Finite momentum at string endpoints

(with application to a non-equilibrium phenomenon in AdS_5)

Steve Gubser (Princeton University)

Based on 1306.6648 with A. Ficnar (Columbia University)

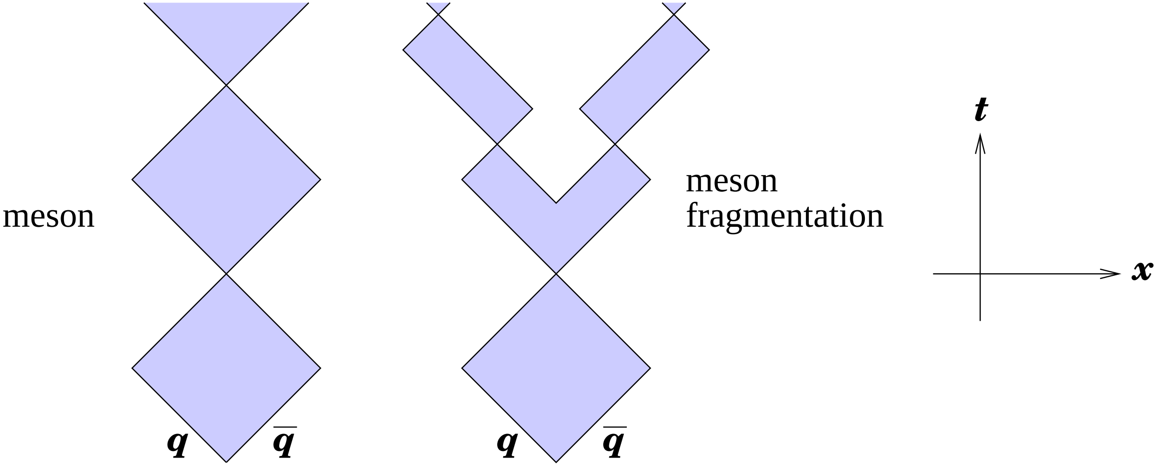
Kyoto, July 3, 2013

Contents

1. Finite endpoint momentum

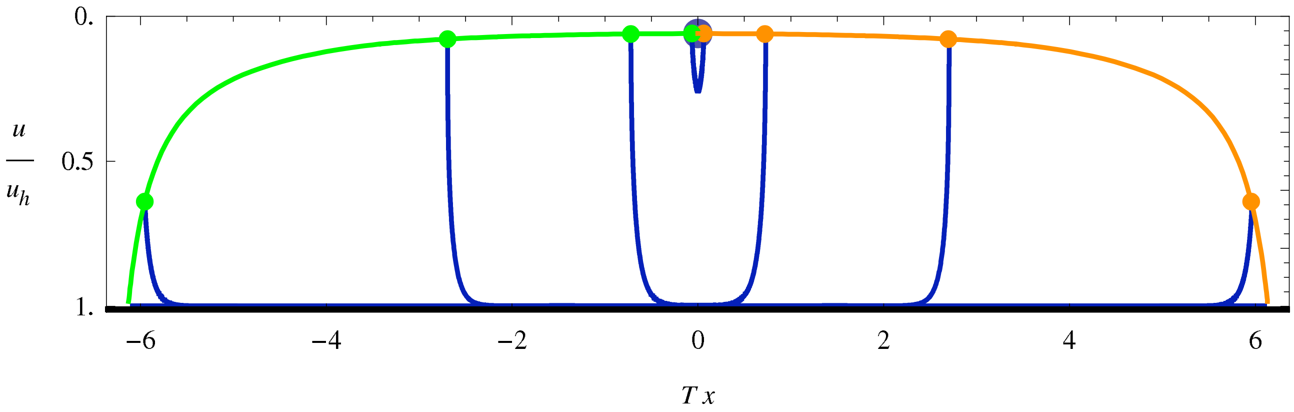
1.1. Why do we need it?

A highly successful phenomenological account of fragmentation (Lund model) starts with energetic quarks moving apart while linked by a string: the "yo-yo" [Andersson et al, 1983; Artru, 1983]. Earlier work goes back to [Bardeen et al, 1976].



- When $g_{\text{str}} = 0$, all that can happen is that the massless quark and anti-quark oscillate in a linear potential. $g_{\text{str}} \neq 0$ allows for fragmentation events.
- Initial energy is *entirely* in q and $\bar{q}$. Sometime later, it's entirely in the string.

To account for the medium in a heavy ion collision, a related strategy was pursued in AdS_5-Schwarzschild: [Chesler et al, 0804.3110], similar to [Gubser et al, 0803.1470].



Standard boundary conditions were applied: $\partial_\sigma X^\mu = 0$.

Initial state is a short string intended to reflect state of a quark-anti-quark pair produced in an energetic scattering event.

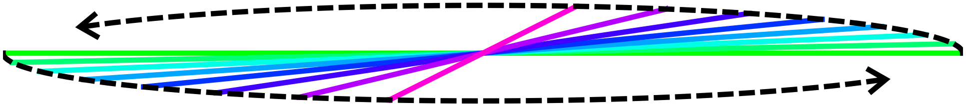
It would be more faithful to the Lund model to have finite momentum at the string endpoints.

To see why finite endpoint momentum makes sense for classical strings, consider an interpolation between Regge and the yo-yo:

$$X^\mu(\tau,\sigma) = \frac{1}{2}Y^\mu(\tau-\sigma) + \frac{1}{2}Y^\mu(\tau+\sigma)\,. \tag{1}$$

where

$$\frac{dY^\mu}{d\xi} = \begin{pmatrix} \sqrt{\ell_1^2 \sin^2\xi + \ell_2^2 \cos^2\xi} \\ \ell_1 \sin\xi \\ \ell_2 \cos\xi \end{pmatrix} \qquad Y^\mu(0) = \begin{pmatrix} 0 \\ -\ell_1 \\ 0 \end{pmatrix}\,. \tag{2}$$



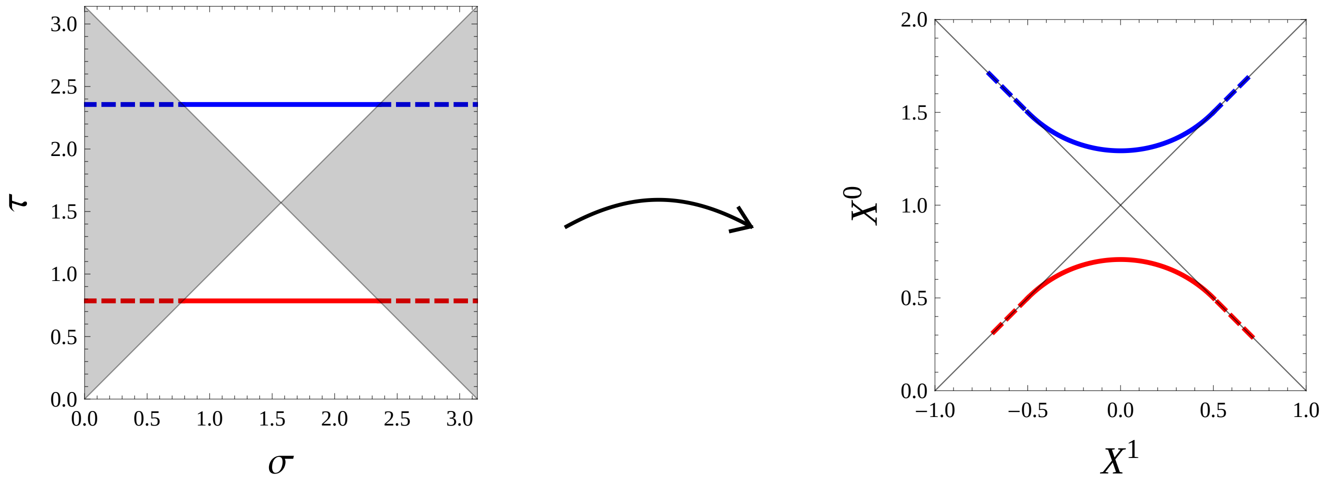
Snapshots at constant τ, with $\ell_2 = \ell_1/10$

Regge case is $\ell_1 = \ell_2$, and then $X^0 = \tau$.

