Entanglement Entropy for Disjoint Intervals in AdS/CFT

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based on arXiv:1303.7221

(see also T.Hartman arXiv:1303.6955)

Entanglement Entropy : Definitions

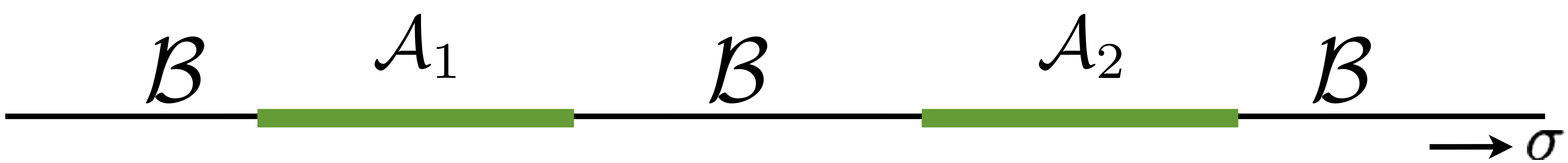
$$\mathcal{H} = \mathcal{H}_{\mathcal{A}} \times \mathcal{H}_{\mathcal{A}^c}$$

$$\rho_{\mathcal{A}} = \mathrm{tr}_{\mathcal{A}^c} \, |0\rangle \langle 0|$$

Vacuum

$$S_{EE} = -\mathrm{tr} \rho_{\mathcal{A}} \ln \rho_{\mathcal{A}}$$

Hilbert subspace chosen geometrically:



Focus on 1+1 dimensions: $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \ldots$

Entanglement Renyi Entropy (ERE)

$$S_n = \frac{1}{1-n} \log \left(\mathrm{tr} \rho_{\mathcal{A}}^n \right)$$

interesting quantity to study

$$\lim_{n \to 1} S_n = S_{EE}$$

Entanglement Entropy : Motivation

- Universal observable (lattice systems, QFT, ...)
- Diagnostic tool:
  - gapped phases (exotic/topological phases in CM)
  - critical systems (RG flow)
  - Fermions (area law violations, oscillating terms,...)
  - non-equilibrium dynamics (quantum quenches)
- Foundations:
  - computational algorithms based on EE
  - underpinnings of AdS/CFT? gravity?

cf: Myers and Polchinski's talk

Entanglement Entropy : Motivation

- However a major drawbacks:

  Really hard to calculate! Even for free fields

- But slowly our tools are becoming better

  e.g. Casini, Huerta, Myers; Nishioka, Yaakov

- This talk - intersection of two tools used to study EE

  2d CFT and the Ryu-Takayanagi formula

  Cardy, Calabrese — Ryu, Takayanagi

Original Goal:

arXiv:1303.7221

$\boxed{\text{Attempt to prove RT formula for some simple yet non-trivial cases}}$

After Juan and Aitor:

Maldacena,Lewkowycz
1304.4926

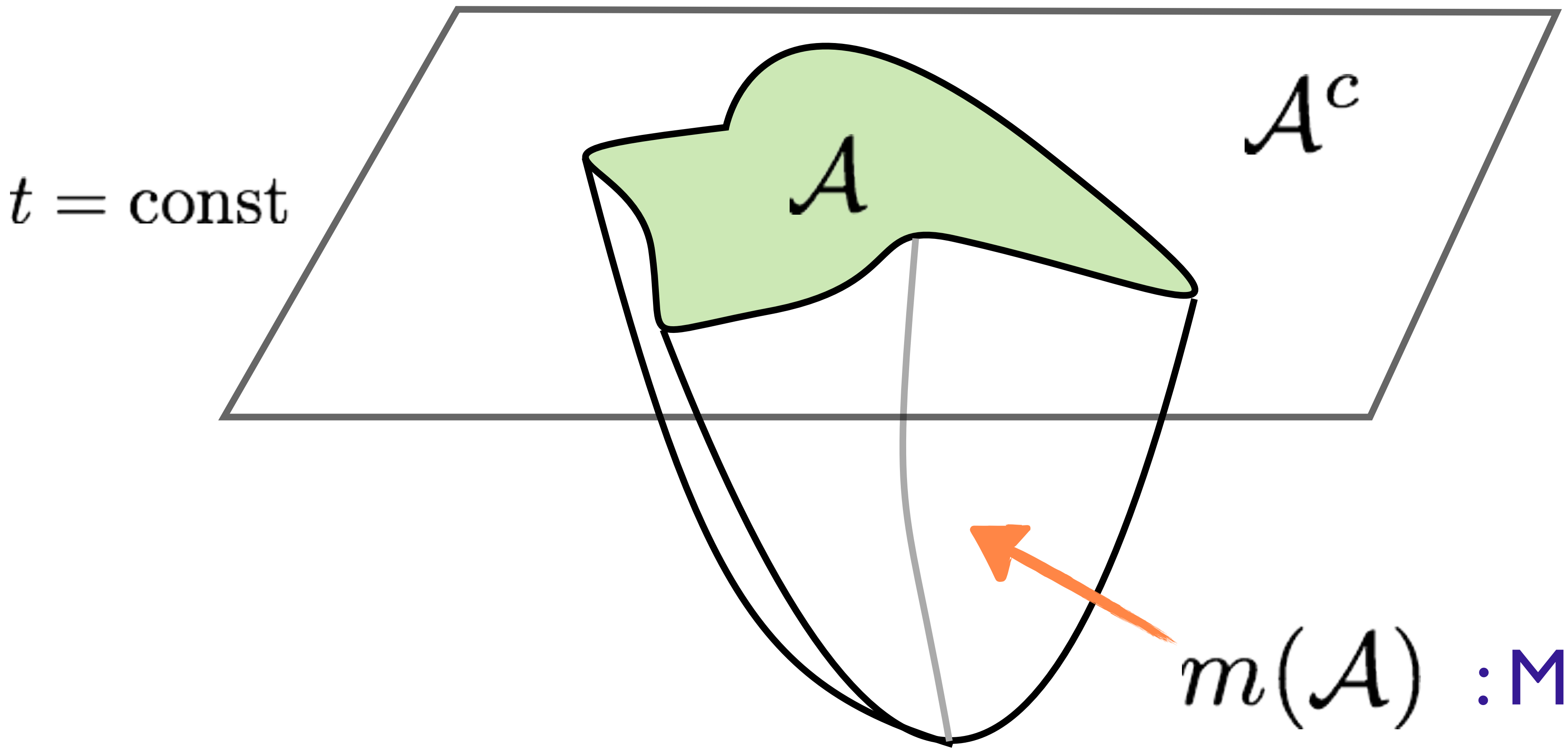
$\boxed{???}$

Maybe some things that made you uncomfortable about their proof should be alleviated here

- Proof does not directly apply here since the boundary manifold has non-trivial topology
- Understand limitations - if there are some?
- Systematically formulate corrections to it

Ryu-Takayanagi formula

Ryu, Takayanagi `06



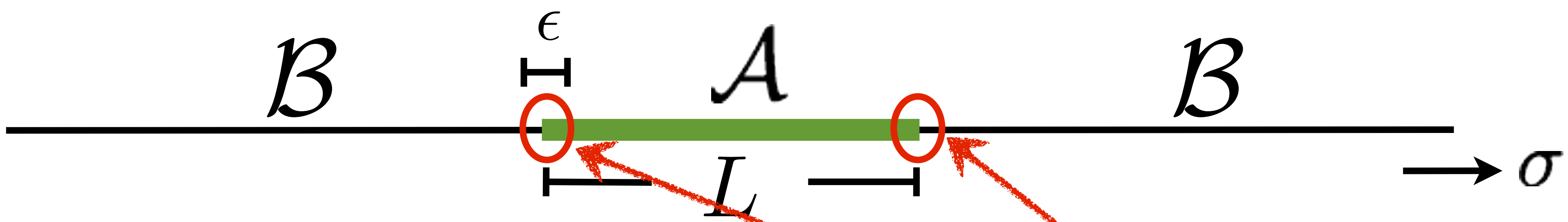
$m(\mathcal{A})$: Minimal surface

$$S_{EE} = \frac{\text{area}\,(m(\mathcal{A}))}{4G_N}$$

Valid for: $G_N \to 0$

Multiple minimal surfaces: take the global minimum

EE in 1+1 CFTs



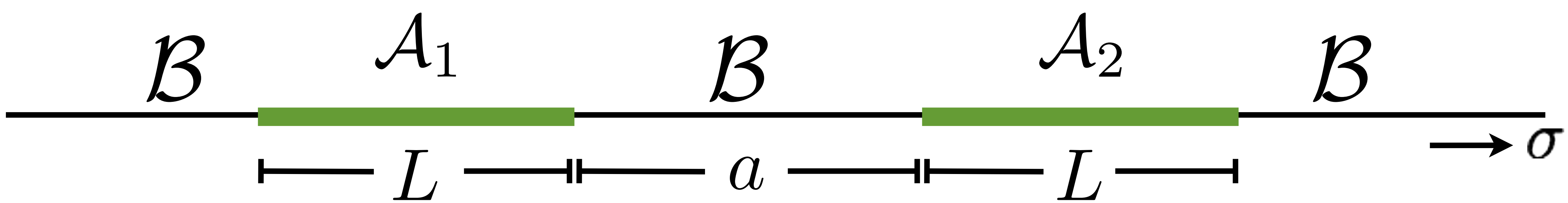
Single interval fixed by conformal invariance.

UV divergence

Holzhey, Larsen, Wilczek: $$S_{EE} = \frac{c}{3}\ln(L/\epsilon)$$

Next non-trivial case: Two intervals

Headrick `10

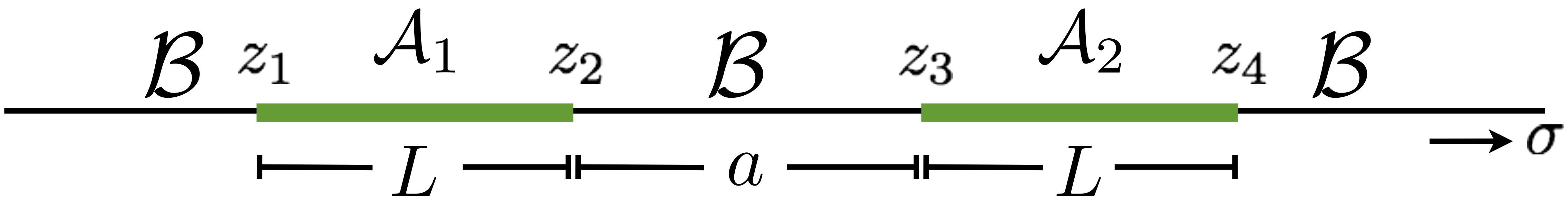


$$S_{EE}(\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2) = ??$$

Testing ground for RT

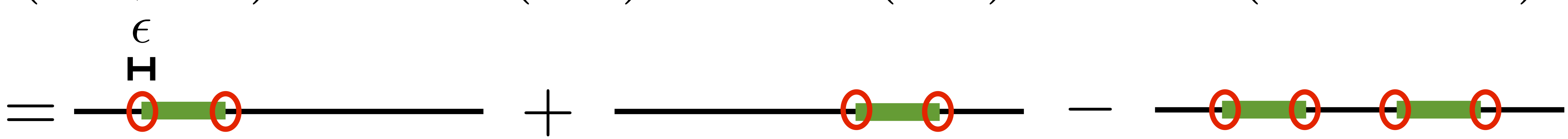
EE known for free fermions but NOT compact bosons!

