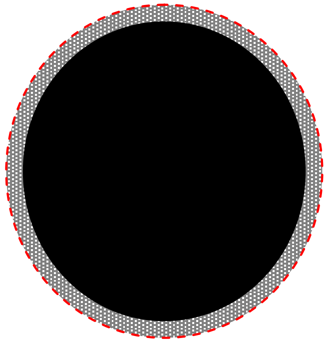
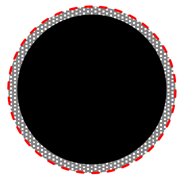
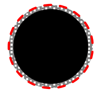
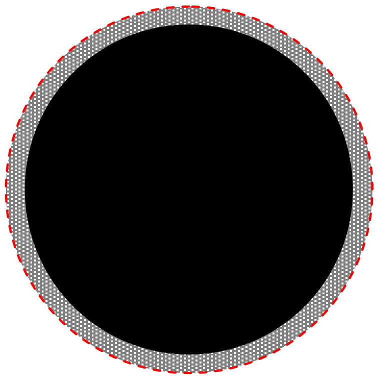
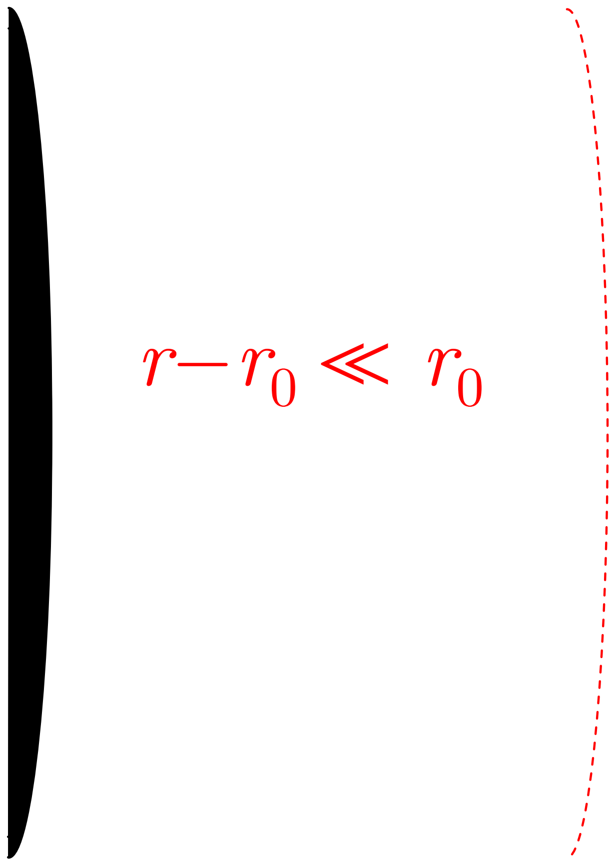





No interaction, confirmed in

- collisions
- radiation from binaries
- scalar emission/absorption

GR @ D $\longrightarrow \infty$: *near view* scale $O(r_0/D)$



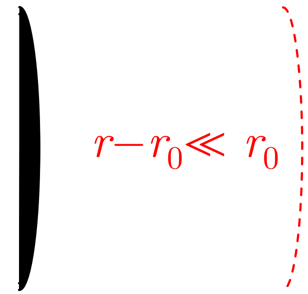
Keep r_0/D finite

Includes ‘sphere of influence’
$r - r_0 < r_0/D$

$\Rightarrow$ Non-trivial dynamics

Overlaps with far region at
$r_0/D \ll r - r_0 \ll r_0$

GR @ D $\to\infty$: *near* view



Near-horizon coordinate: $\mathsf{R} = (r/r_0)^{D-3}$

$$ds^2(\mathrm{Schw}) \to -\frac{\mathsf{R}-1}{\mathsf{R}}dt^2 + \frac{r_0^2}{D^2}\frac{d\mathsf{R}^2}{\mathsf{R}(\mathsf{R}-1)} + r_0^2 d\Omega^2_{D-2}$$