Yo-yo is $\ell_2 = 0$, but now $X^0(\tau,\sigma)$ is complicated because $\dot{Y}^0 = \ell_1|\sin\xi|$.

Observe $X^\mu(\tau,0) = Y^\mu(\tau)$: endpoint prescribes entire motion of string.

The mapping $(\tau, \sigma) \to (X^0, X^1)$ is partially degenerate when $\ell_2 = 0$: a finite region maps to the edge of the string.



More transparent would be to use a static gauge, $X^0 = t$ and $X^1 = x$, and allow each endpoint to carry $E_{\text{endpoint}} = t/(2\pi\alpha')$, so that

$$E_{\text{total}} = \frac{2\ell_1 - 2t}{2\pi\alpha'} + 2 \times \frac{t}{2\pi\alpha'} = \frac{2\ell_1}{2\pi\alpha'}\,. \tag{3}$$

1.2. Endpoints follow geodesics

Now I want to argue that endpoint trajectories naturally follow spacetime geodesics when the endpoint momentum is non-vanishing. Argument proceeds in three steps:

Step 1: Formulate an action that includes finite endpoint momentum.

$$S = -\frac{1}{4\pi\alpha'} \int_M d\tau d\sigma \, \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \int_{\partial M} d\xi \, \frac{1}{2\eta} \dot{X}^\mu \dot{X}^\nu G_{\mu\nu} \,, \qquad (4)$$

where η is the einbein on the edge of the worldsheet.

Step 2: Formulate eom's in terms of endpoint momenta and bulk momentum density.

$$\begin{aligned} P^a_\mu &= -\frac{1}{2\pi\alpha'} \sqrt{-h} h^{ab} G_{\mu\nu} \partial_b X^\nu && \text{bulk momentum density} \\ p_\mu &= \frac{1}{\eta} G_{\mu\nu} \dot{X}^\nu && \text{endpoint momentum} \end{aligned} \qquad (5)$$

$$\begin{aligned} \partial_a P^a_\mu - \Gamma^\kappa_{\mu\lambda} \partial_a X^\lambda P^a_\kappa &= 0 && \text{bulk conservation of momentum} \\ \dot{p}_\mu - \Gamma^\kappa_{\mu\lambda} \dot{X}^\lambda p_\kappa &= \dot{\sigma}^a \epsilon_{ab} P^b_\mu && \text{boundary loses/gains energy from bulk} \end{aligned} \qquad (6)$$

Step 3: Manipulate endpoint equations in a conformal gauge.

Use a metric where $\sqrt{-h}h^{ab} = \text{diag}\{-1,1\}$. Then I claim

$$\dot{\sigma}^a \epsilon_{ab} P^b_\mu \pm \frac{\eta}{2\pi\alpha'} p_\mu = 0\,. \tag{7}$$

or, equivalently,

$$(\epsilon_{ab}\sqrt{-h}h^{bc} \mp \delta^c_a)\dot{\sigma}^a \partial_c X^\nu = 0\,. \tag{8}$$

This is because $M_a{}^c \equiv \epsilon_{ab}\sqrt{-h}h^{bc}$ has eigenvectors $(1,\pm 1)$; and along the worldsheet boundary, we have $\dot{\sigma}^a \propto (1,\pm 1)$.

So

$$\dot{p}_\mu - \Gamma^\kappa_{\mu\lambda}\dot{X}^\lambda p_\kappa = \mp \frac{\eta}{2\pi\alpha'} p_\mu\,, \tag{9}$$

where we take $-$ when the string endpoint is “unrolling.”

We can now see that endpoint moves along a geodesic:

$$\dot{\tilde{p}}_\mu - \Gamma^\kappa_{\mu\lambda}\dot{X}^\lambda \tilde{p}_\kappa = 0 \qquad \text{where} \qquad \tilde{p}_\mu = \frac{1}{\tilde{\eta}} G_{\mu\nu}\dot{X}^\nu\,. \tag{10}$$

1.3. Doubled strings in AdS_5

Yo-yo generalizes easily to global AdS_5, most simply as a doubled string.

$$ds_5^2 = L^2 \left(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_3^2\right) \,, \tag{11}$$

and we embed string into an AdS_2 submanifold:

$$ds_2^2 = L^2 \left(-\cosh^2\rho\, d\tau^2 + d\rho^2\right) \tag{12}$$

with endpoint trajectory determined by

$$\tan\frac{\tau}{2} = \tanh\frac{\rho}{2} \,. \tag{13}$$

The endpoint energy is

$$p_\tau = -\frac{EL}{2} + \frac{L^2}{\pi\alpha'}\sinh\rho \,. \tag{14}$$

so snapback occurs at $\rho_* = \sinh^{-1}\left(\frac{\pi\alpha'}{2L}E\right)$.

What is dual operator? Propose

$$\mathcal{O} = \operatorname{tr} X^I (\nabla_1)^S X^I \,, \tag{15}$$

in same multiplet as the operators $\operatorname{tr} X^I (\nabla_2 + i\nabla_3)^S X^I$ dual to Regge strings.

1.4. Lightcone Green-Schwarz action

Lightcone Green-Schwarz formalism accommodates finite endpoint momentum in an interesting way.

The claim is $S = S_{\text{bulk}} + S_{\text{bdy}}$ where, after requiring $X^+ = \pi q^+ \tau$ and $\Gamma^+ \theta = 0$, we write (with $\alpha' = 1/2$ and assuming a boundary at $\sigma^- = \text{constant}$)

$$S_{\text{bulk}} = \int_M d^2\sigma \left[-\frac{1}{2\pi} \eta^{ab} \partial_a X^i \partial_b X^i + i q^+ \bar{\theta} \Gamma^- \rho^a \partial_a \theta \right] \tag{16}$$

$$S_{\text{bdy}} = \frac{1}{2} \int_{\partial M} d\xi \, \frac{1}{\eta} \left[\dot{X}_i^2 + 2\pi i q^+ \bar{\theta} \rho^- \Gamma^- \dot{\theta} \right] \,, \tag{17}$$

where ρ^α are worldsheet gamma matrices and $\bar{\theta}^{Aa} \equiv \theta^{Bb} \Gamma^0_{AB} \rho^0_{ab}$.