Mutual Information



- In two interval case useful to consider:

$$I(\mathcal{A}_1, \mathcal{A}_2) = S_{EE}(\mathcal{A}_1) + S_{EE}(\mathcal{A}_2) - S_{EE}(\mathcal{A}_1 \cup \mathcal{A}_2)$$



UV divergence cancels out

- Cutoff independent + scale invariance - depends only on the ratio: $I(a/L)$

- More generally conformally invariant:

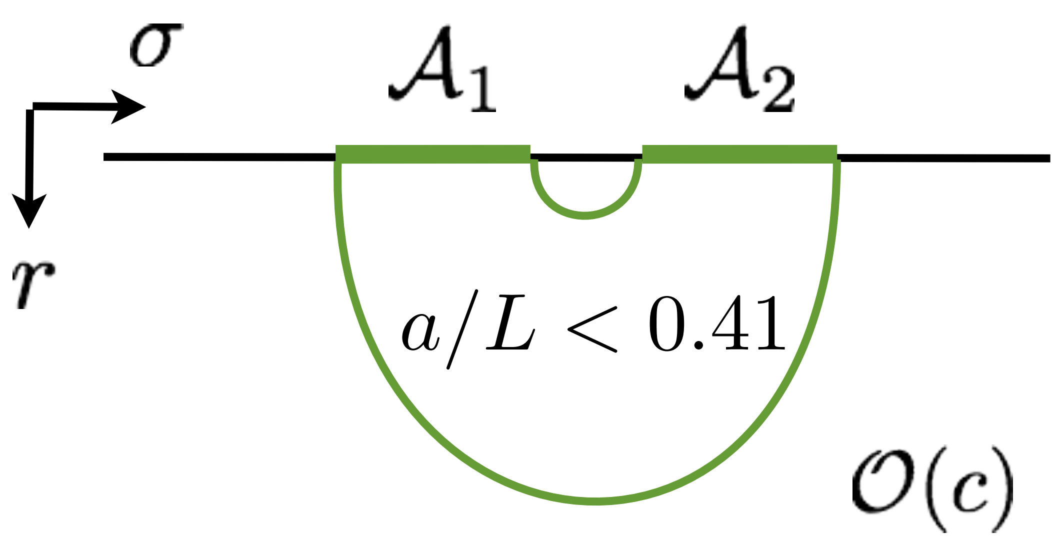
$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_4 - z_2)(z_3 - z_1)} = \frac{1}{(1 + a/L)^2}$$

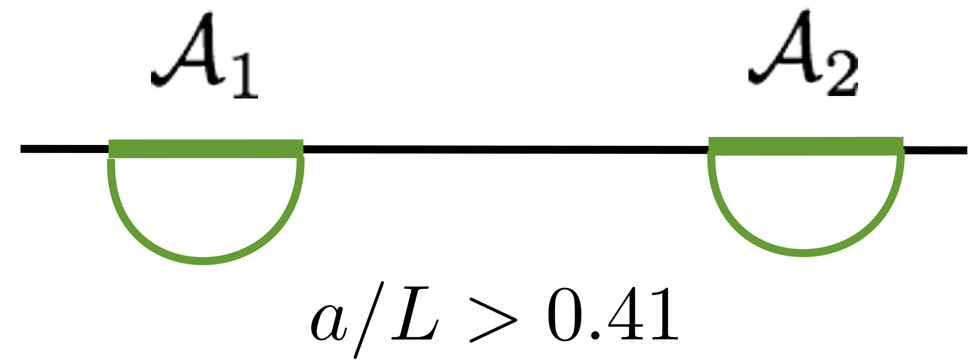
$G_N \propto c^{-1}$

RT prediction:

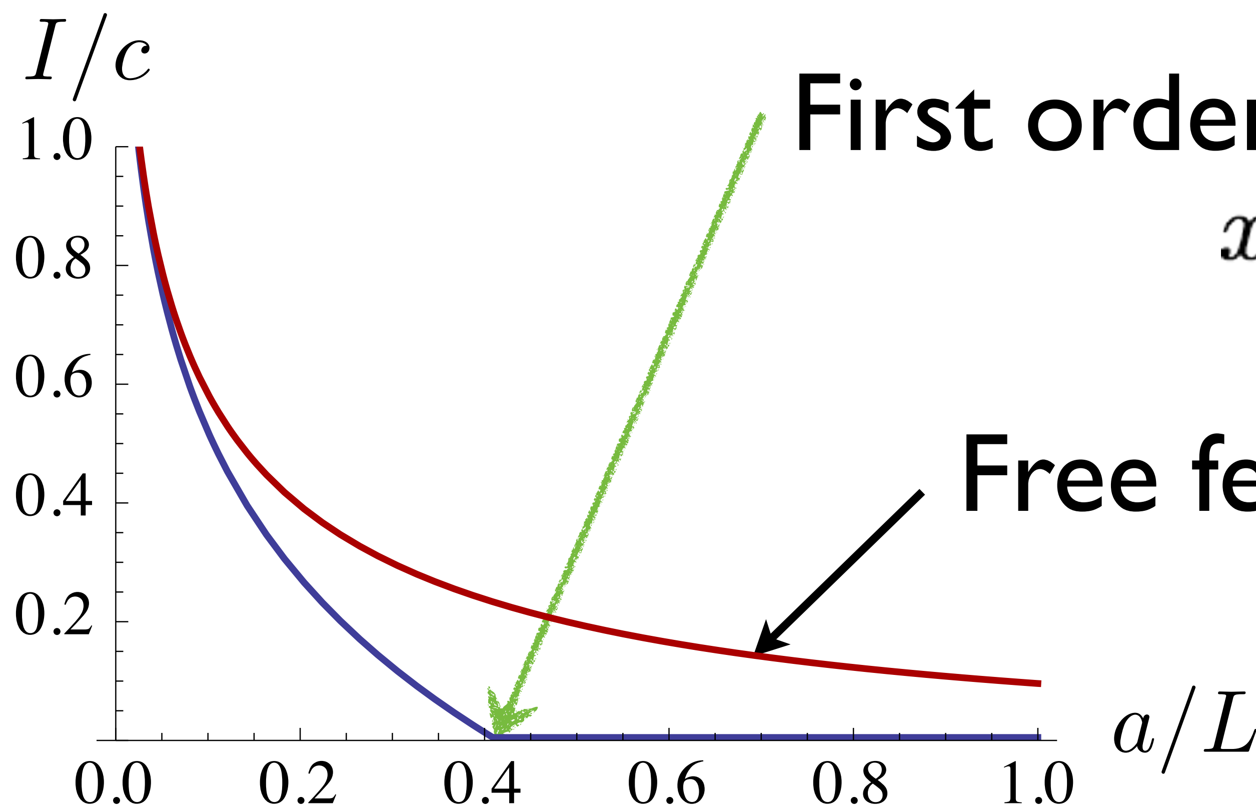
Headrick; Swingle

Two locally minimal “surfaces” (geodesics):





$I = 0$ **really** $\mathcal{O}(c^0)$



First order phase transition!

$x = 1/2$

Free fermions

The replica trick

Main computational tool for EE in QFT

$$S_{EE}(\mathcal{A}) = -\mathrm{Tr}\rho_{\mathcal{A}} \ln \rho_{\mathcal{A}}$$

hard to deal with

Introduce Entanglement Renyi Entropy:

$$S_n(\mathcal{A}) = -\frac{1}{n-1} \ln \mathrm{Tr}\rho_{\mathcal{A}}^n$$

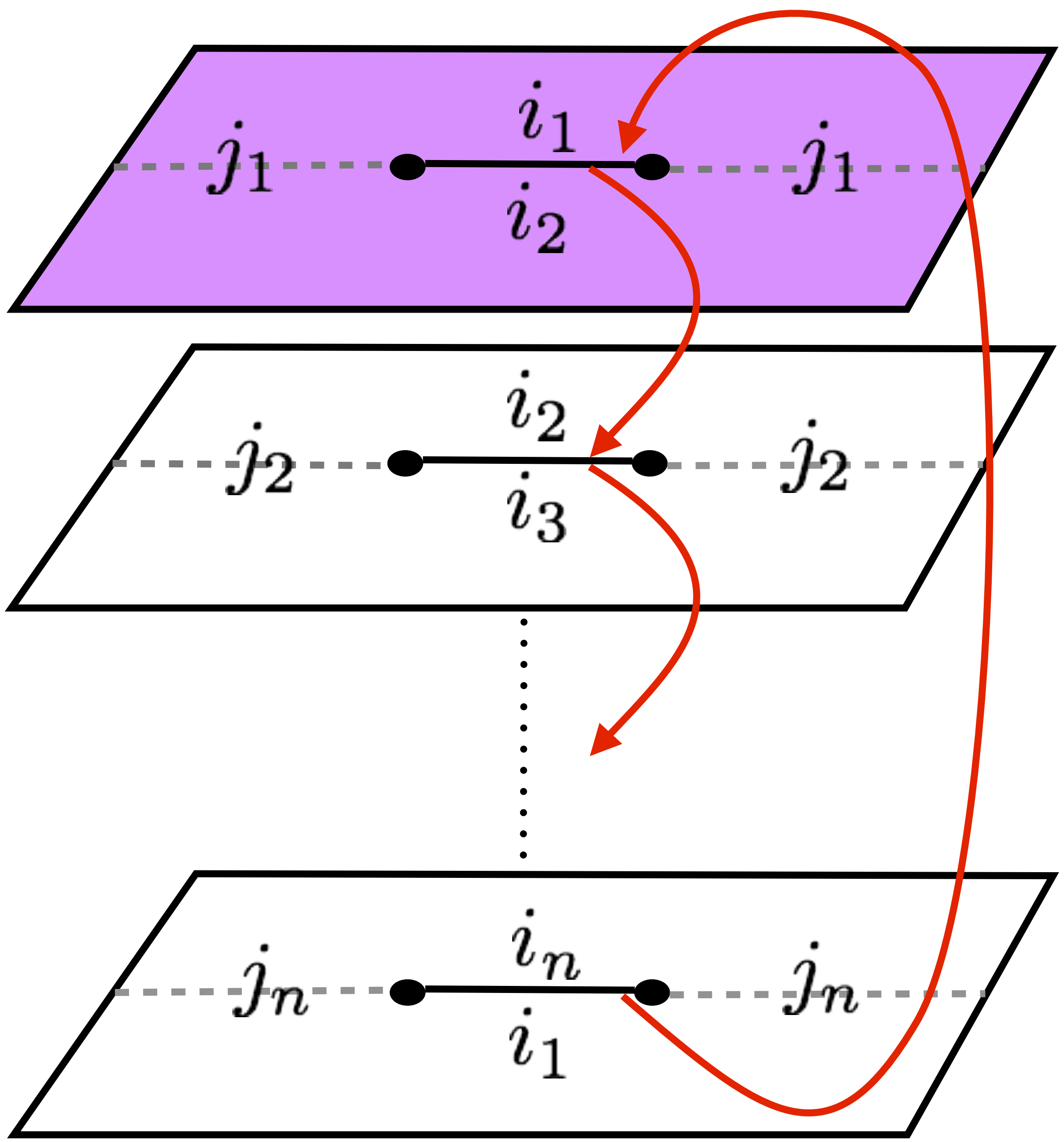
Compute for integer $n \geq 2$ ***attempt*** to continue to non-integer .... take the limit