GR @ D $\to\infty$: *near view* scale $O(r_0/D)$

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Change $\quad t_{\mathrm{near}} = D/(2r_0)\ t_{\mathrm{far}} \qquad \mathsf{R} = \cosh^2\rho$

$\Rightarrow$ 2d string black hole *Witten, Das et al, Elitzur et al*

$$ds_{\mathrm{nh}}^2 = \frac{4r_0^2}{D^2}\left(-\tanh^2\rho\, dt_{\mathrm{near}}^2 + d\rho^2\right) + r_0^2 d\Omega_{D-2}^2$$

GR @ D $\to\infty$: *near view* scale $O(r_0/D)$

- Spherical reduction of Einstein-Hilbert

$$ds^2_{\rm nh} = \frac{4r_0^2}{D^2}\left(g^{(2)}_{\mu\nu}dx^\mu dx^\nu\right) + r_0^2 e^{-4\Phi(x)/(D-2)} d\Omega^2_{D-2}$$

$\Rightarrow$ 2d dilaton gravity $\quad g^{(2)}_{\mu\nu}\,,\;\Phi$

$$I = \int d^2x\,\sqrt{-g}\,e^{-2\Phi}\left(R + 4\frac{D-3}{D-2}(\nabla\Phi)^2 + \frac{(D-3)(D-2)}{r_0^2}e^{\frac{4\Phi}{D-2}}\right)$$

GR @ D $\to\infty$: *near view* scale $O(r_0/D)$

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D $\to\infty$ $\Rightarrow$ 2d string gravity

Soda, Grumiller et al

$$I = \int d^2x \sqrt{-g}\, e^{-2\Phi}\left(R + 4(\nabla\Phi)^2 + 4\lambda^2\right)$$

$$\lambda = \frac{D}{2r_0}$$

GR @ D $\rightarrow\infty$: *near view* scale $O(r_0/D)$

$$I = \int d^2x \sqrt{-g}\, e^{-2\Phi} \left(R + 4(\nabla\Phi)^2 + 4\lambda^2\right)$$

$$\lambda = \frac{D}{2r_0}$$

- String length:

$$l_{\mathrm{string}} = 1/\lambda = 2r_0/D$$

$$\alpha' \sim (r_0/D)^2$$

GR @ D $\longrightarrow \infty$: *near view* scale $O(r_0/D)$

- Minkowski$_D \longrightarrow$ 2d linear dilaton vacuum

- 2d conformal symmetry @ large D
- Quasinormal modes: string-scale

$$\omega_{\mathrm{qnm}} \sim \omega_{\mathrm{string}} = D/r_0$$

(long lived: $\mathrm{Im}\,\omega_{\mathrm{qnm}} \ll \mathrm{Re}\,\omega_{\mathrm{qnm}}$)

Entropy, *far* view

$$S \sim M^{1+1/(D-3)} \to S \sim M$$

$\Rightarrow$ Black holes merge w/ no entropy gain

Could also break up at no entropy cost

(horizon becomes singular at $D \to \infty$)

$\Rightarrow$ Absence of interactions

Entropy, *near* view

$$S \sim M^{1+1/(D-3)} \longrightarrow S \sim M$$

$\Rightarrow$ Hagedorn string entropy

$$S = T_{\mathrm{string}} M$$

$$T_{\mathrm{string}} = D/(2r_0)$$

GR @ $D \to \infty$

Picture is very generic

- Outside (far) of horizon: *holes* that do not interact

  for essentially all bhs:

  charged, rotating, in AdS, extremal, etc

- Near-horizon: 2d string bh for all neutral non-extremal bhs

  other: 3d string-theory black string (*Horne+Horowitz*)