S_{bdy} is *not* the light-cone superparticle, which would involve ρ^τ not ρ^-, and would be supersymmetric by itself.

The supersymmetry variations are

$$\delta X^i = 2\bar{\theta} \Gamma^i \epsilon \qquad \delta\theta = \frac{1}{2\pi i q^+} \Gamma^+ \Gamma^i \rho^a \partial_a X^i \epsilon \,. \tag{18}$$

A straightforward calculation leads to

$$\delta S_{\text{bulk}} = \int_M d^2\sigma\, \partial_a \left[\frac{1}{\pi} \bar\theta \rho^b \rho^a \Gamma^i \partial_b X^i \epsilon \right] . \tag{19}$$

The standard setup is to require $\partial_\sigma X^i = 0$ at a boundary $\sigma = 0$: then

$$\delta S_{\text{bulk}} = \int_{\partial M} d\xi\, \frac{1}{\pi} \bar\theta \rho_3 \Gamma^i \dot X^i \epsilon \qquad \text{where} \qquad \xi = \tau \quad \text{and} \quad \rho_3 = \rho^\tau \rho^\sigma ; \tag{20}$$

and by requiring

$$\theta = -i\rho^\sigma \theta \qquad \epsilon = i\rho^\sigma \epsilon \tag{21}$$

we get $\delta S_{\text{bulk}} = 0$.

Life is not so simple for a null boundary, say at $\sigma^- = 0$: Now

$$\delta S_{\text{bulk}} = -\int_{\partial M} d\xi\, \frac{1}{\pi} \bar\theta \rho^+ \rho^- \dot X^i \Gamma^i \epsilon = -\int_{\partial M} d\xi\, \frac{1}{\pi} \bar\theta (1 - \rho_3) \dot X^i \Gamma^i \epsilon \tag{22}$$

and we need boundary term to cancel this non-vanishing variation from the bulk.

First trick: Computation of p^+ and use of eom for $\dot p^+$ leads to

$$\frac{d}{d\xi} \left(\frac{1}{\eta} \right) = -\frac{1}{\pi} \qquad \text{where} \qquad \xi = \sigma^+ = \frac{\tau + \sigma}{\sqrt{2}} . \tag{23}$$

Second trick: $\delta\theta$ contains $\rho^a \partial_a X^i$, but because we only use $\rho^- \delta\theta$ on boundary, we only need $\rho^+ \partial_+ X^i = \rho^+ \dot{X}^i$.

Now we just need a couple of partial integrations wrt ξ to get

$$\delta S_{\mathrm{bdy}} = \int_{\partial M} d\xi \, \left[\frac{2}{\eta} \bar{\theta} \rho_3 \ddot{X}^i \Gamma^i \epsilon + \frac{1}{\pi} \bar{\theta} (1 - \rho_3) \dot{X}^i \Gamma^i \epsilon \right] . \tag{24}$$

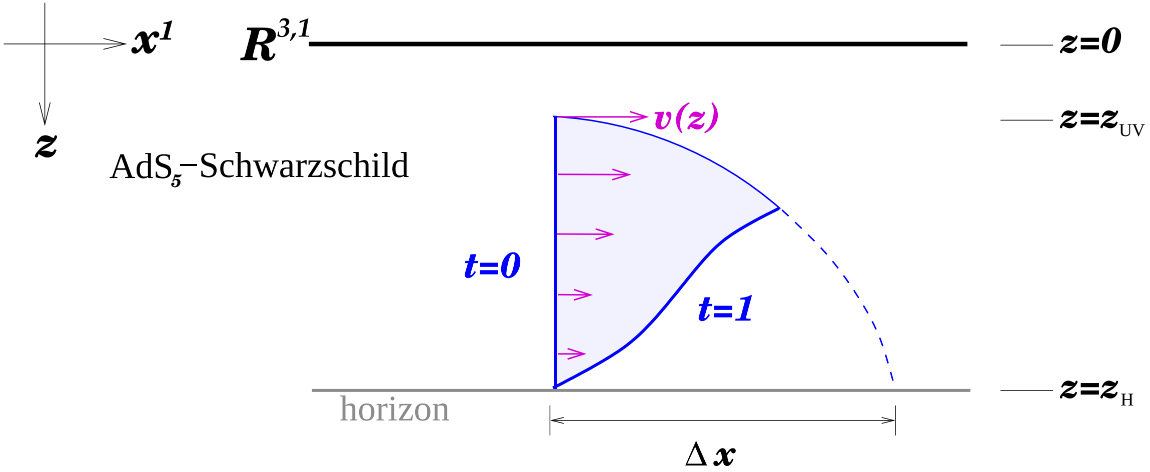
Red term cancels δS_{bulk}, and remaining term vanishes using (21).

Interesting questions remain:

- Could we have gotten boundary term from bulk GS action using a non-injective worldsheet embedding?
- What is the covariant, kappa symmetric action with boundary term?
- Could we consider localized momentum more generally on higher branes?

2. Application to light quark energy loss

Single quark setup: [Gubser et al, 0803.1470]

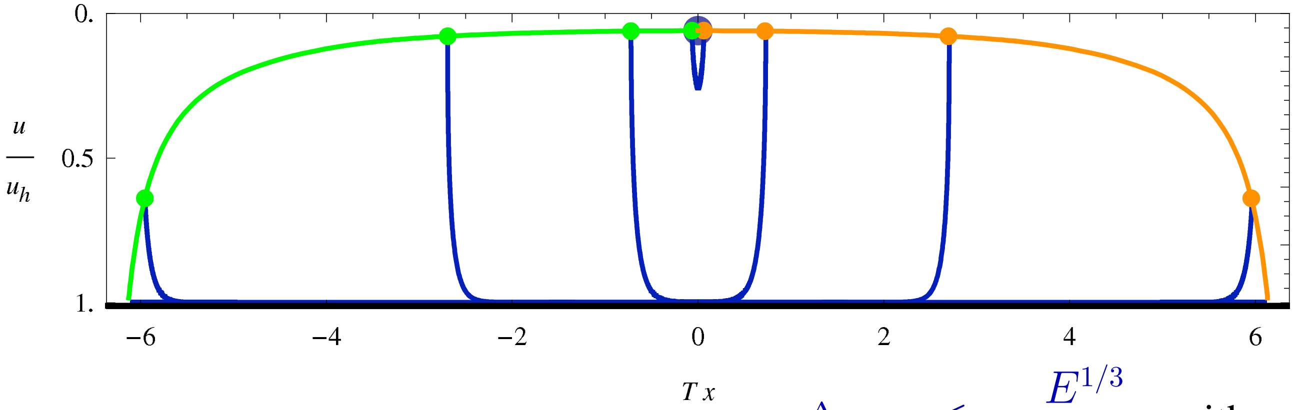


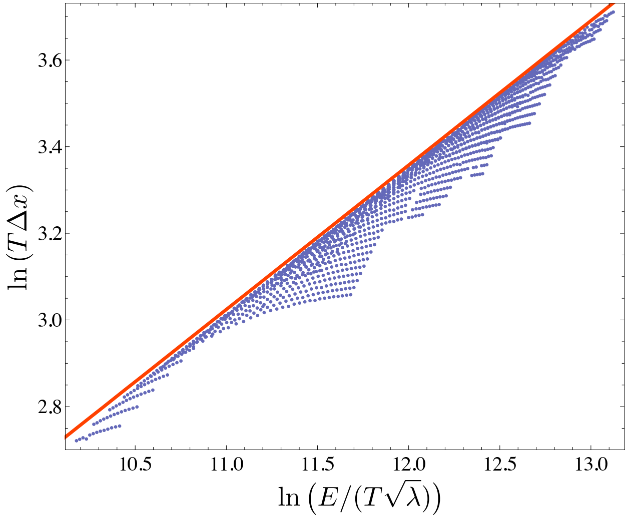
If a string starts at $t = 0$ with one end through the horizon and the other on a flavor brane, how far can it get before it falls through the horizon?