$$\lim_{n\to 1} S_n(\mathcal{A}) = S_{EE}(\mathcal{A})$$

Why? One can formulate $\mathrm{Tr}\rho_{\mathcal{A}}^n$ as a *euclidean* path-integral

The replica trick

$$\mathrm{tr}\rho_{\mathcal{A}}^n = \sum_{i_1\in\mathcal{A}} \underline{\langle i_1|\,\rho_{\mathcal{A}} \sum_{i_2\in\mathcal{A}} |i_2\rangle} \langle i_2|\,\rho_{\mathcal{A}} \ldots \sum_{i_n\in\mathcal{A}} |i_n\rangle \langle i_n|\,\rho_{\mathcal{A}}\,|i_1\rangle$$



$$\boxed{\langle i_1|\,\rho_{\mathcal{A}}\,|i_2\rangle = \sum_{j_1\in\mathcal{A}^c} \langle i_1| \otimes \langle j_1|\,0\rangle\,\langle 0|\,i_2\rangle \otimes |j_1\rangle}$$

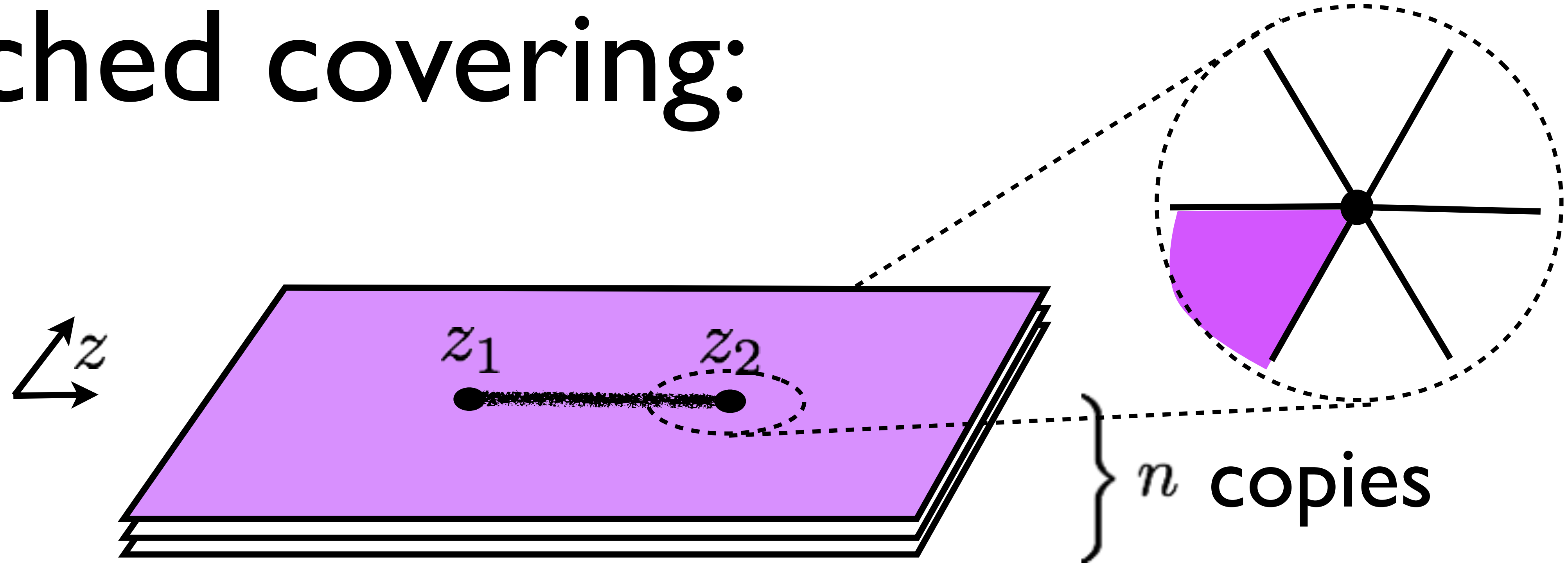
Partition function on this manifold:

$$\mathrm{tr}\rho_{\mathcal{A}}^n = \frac{Z_{\mathcal{M}_n}}{Z_1^n}$$

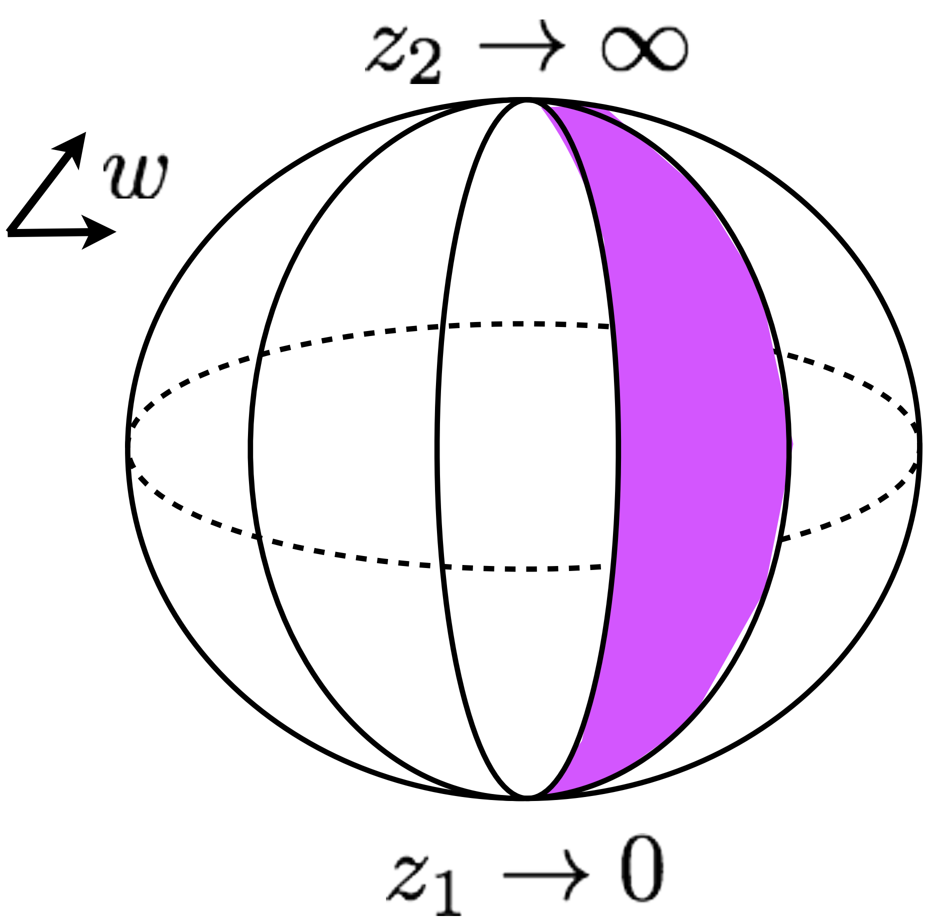
$$S_n(\mathcal{A}) = -\frac{1}{n-1}\left(\ln Z_{\mathcal{M}_n} - n\ln Z_1\right)$$

Branched covering:

$\mathcal{M}_n =$



For a single interval this is topologically a sphere:



Conformal transformation:

$$w^n = \left(\frac{z - z_1}{z - z_2} \right)$$

$$Z(\mathcal{M}_n) \sim Z(\text{sphere}) \quad ??$$

Almost but not quite!

Remember the Weyl anomaly in 2d CFTs:

$$ds^2 \equiv dzd\bar{z} = e^{2\phi} dwd\bar{w} \equiv e^{2\phi} d\hat{s}^2$$

$$Z(ds^2) = e^{S_L(\phi)} Z(d\hat{s}^2)$$

conformal factor

Liouville action

$$S_L = \frac{c}{6}\left(n - \frac{1}{n}\right) \ln(|z_1 - z_2|/\epsilon)$$

Gives ERE of a single interval:

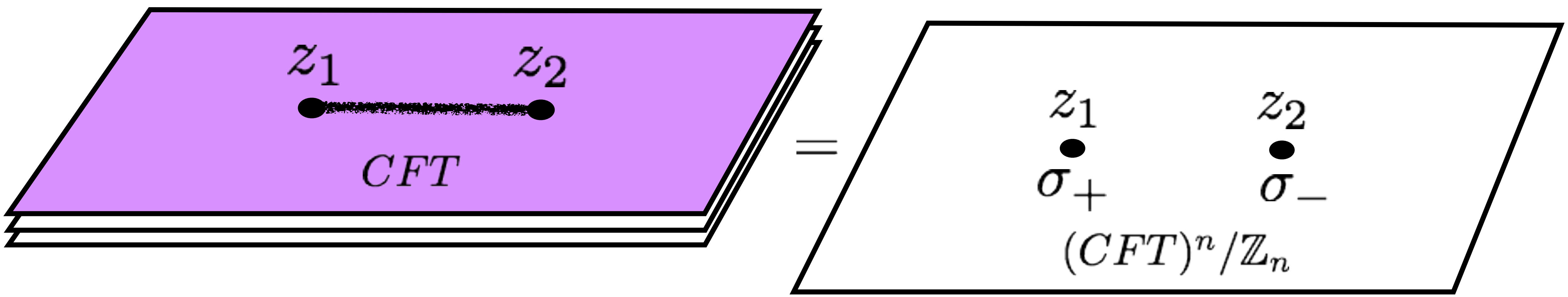
$$S_n = \frac{c}{6}(1 + n^{-1}) \ln(|z_1 - z_2|/\epsilon)$$