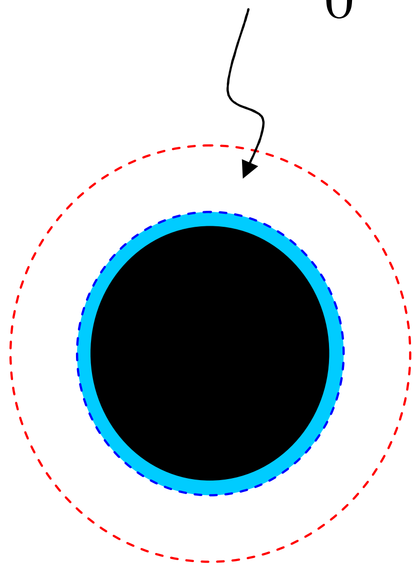
Large-D expansion at work

1. Scalar propagation
   - absorption probability
   - quasinormal modes

2. Black brane instability
   - spectrum of unstable modes

Scalar field propagation

$$r_0/D \ll r - r_0 \ll r_0$$



Far-zone d.o.f.'s: *waves in flat space*

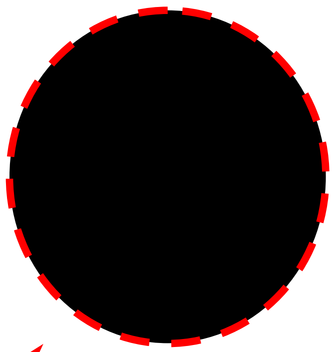
Near-zone d.o.f.'s: *black hole excitations*

They interact in overlap region

Effective theory

Propagation in flat space

$$\Box_{\text{flat}}\phi = 0$$



flat space

w/ bdry conds at holes

$$\left.\frac{\partial_r \phi_{\omega l}}{\phi_{\omega l}}\right|_{r=r_0} = F(r_0, \omega, l)$$

determined from near solution

Black hole absorption

- Analytic calculation of absorption probability
-

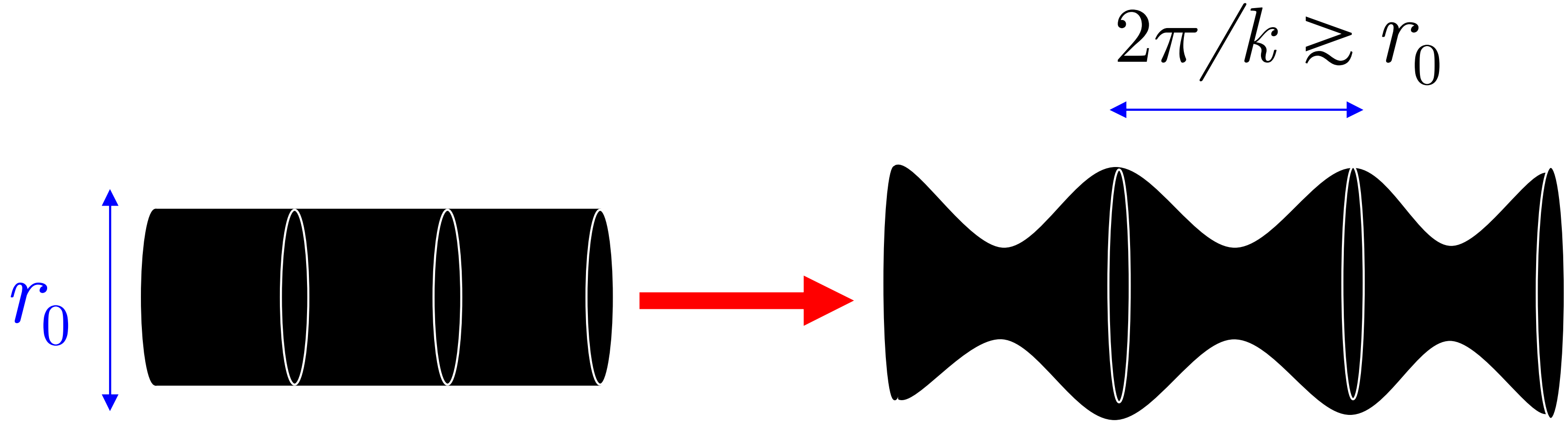
$\rightarrow 0$ for $\omega < \omega_{crit} \sim \omega_{\text{string}}$: perfect mirror