There's a big range of choice of initial conditions.

$\Delta x_{\mathrm{stop}} \lesssim \kappa \frac{E^{1/3}}{\lambda^{1/6} T^{4/3}}$ with an estimate $\kappa \in (0.35, 0.41)$.

Dissociating meson setup: [Chesler et al, 0804.3110]





- $\Delta x_{\text{stop}} \leq \kappa \frac{E^{1/3}}{\lambda^{1/6} T^{4/3}}$ with $\kappa = 0.526$ from extensive numerical study.
- Reminiscent of perturbative BDMPS result (e.g. [Baier et al, hep-ph/9608322])
$$\Delta E_{\text{BDMPS}} = \frac{1}{4}\alpha_s C_R \hat{q} (\Delta x)^2 .$$
- Recent PHENIX study [Adare et al, 1208.2254] actually favors $\Delta E \propto \ell^3$ over ℓ^2.

Our plan:

- Show how $\kappa = 0.526 = \frac{2^{1/3}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$ comes out of spacetime geodesics plus a slightly tricky accounting of initial energy.
- Show how finite endpoint momentum gives $\kappa = 0.624$.
- Show how single quark can approach $\kappa = 0.990$.
- Propose a new account of instantaneous energy loss based on endpoint $\dot{p}_\mu$.

2.1. No endpoint momentum

When a string has a lot of momentum in x^1 direction, it quickly settles into a segment of trailing string with velocity $v = \sqrt{f(z_*)}$, where AdS_5-Schwarzschild metric is

$$ds^2 = \frac{L^2}{z^2}\left(-f(z)dt^2 + d\vec{x}^2 + \frac{dz^2}{f(z)}\right) \qquad \text{with} \qquad f(z) = 1 - \frac{z^4}{z_H^4} . \tag{25}$$

So we evaluate energy (half the total energy of the meson) as

$$E_* = \frac{L^2}{2\pi\alpha'}\frac{1}{\sqrt{1-v^2}}\left[\frac{1}{z_*} - \frac{1}{z_H}\right] + \frac{1}{v}\frac{dE}{dt}\Delta x(z_*, z_H) \approx \frac{L^2}{2\pi\alpha'}\frac{1}{\sqrt{1-v^2}}\frac{1}{z_*} . \tag{26}$$

The endpoint subsequently stays *close* to a null geodesic, which is a solution to

$$\frac{dx_{\text{geo}}}{dz} = \frac{1}{\sqrt{f(z_*) - f(z)}} = \frac{z_H^2}{\sqrt{z^4 - z_*^4}} \,. \tag{27}$$

So we find Δx_{stop} by intersecting geodesic with horizon:

$$\Delta x_{\text{stop}} = \frac{z_H^2}{z_*} \frac{\sqrt{\pi}\Gamma(\frac{5}{4})}{\Gamma(\frac{3}{4})} - {}_2F_1\left(\frac{1}{4}, \frac{1}{2}, \frac{5}{4}, \frac{z_*^4}{z_H^4}\right) z_H \,, \tag{28}$$

and in the high-energy limit where $z_* \ll z_H$

$$\Delta x_{\text{stop}} = \frac{2^{1/3}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} \frac{1}{T} \left(\frac{E_*}{\sqrt{\lambda} T}\right)^{1/3} \,, \tag{29}$$

2.2. Including endpoint momentum

Following spirit of Lund, assign all energy to the endpoints initially. Also require $E_{\text{endpoint}} \to 0$ just as string crosses horizon.

Calculate evolution of $E_{\text{endpoint}} = -p_t$ using

$$\dot{p}_t = -\frac{\eta}{2\pi\alpha'} p_t = \frac{\sqrt{\lambda}}{2\pi} \frac{f}{z^2} \frac{dt}{dz} \,. \tag{30}$$

Arrive at

$$E_* \approx \frac{\sqrt{\lambda}}{2\sqrt{\pi}} \frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)} \frac{z_H^2}{z_*^3} \sqrt{f(z_*)} \,. \tag{31}$$

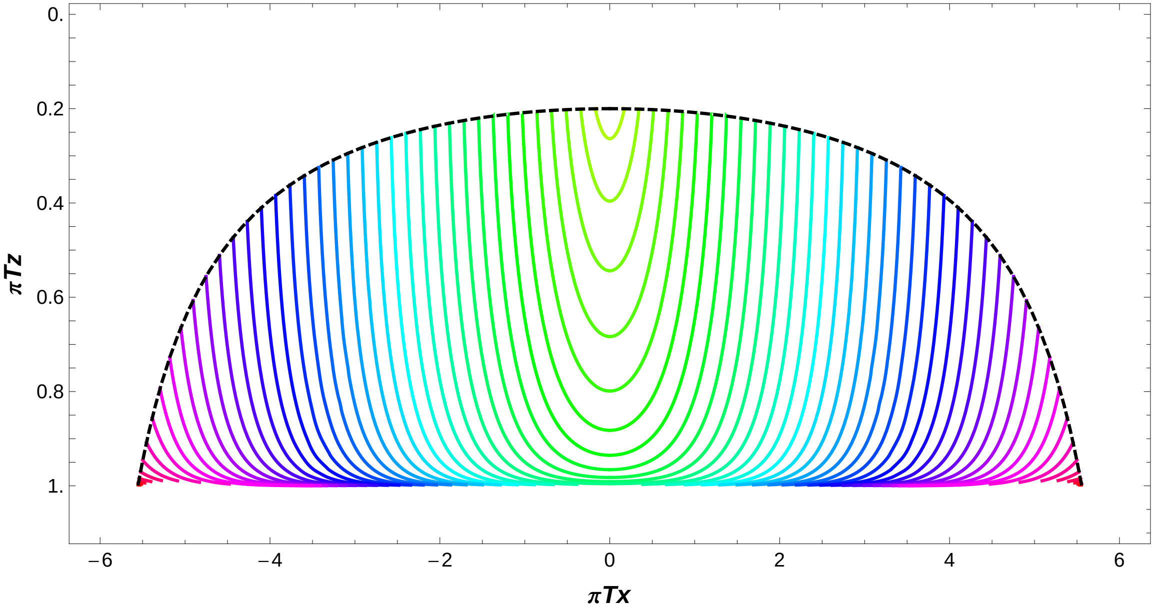
The *same* spacetime geodesic calculation as before now leads to

$$\Delta x_{\text{stop}} = \frac{2^{1/3}}{\pi^{2/3}} \frac{\Gamma\left(\frac{5}{4}\right) \Gamma\left(\frac{1}{4}\right)^{1/3}}{\Gamma\left(\frac{3}{4}\right)^{4/3}} \frac{1}{T} \left(\frac{E_*}{\sqrt{\lambda} T} \right)^{1/3} = \frac{0.624}{T} \left(\frac{E_*}{\sqrt{\lambda} T} \right)^{1/3} \,, \tag{32}$$

as before with $z_* \ll z_H$.