Twist operators:

Replica trick = computation in product orbifold:

$$(CFT)^n/\mathbb{Z}_n$$

Branch points = twist operators:



$$Z_{\mathcal{M}_n}(ds^2) = \langle \sigma_+(z_1)\sigma_-(z_2)\rangle|_{(CFT)^n/\mathbb{Z}_n}$$

More efficient way to calculate ERE:

$$w^n = \left(\frac{z - z_1}{z - z_2}\right)$$

$$\langle T(w)\rangle_{w-\text{plane}} = 0 \quad \rightarrow \quad \langle T(z)\rangle_{z-\text{plane}} = \frac{c}{12}\{w(z), z\}$$

Schwarzian: $\{w(z), z\} = \dfrac{w'''}{w'} - \dfrac{3}{2}\left(\dfrac{w''}{w'}\right)^2$

Famous example: Casimir energy of CFT on circle

$$\left(w = e^{iz/L} \rightarrow \langle T_{tt}\rangle_{\text{cyl}} \sim -c/L^2\right)$$

Manifestation of the Weyl Anomaly!

$$n\,\langle T(z)\rangle_{z-\text{plane}}\Big|_{CFT} = \frac{\langle T_{\text{orb}}(z)\sigma_+(z_1)\sigma_-(z_2)\rangle}{\langle\sigma_+(z_1)\sigma_-(z_2)\rangle}\Big|_{CFT^n/\mathbb{Z}_n}$$

$$= \sum_{i=1,2}\frac{h_n}{(z - z_i)^2} + \frac{p_i}{(z - z_i)}$$

Where: $h_n = \frac{c}{12}(n - n^{-1})$

$$p_1 = -p_2 = \frac{2h_n}{z_1 - z_2}$$

More efficient way to calculate ERE:

Conformal Ward Identity applied to the twist operators in orbifold theory:

$$\langle T_{\rm orb}(z)\sigma_+(z_1)\sigma_-(z_2)\rangle = \left(\sum_i \frac{h_n}{(z-z_i)^2} + \frac{1}{(z-z_i)}\frac{\partial}{\partial z_i}\right)\langle \sigma_+(z_1)\sigma_-(z_2)\rangle$$

conformal primaries

From this we can identify:

- twist operator dimension: $h_n = \frac{c}{12}\frac{n^2-1}{n}$
- accessory parameter:

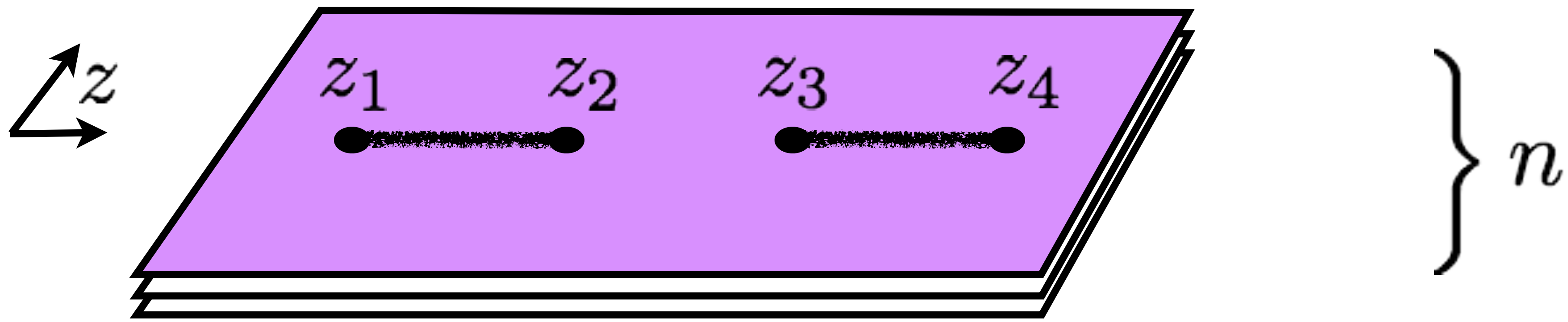
$$p_1 = \frac{\partial}{\partial z_1}\langle \sigma_+(z_1)\sigma_-(z_2)\rangle = \frac{2h_n}{z_1-z_2}$$

Integrating the accessory parameter gives us the ERE!

- this will hold for the multiple intervals case as well

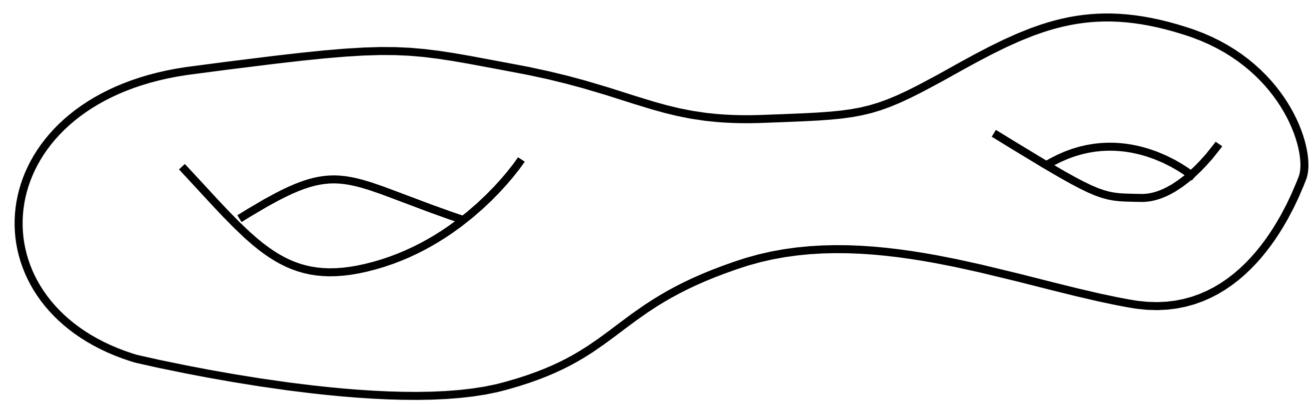
Two intervals leads to more complicated surface:

Riemann surface:

$\mathcal{M}_n =$ 

$$y^n = \frac{(z-z_1)(z-z_3)}{(z-z_2)(z-z_4)} \qquad ds^2 = dz d\bar{z}$$

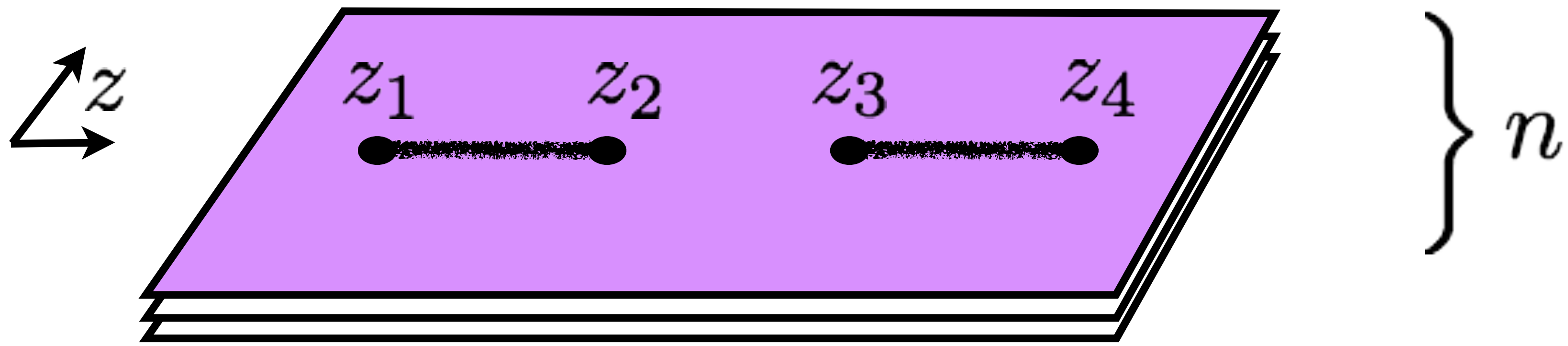
$\boxed{\text{Genus (n-1)}}$



- Goal: use usual rules of AdS/CFT to compute the partition function: $Z_{\rm CFT}(\mathcal{M}_n)$

Riemann surface:

$\mathcal{M}_n =$

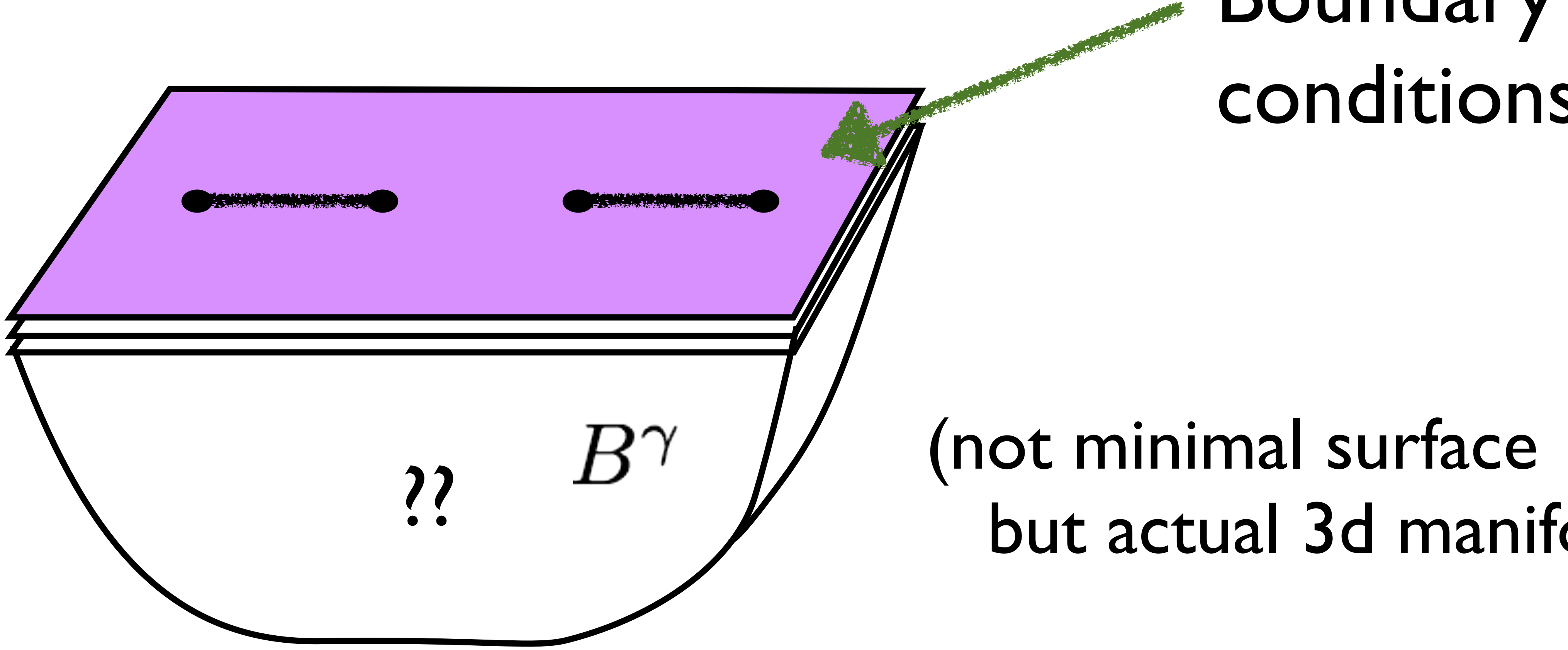


$$y^n = \frac{(z - z_1)(z - z_3)}{(z - z_2)(z - z_4)} \qquad ds^2 = dz d\bar{z}$$