$\rightarrow 1$ for $\omega > \omega_{crit}$: perfect absorber

Low-frequency limit

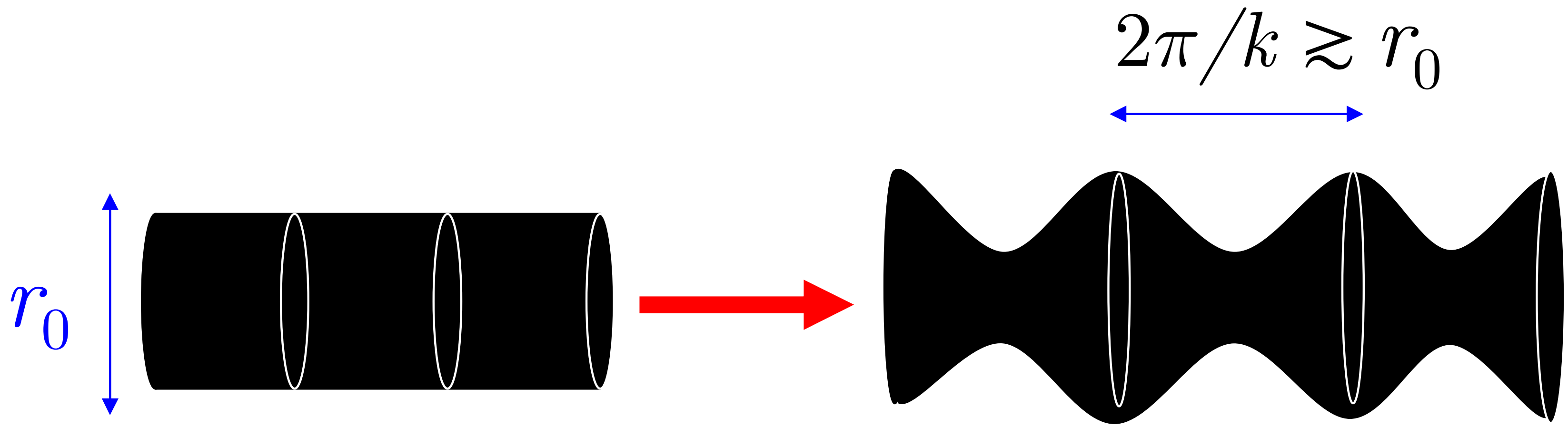
$$\sigma^{\mathrm{s-wave}} = A_{\mathrm{Hor}} \left(1 - \frac{\omega^2 r_0^2}{D} + \dots \right)$$