Only the energy calculation changed.

One can numerically determine the shape of the bulk of the string:



String goes further because we budgeted initial energy differently: no initial downward motion, only longitudinally outward.

2.3. Single quarks and instantaneous energy loss

How far a string can go if one end passes through the horizon and total energy E outside horizon is fixed?

Argument from spacetime geodesics is now familiar: start near the horizon moving upward; require $E_{\text{endpoint}} \to 0$ only when we fall completely into the horizon; and use $\dot{p}_\mu$ equation to evolve E_{endpoint} along endpoint geodesic. Answer:

$$\Delta x_{\text{stop}} = \frac{2}{\pi^{2/3}} \frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(\frac{1}{4}\right)^{1/3}}{\Gamma\left(\frac{3}{4}\right)^{4/3}} \frac{1}{T} \left(\frac{E}{\sqrt{\lambda}T}\right)^{1/3} = \frac{0.990}{T} \left(\frac{E}{\sqrt{\lambda}T}\right)^{1/3} \tag{33}$$

To find motion of the bulk of the string, it helps a lot to use Eddington-Finkelstein coordinates:

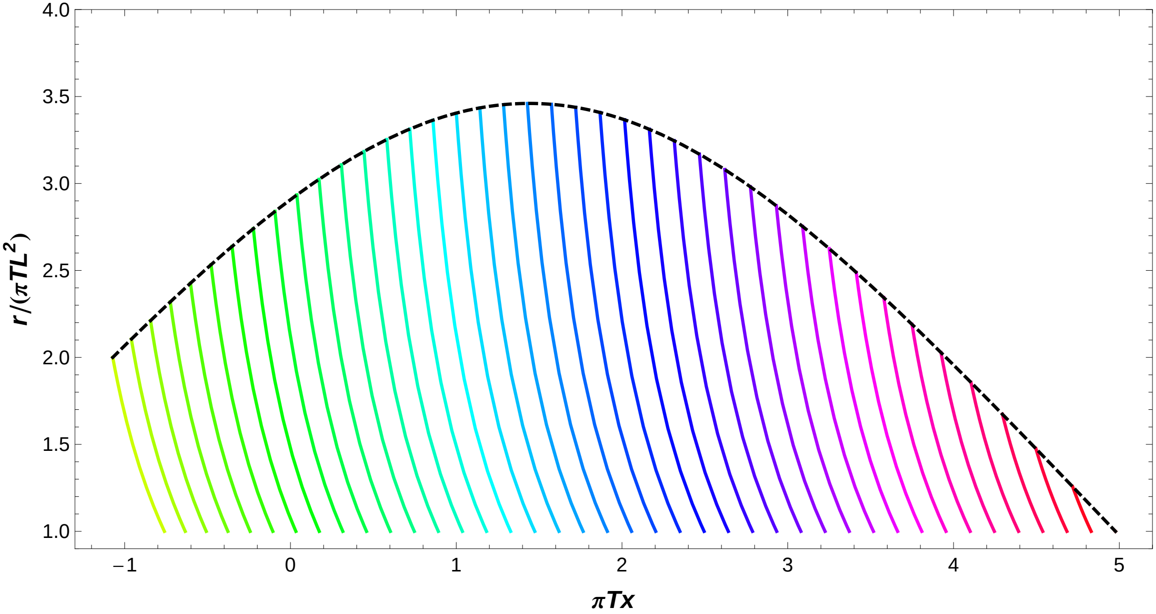
$$ds^2 = -\frac{r^2}{L^2}\left(1 - \frac{r_H^4}{r^4}\right) dv^2 + 2dvdr + \frac{r^2}{L^2} d\vec{x}^2 \,. \tag{34}$$

Initializing with a segment of the trailing string,

$$x_{\text{trailing}} = \beta \left(v - \frac{L^2}{r_H} \tan^{-1} \frac{r}{r_H}\right) \,, \tag{35}$$

one finds—qualitatively—a trailing string truncated by the null geodesic.

Amusing feature: at fixed E-F time, “trailing” string *leads* the endpoint (known to [Casalderrey-Solana and Teaney, hep-th/0701123]).

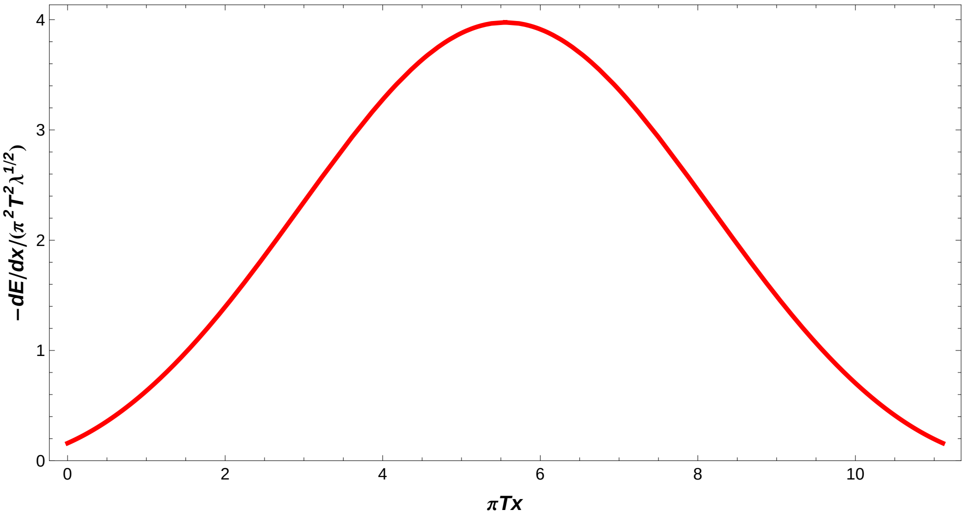


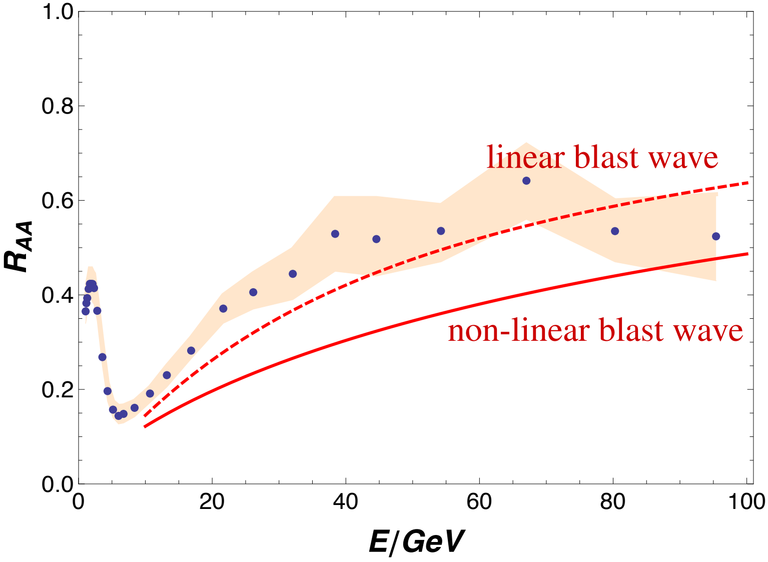
Starting from $\dot{p}_\mu$ for the endpoint, can derive

$$\frac{dE}{dx} = -\frac{\sqrt{\lambda}}{2\pi}\frac{\sqrt{f(z_*)}}{z^2}\,, \tag{36}$$

where z is determined as the height of the endpoint geodesic at position x.