- A genus $(n-1)$ surface has a $3(n-2)$ space of complex structures ⟵ (generalization of τ)
- This surface has only one real parameter
- This is a one dimensional slice of moduli space

Characterized as those with $\mathbb{Z}_n$ replica symmetry

Bulk Solution?



(not minimal surface but actual 3d manifold)

Solve Einstein's equations subject to boundary conditions and bulk regularity. $\partial B^\gamma = \mathcal{M}_n$

Many solutions! $\mathcal{O}(c)$

$$Z_{\mathcal{M}_n} = \sum_\gamma \exp\left(-S^\gamma_{\mathrm{grav}} + \mathcal{O}(c^0)\right) \qquad G_N \propto c^{-1}$$

Classical gravity limit: only need least action solution

$\boxed{\text{Key- magic of } AdS_3 \text{ !!}}$

All solutions of 3d gravity:

Quotients: $B_\gamma = \mathbb{H}_3/\Gamma_\gamma$

isometries

Γ_γ subgroup of $SL(2,\mathbb{C})$

still a huge number of these!

Simplifying assumptions:

1. Least action solution is a handlebody
2. This handlebody preserves the boundary symmetries:

$\boxed{\mathbb{Z}_n \text{ replica symmetry not spontaneously broken}}$

Find two solutions: $\gamma = \alpha, \beta$

TF `13

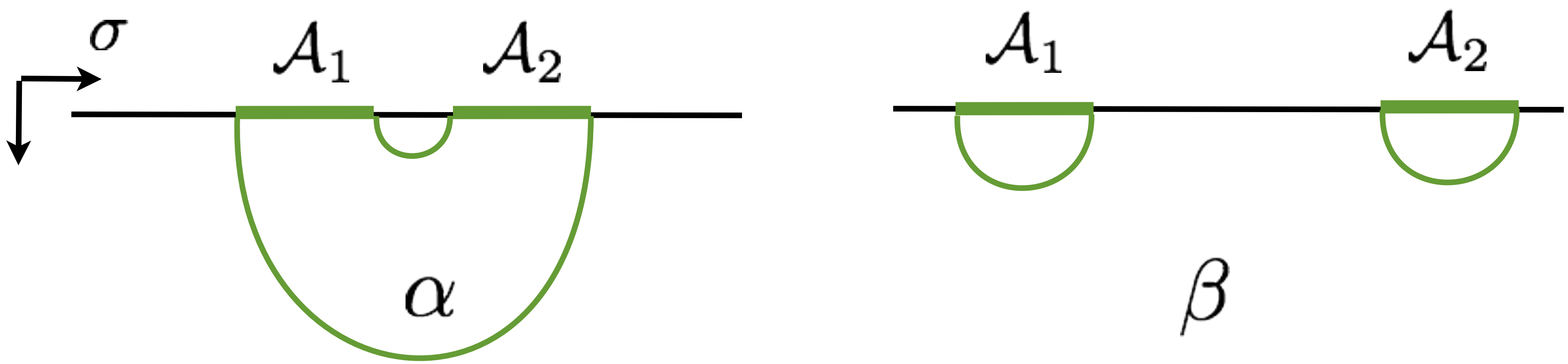
at fixed: (x, n)

To summarize:

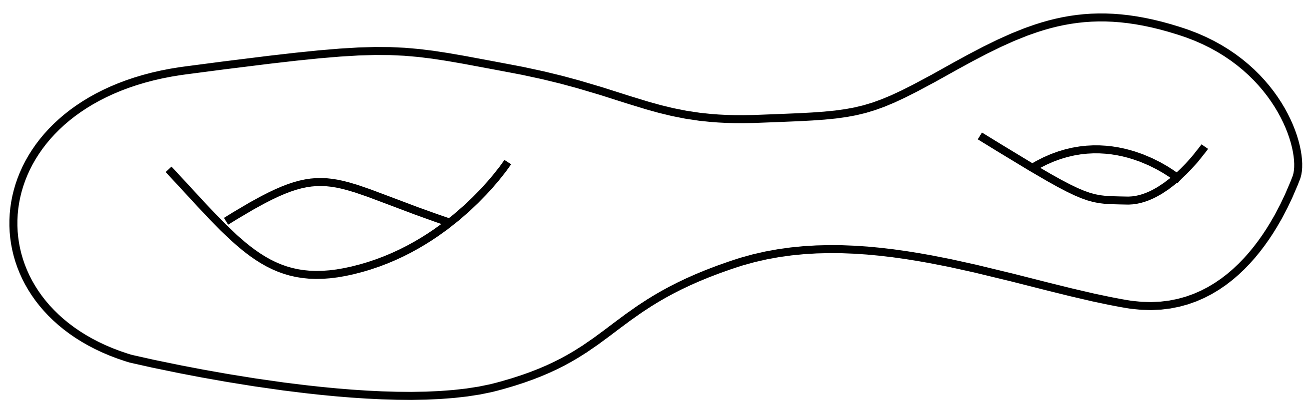
- α, β exchange dominance at $x = 1/2$
- Bulk action can be analytically continued in n

$$\lim_{n\to 1} \frac{1}{n-1}(S^{\alpha,\beta}_{\text{grav}} - S^1) =$$

Lengths of following geodesics:



Handlebody solution:



Fill in to make a solid torus

Handlebody solution:

Actually many different ways to fill in!

General handlebody solution:

$$\mathbb{H}_3 \; : \;\; ds^2 = \frac{dr^2 + dw d\bar{w}}{r^2}$$

Action of the quotient Γ_γ: $(r \approx 0)$

$$w \rightarrow \frac{aw+b}{cw+d}$$

$\rightarrow$ Conformal isometries of the boundary

Γ_γ acts on $\mathbb{C}$ such that $\mathbb{C}/\Gamma_\gamma = \mathcal{M}_n$

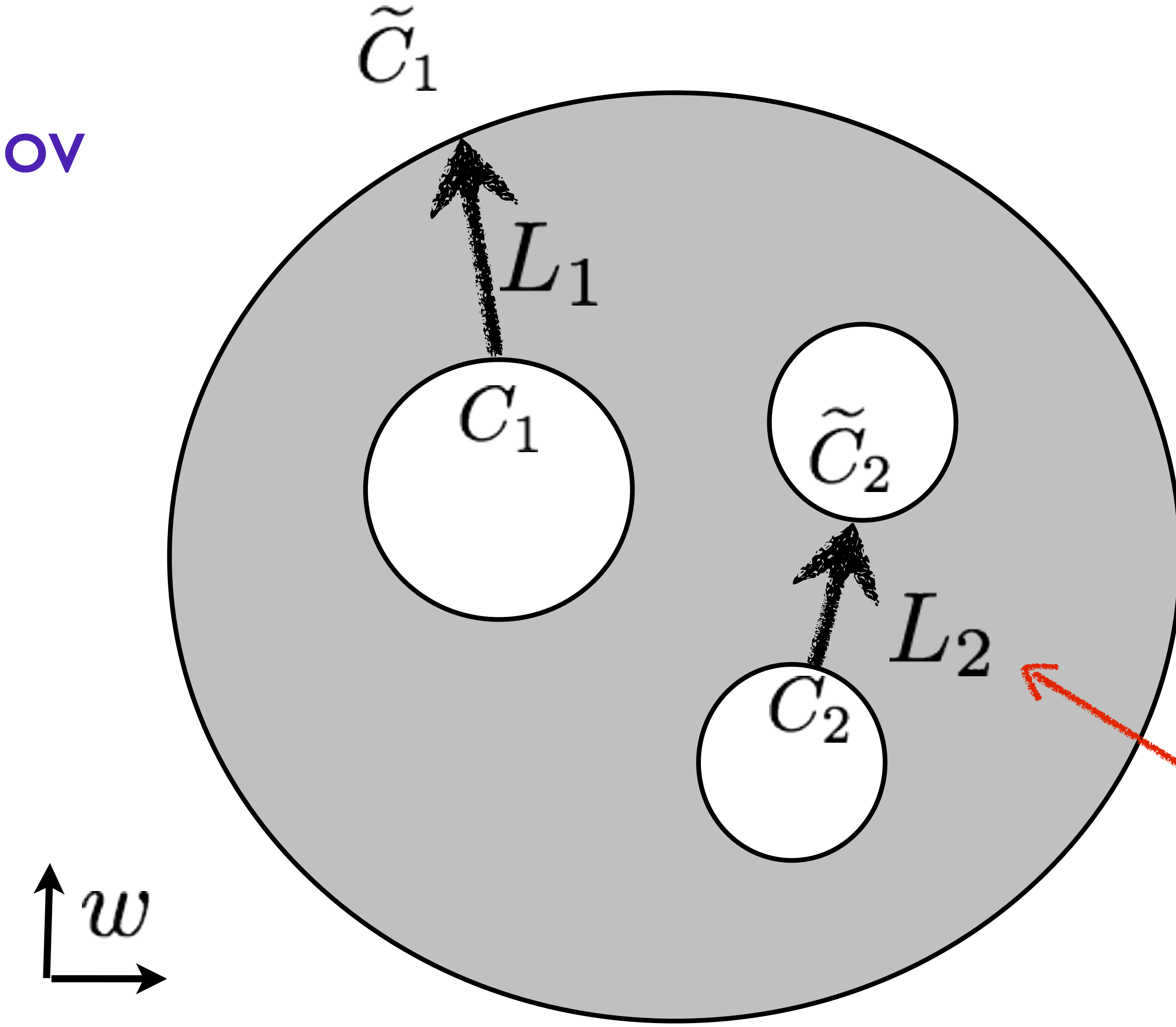
So let's start by describing: $\mathbb{C}/\Gamma_\gamma$