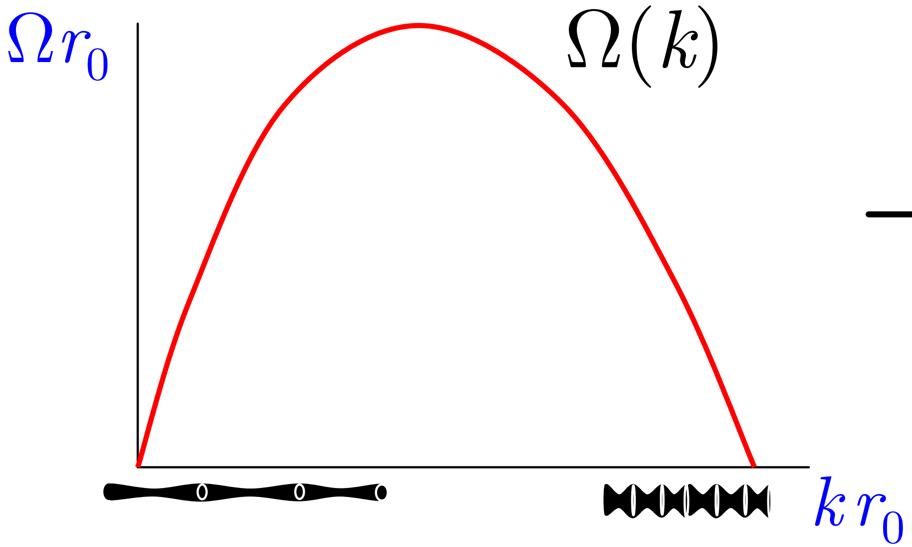
Gregory-Laflamme instability



$$\delta r_0 \sim e^{\Omega t + ikz}$$

Gregory-Laflamme instability





$$\delta r_0 \sim e^{\Omega t + ikz}$$

$\longrightarrow$ computed numerically from linearized perturbation

GL instability @ large *D*

- Near / far match
  up to fourth order:

$$\Omega = \hat{k} - \hat{k}^2 - \frac{\hat{k}}{2n}(1 + 2\hat{k} - 2\hat{k}^2) + \frac{\hat{k}}{24n^2}(9 + 24\hat{k} + 12\hat{k}^2 - 8\pi^2\hat{k}^2 + 8\pi^2\hat{k}^3 - 12\hat{k}^4)$$