The result is a bell-shaped dE/dx, different from usual ansatz $dE/dx \sim E^\alpha x^\beta T^\gamma$.





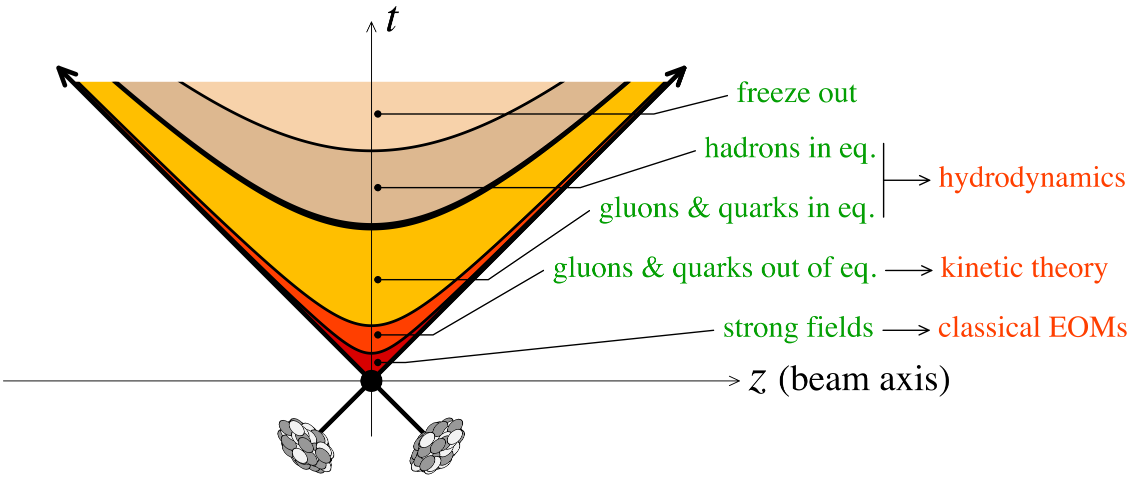
- Preliminary results (red) suggest that R_{AA} at LHC is underpredicted by this model: $\lambda = 1$ here!
- R_{AA} is number of high-energy particles observed divided by expectations from pp.
- Probably need to go beyond conformal models—running coupling is important.

3. Summary

- Finite endpoint momentum is part of classical string theory.
- Generally covariant superstring action including finite endpoint momentum must exist, but I don't understand the details.
- Endpoints with finite momentum follow spacetime geodesics except for abrupt changes in direction.
- Generalizations to finite momentum localized on a higher dimensional brane seem natural and interesting.
- Could try to replay Lund model in AdS_5: starts with strings localized to AdS_2.
- Finite endpoint momentum helps identify trajectories that maximize transverse distance traveled in AdS_5-Schwarzschild with fixed energy.
- Heavy-ion applications of $\Delta x_{\text{stop}} \propto E^{1/3}$ and bell-shaped dE/dx are under consideration.
- Interesting to consider also the *charge* of the endpoints: e.g. with a strong electric field, could we get endpoints to spontaneously rise up out of a black hole?

4. The whole history of a heavy ion collision

I'd like to offer a few thoughts (somewhat disconnected) about the whole history of a heavy ion collision, based partly on 1210.4181.



freeze out

hadrons in eq.

gluons & quarks in eq.

→ hydrodynamics

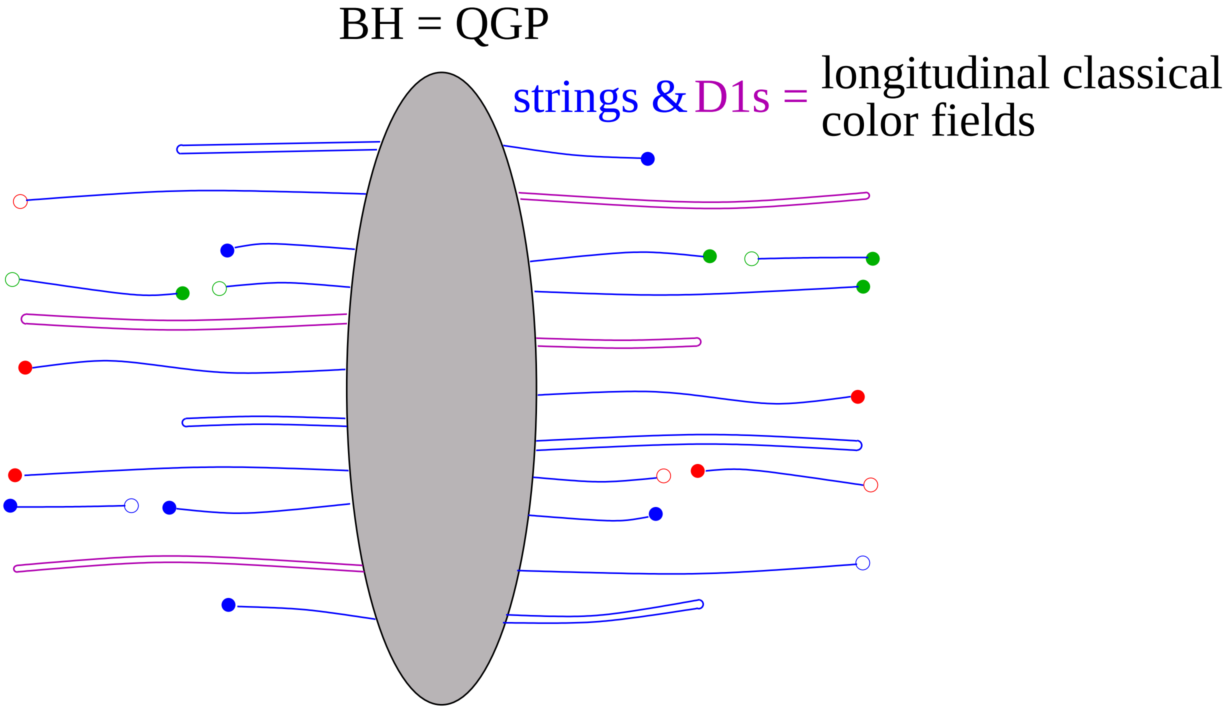
gluons & quarks out of eq. ⟶ kinetic theory

strong fields ⟶ classical EOMs

Conventional understanding of the history of a heavy ion collision relies on beam-axis boost invariance and a series of "phases" of hadronic matter.