Handlebody requires Γ_γ to be a Schottky group

General handlebody solution:

draw 2g circles and identify pair wise:

Krasnov



$SL(2,\mathbb{C})$

$$\widetilde{C}_m = L_m(C_m)$$

Γ_γ : freely generated by $L_1, L_2, \ldots L_g$

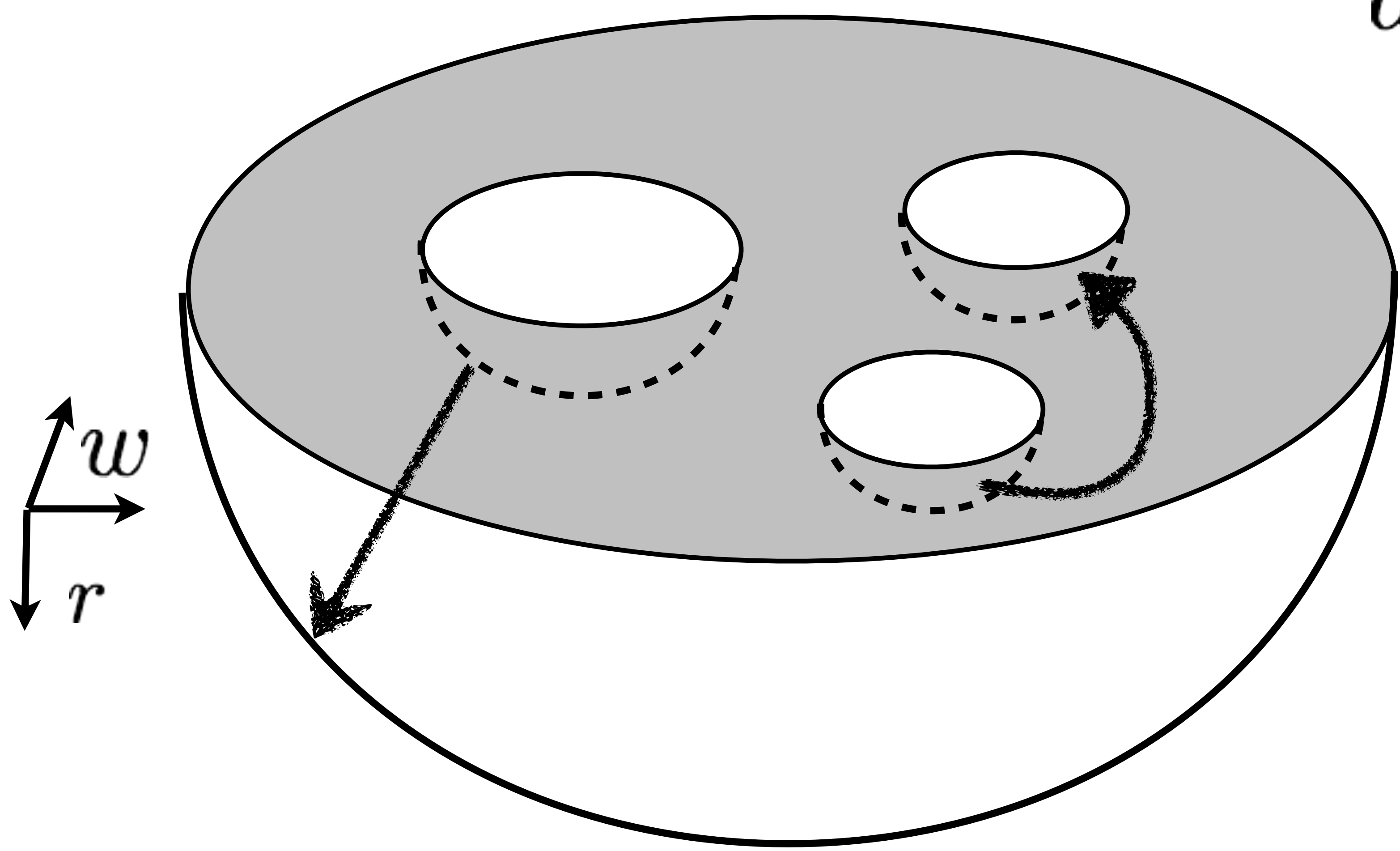
Fundamental domain of quotient

How do we describe the bulk?

General handlebody solution:

Super easy - extend circles into bulk hemispheres and identify hemispheres:

$$ds^2 = \frac{dr^2 + dw d\bar{w}}{r^2}$$



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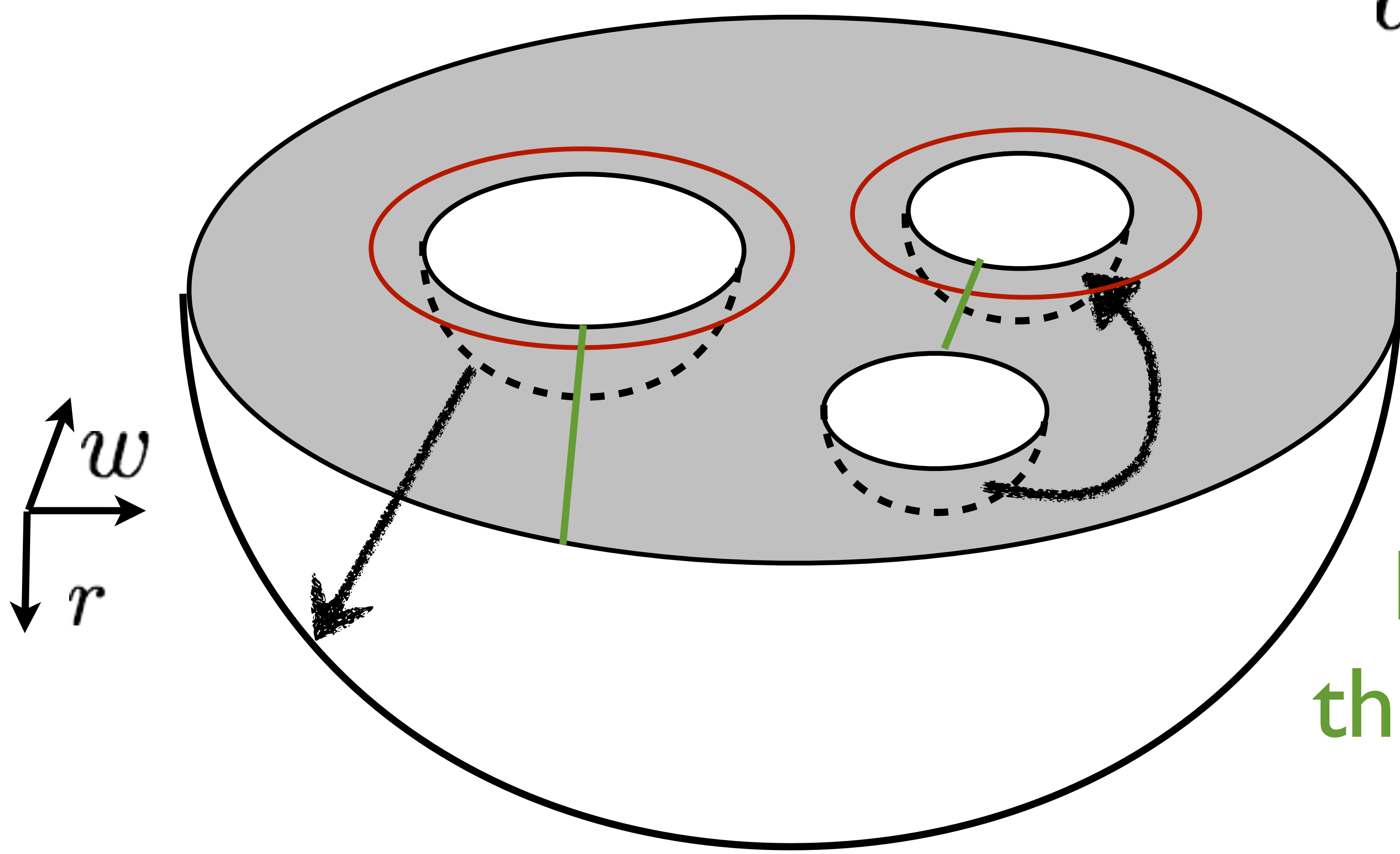