$$\hat{k} = \frac{k}{\sqrt{n}}$$

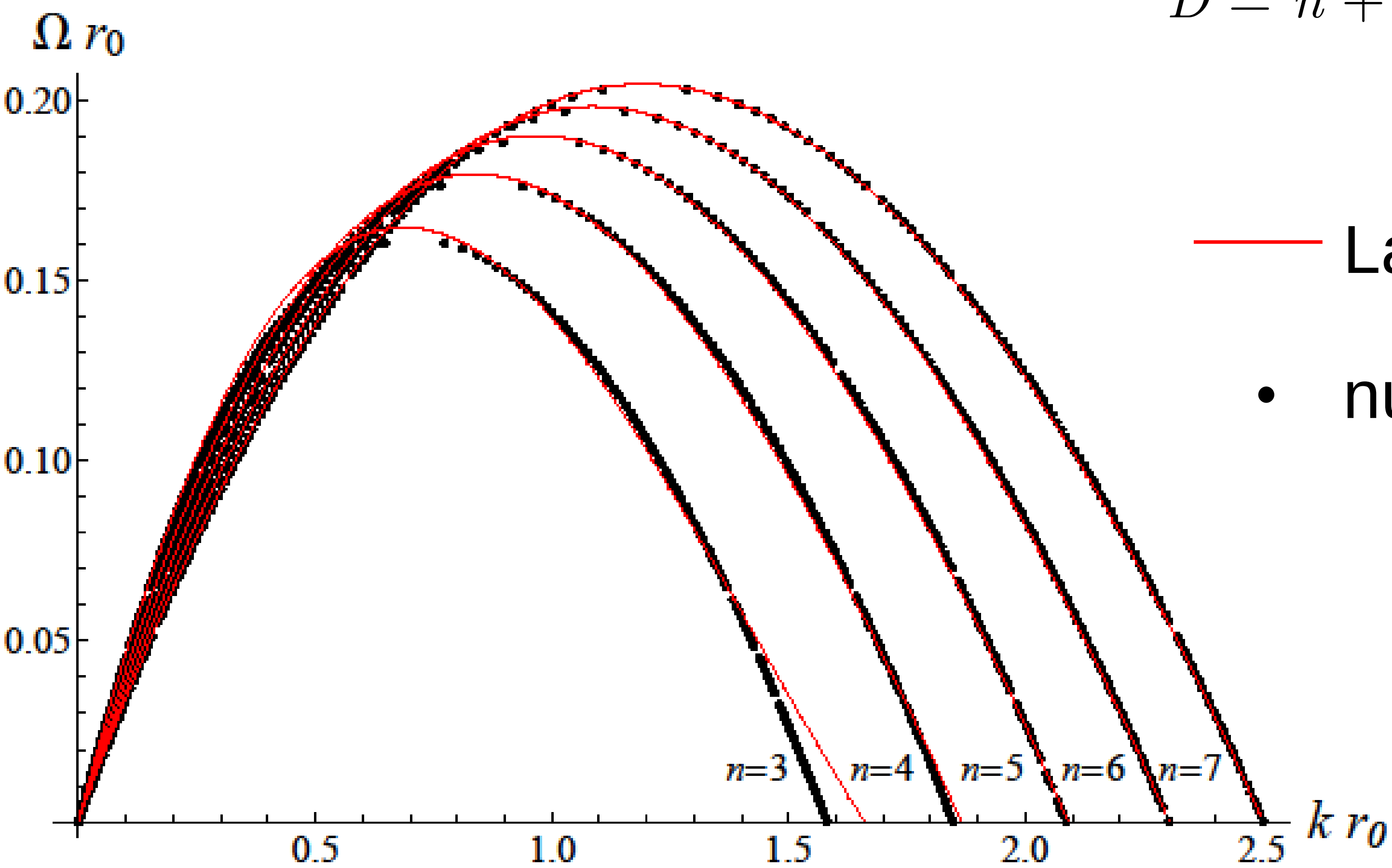
$$n = D-4$$

NB: zero-mode done by *Kol et al*

GL instability @ large *D*

- Compare to numerical results:

$D = n + 4$



Large D, analytic

- numerics (*Figueras*)

Outlook

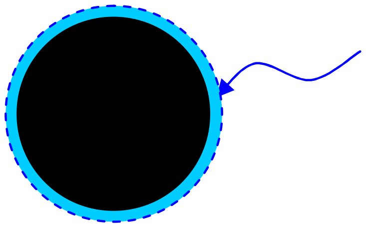
1. Practical method for solving GR

- *Any* problem that can be formulated in arbitrary D is amenable to large D expansion

  simpler, even analytically solvable

  Encouraging example: black string instab

2. Reformulate GR around large D?



$r - r_0 \lesssim \frac{r_0}{D}$ $\qquad$ $\alpha' \sim (r_0/D)^2$

- Far: $l > (\alpha')^{1/2}$: grav waves in flat space w/ holes
- Near: $l \sim (\alpha')^{1/2}$: ‘string’ physics
- Moonshine? or really *string theory* ? What kind?

Quantum effects

- Dimensionful scale

$$L_{\text{Planck}} = (G\hbar)^{1/(D-2)}$$

- Quantum effects governed by r_0 / L_{planck}
- Scaling: how large are the black holes, which quantum effects at large D

Finite entropy: $r_0 / L_{\text{planck}} \sim D^{1/2}$

Finite temperature: $r_0 / L_{\text{planck}} \sim D$

Finite energy of Hawking radn: $r_0 / L_{\text{planck}} \sim D^2$

Mass-length

$GM =$ measure of extrinsic curvature

$$GM = \frac{(D-2)\Omega_{D-2}}{16\pi} r_0^{D-3}$$

$$l_{\mathrm{mass}} \sim (GM)^{1/(D-3)} \sim r_0 / D^{1/2}$$

$$\ll l_{\mathrm{far}} \sim r_0$$

Anti-deSitter$_D$

- Must choose how to scale the cosmo-radius

  $\Lambda = -1/L_\Lambda{}^2 = -(D-1)(D-2)/2L^2$

- Keep L fixed: $\Lambda \to -\infty$

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_{D-2}, \quad f = 1 - \left(\frac{r_0}{r}\right)^{D-3} + \frac{r^2}{L^2}$$

- $r_H = r_0\big(1 + O(1/D)\big)$, $\qquad f \to 1 + r^2/L^2$
- Fields localized near horizon
- Black brane:

  $P = \varepsilon/(D-2) \to 0, \quad T_H \to D r_0/L^2$: hot dusty brane