But boost invariance is actually not a very good symmetry of the final state.

A cluster of *many* quarks and gluons (known as a nucleus) collides with another such cluster. What is this like in AdS_5?



BH = QGP

strings & D1s = longitudinal classical color fields

This is very boost-non-invariant. Is there nevertheless some interesting averaged description?

4.1. Bjorken flow

Bjorken flow is simple because it respects a four-parameter symmetry group, $SO(1,1)$ (along beam axis) $\times ISO(2)$ (in transverse plane).

$$\epsilon = \epsilon(\tau) \qquad \text{where} \qquad \tau = \sqrt{t^2 - x_3^2} \tag{37}$$

because τ is the unique combination of the x^μ invariant under $SO(1,1) \times ISO(2)$.

The velocity field is also determined by symmetry:

$$u_\mu = \frac{\partial_\mu \tau}{\sqrt{\partial_\mu \tau \partial^\mu \tau}} \,. \tag{38}$$

Dynamics only enter into determining the form of $\epsilon(\tau)$: $\nabla^\mu T_{\mu\nu} = 0$ where

$$T_{\mu\nu} = \epsilon u_\mu u_\nu + \frac{\epsilon}{3}(\eta_{\mu\nu} + u_\mu u_\nu) \,, \tag{39}$$

leads to $\epsilon = \epsilon_0 / \tau^{4/3}$.

Janik-Peschanski construction of a boost-invariant black hole in AdS_5 generalizes Bjorken flow to include many viscous corrections.

4.2. A modified boost symmetry

$ISO(2)$ is an acceptable idealization, but boost-invariance is the enemy.

How about altering boost generator to something more general in $SO(4,2)$?

If we insist $[b,g]=0$ for $g \in ISO(2)$, then the only sensible choice is $SO(1,1)_{\rm C}$ generated by

$$b = \underbrace{B_{(3)}}_{\text{beamline boost}} + \underbrace{t_3 T_{(3)}}_{\text{beamline translation}} + \underbrace{k_3 K_{(3)}}_{\text{special conformal}} . \tag{40}$$

If $t_3 \neq 0$ and $k_3 = 0$, then the $SO(1,1)_{\mathbf{C}} \times ISO(2)$-invariant combination of x^μ is

$$\tau^{\mathbf{C}} = \sqrt{(t+t_3)^2 - x_3^2} \, . \tag{41}$$

For t_3 real, we can trivially repeat Bjorken flow story:

$$u_\mu^{\mathbf{C}} = -\frac{\partial_\mu \tau^{\mathbf{C}}}{\sqrt{\partial_\mu \tau^{\mathbf{C}} \partial^\mu \tau^{\mathbf{C}}}} \qquad T_{\mu\nu}^{\mathbf{C}} = \epsilon^{\mathbf{C}} u_\mu^{\mathbf{C}} u_\nu^{\mathbf{C}} + \frac{\epsilon^{\mathbf{C}}}{3}(\eta_{\mu\nu} + u_\mu^{\mathbf{C}} u_\nu^{\mathbf{C}}) \qquad \epsilon^{\mathbf{C}} = \frac{\epsilon_0^{\mathbf{C}}}{(\tau^{\mathbf{C}})^{4/3}} \tag{42}$$

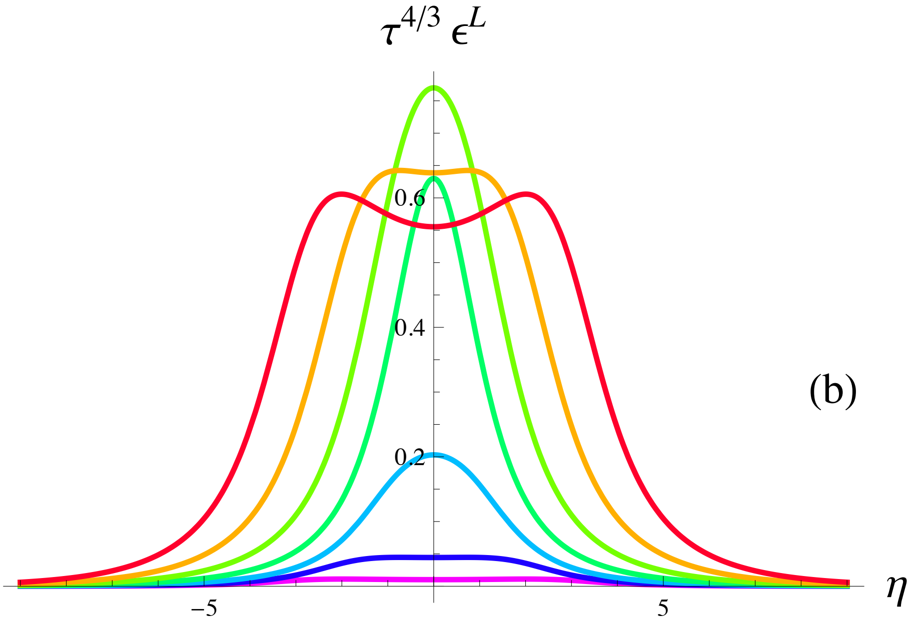
$\nabla^\mu T^{\mathbf{C}}_{\mu\nu} = 0$ is automatic because we're just translating Bjorken flow in time.

What if we translated by *imaginary* t_3? All quantities that were supposed to be real now become complex!

No problem, define $T_{\mu\nu} = \mathrm{Re}\{T^{\mathbf{C}}_{\mu\nu}\}$. Clearly, $\nabla^\mu T_{\mu\nu} = 0$.

Positive energy condition: If we require $T_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ inside future light-cone for null or timelike ξ^μ, it follows that $\arg \epsilon_0^{\mathbf{C}} = \pi/3$.

So "complexified" Bjorken flow is essentially unique.



$T_{\mu\nu}$ generally does *not* obey hydrodynamic ansatz.

We can still define local 4-velocity by

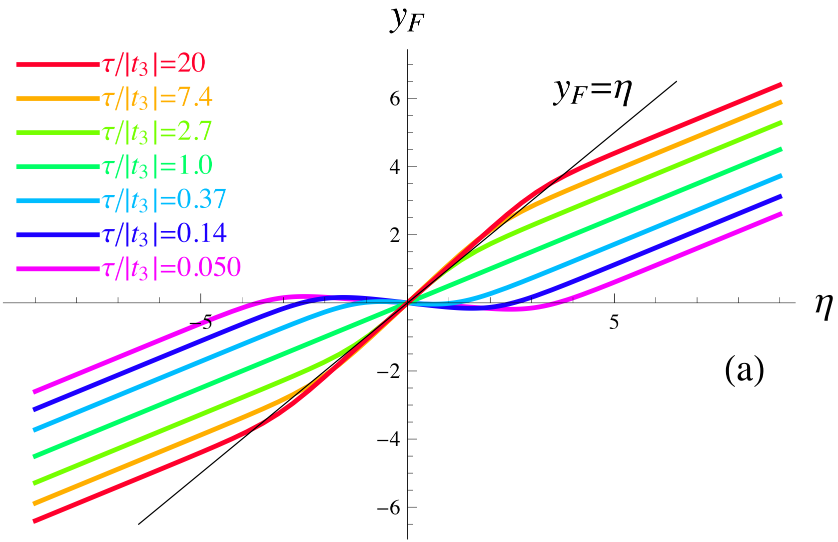
$$T_{\mu\nu}u^\nu = -\epsilon_L u_\mu$$

with $\epsilon_L > 0$.