A cycles contractible in bulk

B cycles generate the L_m identifications

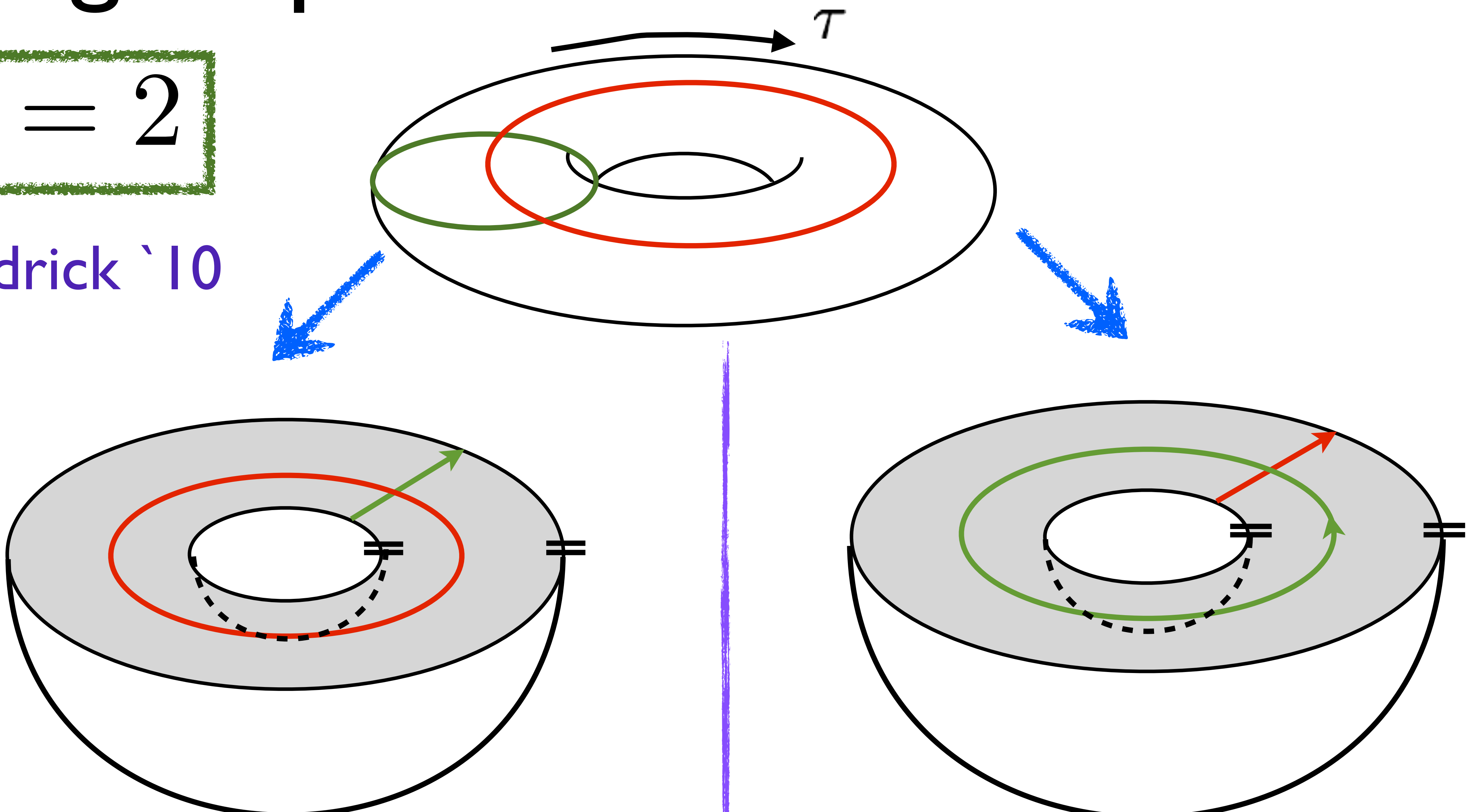
Solution distinguishes A and B cycles

$\mathcal{M}_n$ has 2g cycles

e.g. Representations of the torus

$$\boxed{n = 2}$$

Headrick `10



BTZ black hole

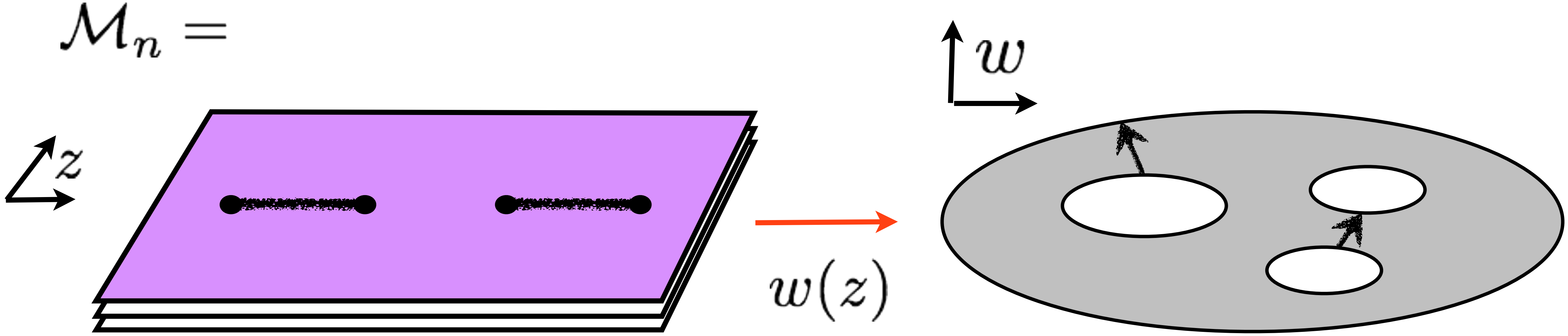
Thermal AdS

α $\longleftrightarrow$ β

Hawking page phase transition!

How to construct?

$\mathcal{M}_n =$



- Want to find w(z)
- Stress tensor on w-plane: $\langle T_{ww} \rangle^\gamma = 0$ ← empty AdS solution
- On the branched z-plane: $\langle T_{zz} \rangle^\gamma = \frac{c}{12}\left(\frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2\right)$
- Differential equation for w(z)! Construct $\langle T_{zz} \rangle^\gamma$ independently
- Better:

$$\psi''(z) + \hat{T}_{zz}\psi(z) = 0 \qquad w(z) = \frac{\psi_1(z)}{\psi_2(z)}$$

How to construct?

Stress tensor takes the form:

$$\hat{T}_{zz} = \sum_{i=1}^{4} \frac{h_n}{(z-z_i)^2} + \frac{p_i}{z-z_i}$$

twist operator dimension

4 unknown accesory parameters

- three conditions on p_i such that ∞ is regular point
- so only one unknown in $\hat{T}_{zz}$!!! call this: p_x
- more generally:

quadratic differential

$$\hat{T}_{zz} \to \hat{T}_{zz} + \sum_{s=1}^{3g-4} c_s \omega_{zz}$$

more accesory parameters

- but all these transform under $\mathbb{Z}_n$ replica symmetry so by assumption: $c_s = 0$

Monodromies:

But how do we fix the remaining accesory parameter?

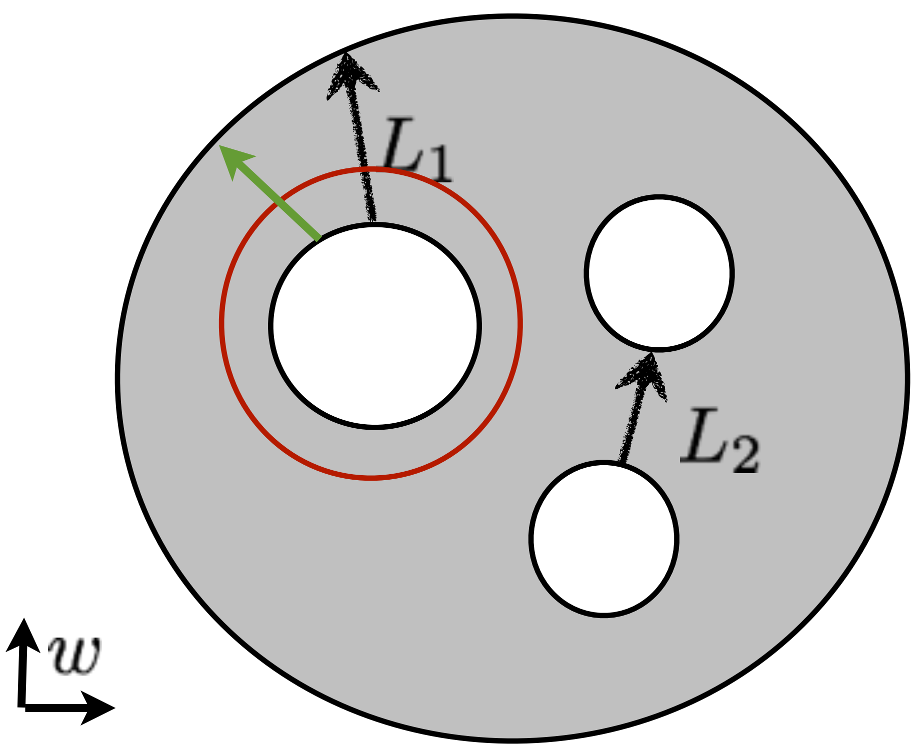
$$\psi''(z) + \hat{T}_{zz}\psi(z) = 0$$

$SL(2,\mathbb{C})$

$$w(z) = \frac{\psi_1(z)}{\psi_2(z)} \xrightarrow{C} L_C(w(z))$$

C closed path on $\mathcal{M}_n$

Tune p_i so this is true



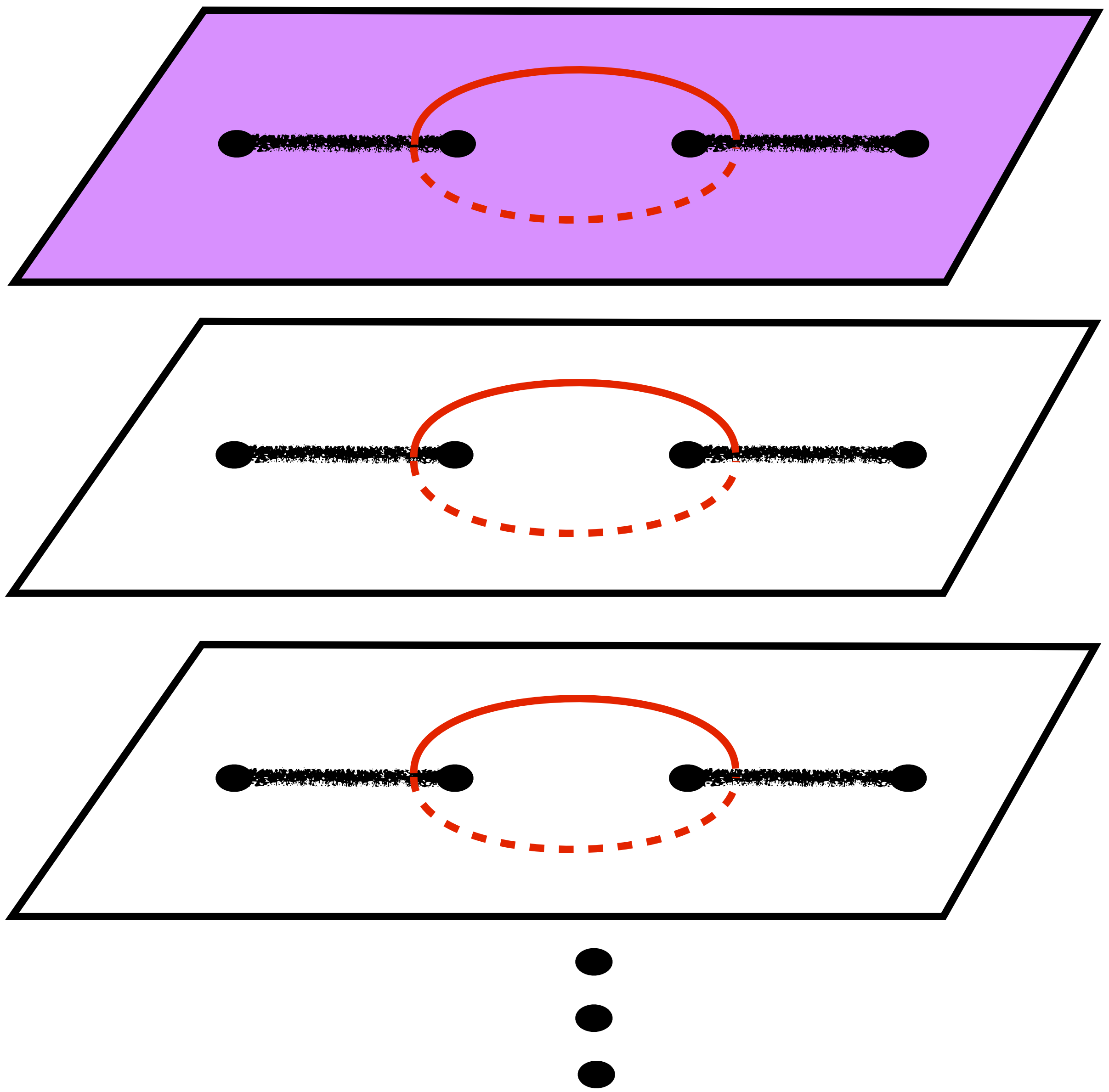
$$L_{A-\text{cycles}} = 1$$

$$L_{B-\text{cycles}} = L_m$$

Monodromies gives the quotient!