Define y_F through

$$u^\mu = (\cosh y_F, 0, 0, \sinh y_F)$$

in lab frame.



Then $y_F \approx \eta$ at mid-rapidities, but $dy_F/d\eta \approx 1/2$ at forward rapidities.

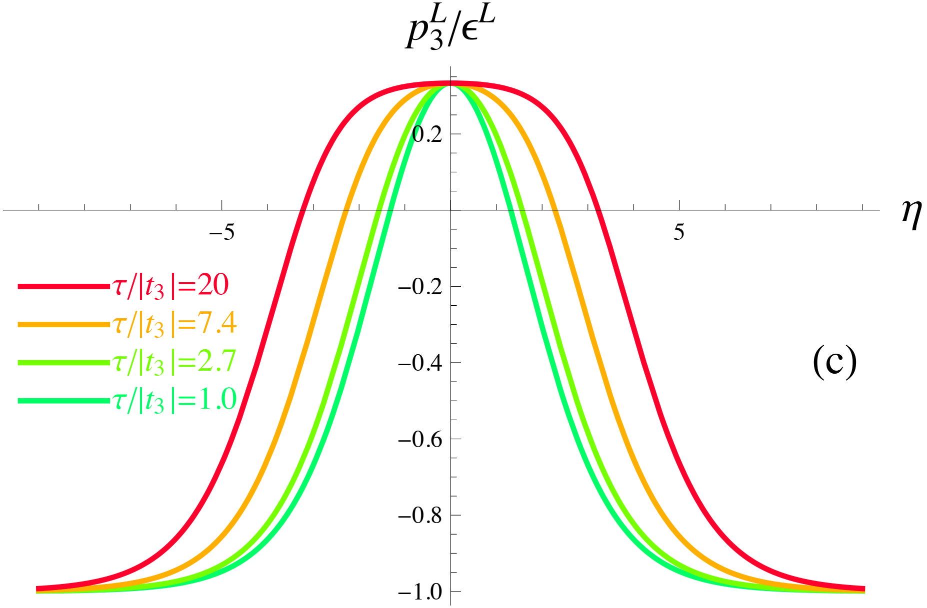
In local frame where $u^\mu = (1, 0, 0, 0)$, find

$$T^\mu{}_\nu = \mathrm{diag}\{-\epsilon_L, p_\perp, p_\perp, p_L\} \tag{43}$$

Landau frame energy density

transverse pressure

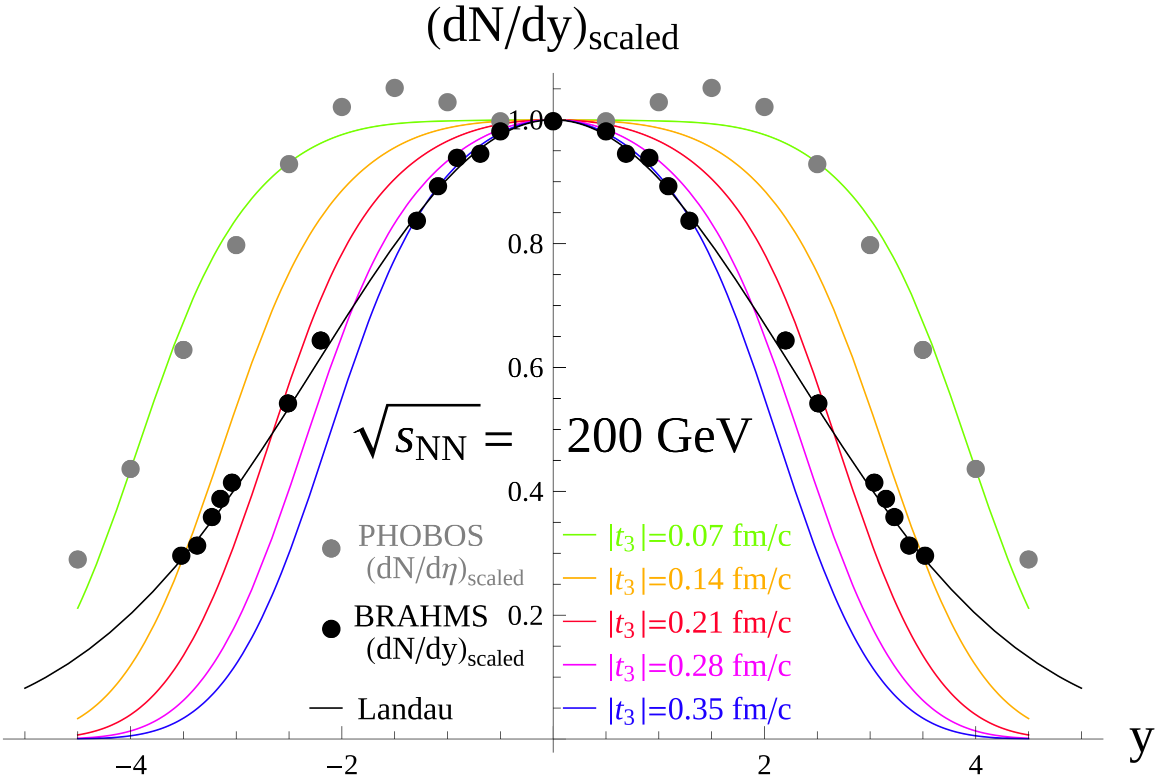
longitudinal pressure



Then $p_\perp \neq p_L$ is a measure of how far we are from inviscid hydro.

In forward region, $p_L \approx -\epsilon_L$, similar to glasma.

Cooper-Frye freezeout over hydrodynamic region results in dN/dy somewhat too square too match BRAHMS data at $\sqrt{s_{NN}} = 200\,\mathrm{GeV}$—but not too bad for $t_3 \approx 0.2\,\mathrm{fm}/c$.



Cooper-Frye converts each fluid element to free-streaming particles at $T \approx 130\,\mathrm{MeV}$.

4.3. Thoughts for the future

- What's going on microscopically or holographically in $SO(1,1)_{\mathbf{C}}$ flow?

  Maybe some combination of longitudinal color fields and quasiparticles, i.e. a QCD-Boltzmann equation?

  What is the AdS dual of complex time deformation? $\mathrm{Re}\{g_{\mu\nu}\}$?? Strings & BH??

- Can we look at perturbations?

  Easy in principle, linearized eom's are often complexified and you take $\mathrm{Re}\{\}$ at the end.

- How should we handle hadronization?

  Maybe something like Schwinger production? Currently, we don't have $\vec{E}$, only $T_{\mu\nu}$.

- Why would $SO(1,1)_{\mathbf{C}}$ be a good symmetry?

- Can we consider finite transverse size?

  $SO(1,1)_{\mathbf{C}} \times SO(3)_q$ symmetry fits the bill, and one can work out the invariant coordinate; but region where positivity constraints should be applied is confusing me, as are branch cuts.