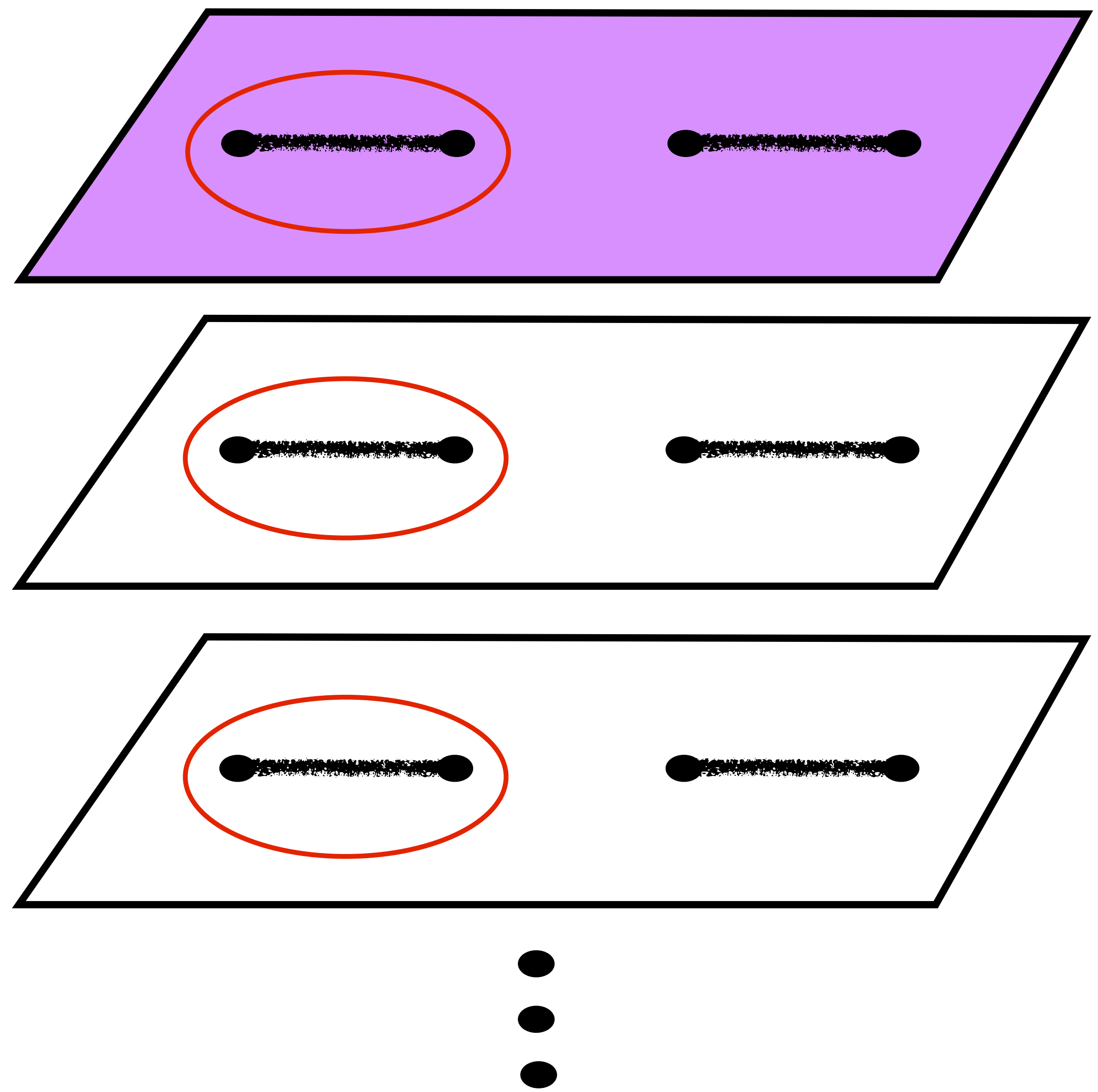
Pick A-cycles symmetrically:

$\alpha =$



OR

$\beta =$



Equivalent to fixing $c_s = 0$

Remaining p_x fixed by above monodromy condition

Computing the bulk action:

Use the same trick as for a single interval

Conformal Ward Identity in orbifold theory:

$$\langle T_{\text{orb}}(z)\sigma_+\sigma_-\sigma_+\sigma_-\rangle \approx \frac{cn}{12}\sum_\gamma \hat{T}^\gamma_{zz} e^{-S^\gamma_{\text{grav}}}$$

$$= \left(\sum_i \frac{h_n}{(z-z_i)^2} + \frac{1}{(z-z_i)}\frac{\partial}{\partial z_i}\right)\underline{\langle\sigma_+\sigma_-\sigma_+\sigma_-\rangle}$$

$$\approx \sum_\gamma e^{-S^\gamma_{\text{grav}}}$$

Comparing on a saddle by saddle basis:

$$(z_1, z_2, z_3, z_4) = (0, x, 1, \infty)$$

$$\boxed{\frac{dS^\gamma_{\text{grav}}}{dx} = \frac{cn}{12}p^\gamma_x}$$

Can integrate this to find bulk action

Comments:

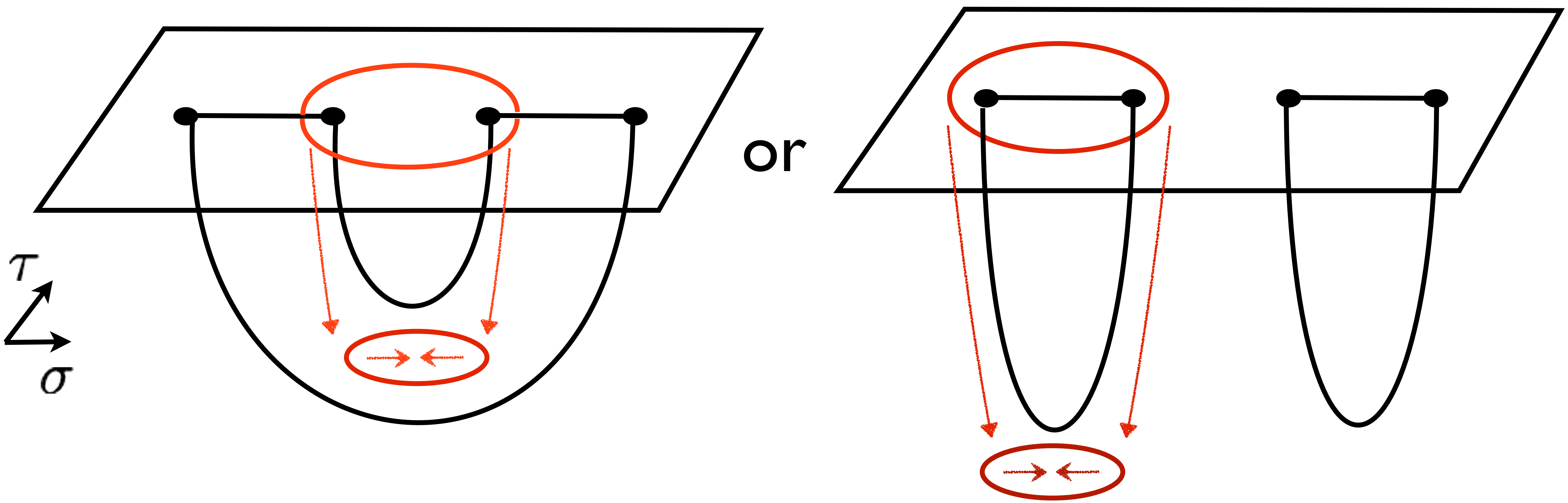
- Prescription: 1. solve the monodromy condition on the α or β cycles by tuning p_x
  2. Integrate to find the bulk action
  3. ERE is related to the minimal action:

$$S_n \sim \frac{1}{n-1} \min_{\gamma=\alpha,\beta} S^{\gamma}_{\mathrm{grav}}$$

cf: Juan and Aitor

- Numerical prescription!
- Prescription can be formulated for non-integer n
- Bulk solution makes no sense for non-integer n
- Limit $n \to 1$ can be found and gives RT answer
- More than two intervals can also be worked out and agrees with RT

More Comments:



Monodromy conditions can be understood in terms of the RT geodesics

Generalizations and applications:

Barrella, Dong, Hartnoll, Martin

- Can also compute ERE for a single interval in a thermal state using these ideas
- ODE is different and monodromy conditions are different
- With the bulk solution in hand these authors calculated the leading quantum corrections by computing 1-loop determinants:

$$Z^{H_3/\Gamma} = \prod_{\gamma \in \mathcal{P}} \left(\prod_{l,l'} \left(1 - q_\gamma^{l+h} \bar{q}_\gamma^{l'+h}\right)^{-1/2} \right)$$

Schottky

Multiplier of group element

- Corrections agreed with general CFT expectation

CFT derivation: Hartman

- Exact same prescription can be arrived at in a completely different way for large-c CFTs

$$\langle\sigma_+\sigma_-\sigma_+\sigma_-\rangle = \sum_{p} C^p_{+-} C^p_{+-} F(h_n, h_p, c; x)$$

primaries OPE coefficients conformal blocks

Zamolodchikov

- At large-c the relevant F's are computed by the same monodromy problem as for the handlebodies
- Assuming nice behavior of the spectrum of primaries as well as for the OPE coefficients one arrives at the same result

Conclusions

- Developed methods to compute EREs in 2d CFTs with a gravity dual
- Found agreement with RT
- Understood some of the assumptions that go into RT. Interesting to find situations where assumptions break down ... new non-geometric phases of gravity?

e.g: $\mathbb{Z}_n$ replica symmetry breaking??

- Methods have forged a way to calculate quantum corrections to RT
- Lots more to do!

Extras: