General Relativity in AdS

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Based on work 2012 w/ Kengo Maeda

w/ Norihiro Iizuka, Kengo Maeda - work in progress

Plan

1. **Classical General Relativity:**
   -- A brief overview of basic results about asymptotically flat black holes in general relativity

2. **General Relativity in AdS:**
   -- Generalizations to asymptotically AdS

3. **Singularities in AdS:**
   -- Motivations and attempts towards a generic theorem

1. A TRIUMPH OF CLASSICAL GENERAL RELATIVITY

BH Uniqueness (No hair) theorem

Stationary electro-vacuum BH solution is uniquely specified by mass M, angular momentum J, and charge Q

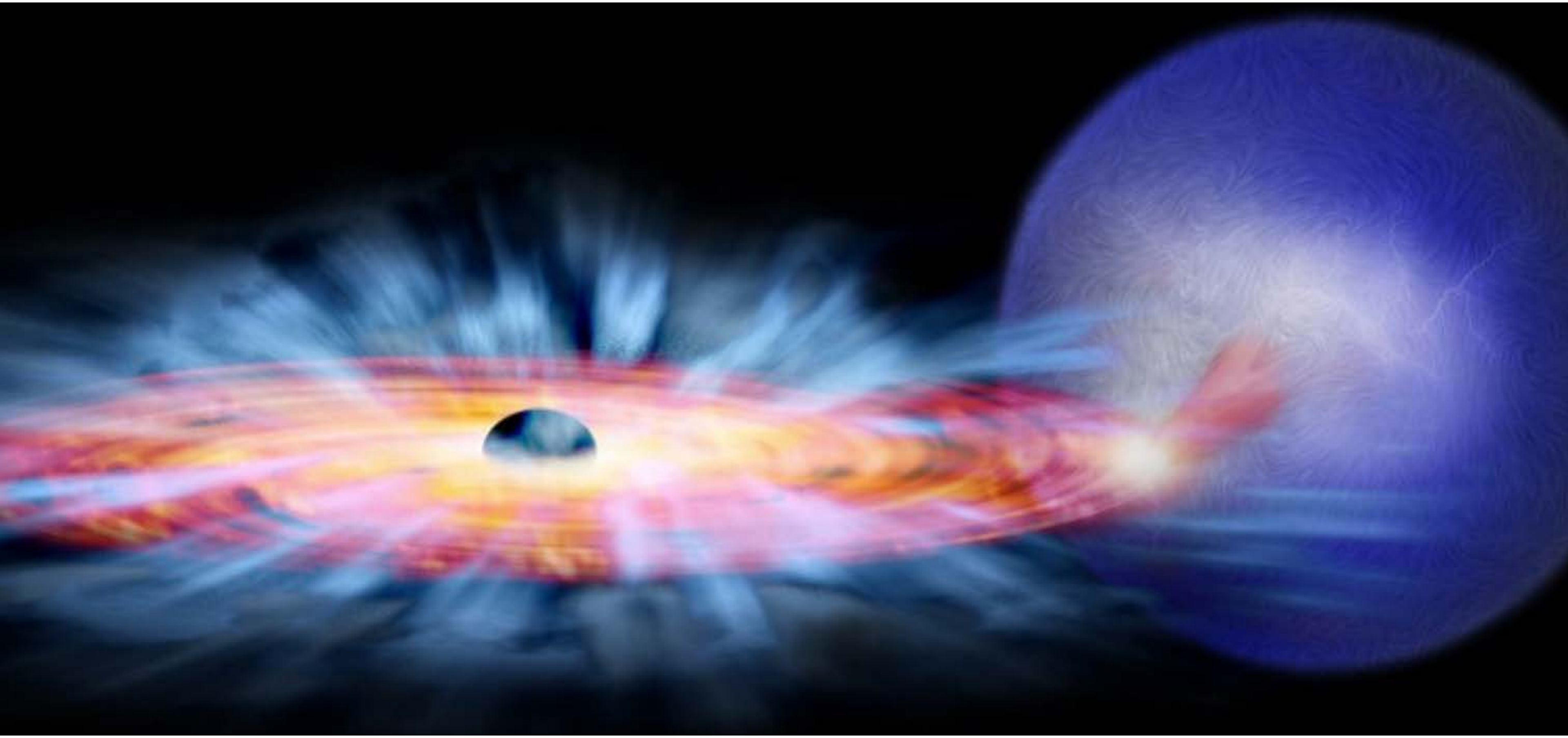
Remark: Vacuum rotating BH ⇒ **Kerr metric**

Physical implication:

- **Singularity Theorem** (Hawking-Penrose)
  \+ **Weak Cosmic Censorship** (Penrose)
  ⇒ gravitational collapse always forms a black hole

- **Stability** of the **Kerr metric** (Press-Teukolsky 73 Whiting 89)
  ⇒ final state of dynamics

A tremendous number of BHs in our observable universe --



-- can be accurately described by the Kerr metric, which possesses only two parameters.

Uniqueness theorem is based on a number of results, in particular:

1. Topology theorem Hawking 1972

X-section of the even horizon is topologically 2-sphere

2. Symmetry (Rigidity) theorem Hawking 1972

The event horizon of a stationary BH must be a Killing horizon.
If rotating, the BH spacetime must be axisymmetric

Theorem 1. helps to show **Theorem 2.**

→ Reduce the Einstein equations to a **2D nonlinear sigma model**

→ Uniqueness proof is cast into a **boundary value problem**

These results also give us deep insights into thermodynamic aspects of BHs

Stationary BHs in 4-dim. Asympt. Flat spacetimes

- **Exact solutions** --- Kerr family
- **Stability** --- Stable ⟹ final state of dynamics
- **Topology** --- Horizon cross-section = 2-sphere
- **Symmetry** --- Static or axisymmetric (rigid rotation)
- **Uniqueness** --- Vacuum rotating ⟹ Kerr metric

Related results/hypothesis

- **Singularity theorem** --- Singularity must form under strong gravity
- **Cosmic censorship** --- Horizon should form ⟹ Predictable
- **Positive energy theorem** --- An isolated system must be stable

Which properties fail / to be modified for asympt. AdS spacetimes?

What are possible consequences in dual theory on AdS boundary?

2. GENERAL RELATIVITY IN ADS

BHs in AdS

- **Exact solutions** --- Kerr-AdS family
- **Stability** --- Linear vs Nonlinear instability
- **Topology** --- Need not be spherical in D> 4 even if compact
- **Symmetry** --- Rigid
- **Uniqueness** --- does not appear to hold

Related results/hypothesis

- **Singularity theorem** --- holds in wider circumstances?
- **Cosmic censorship** --- fails ?
- **Positive energy theorem** --- generalized Abott–Deser 82, Gibbons et al 83

--- also depend **on spacetime dimensions**

TOPOLOGY IN ADS

Topology in AdS

Combine variational analysis $\delta\theta/\delta\lambda$ and fact that outer-trapped surface must be inside BH, to show

$$\int_\Sigma \mathcal{R} > 0 \qquad \mathcal{R}\text{: scalar curvature of } \Sigma\text{: cross-section}$$

$\Longrightarrow \Sigma \approx S^2$ in $4D$ via Gauss-Bonnet Theorem

and **Dominant Energy Conditions** in $D \geq 4$ Galloway – Schoen 05

AdS violates the dominant energy conditions

$\Longrightarrow$ Hawking's and Galloway-Schoen's proof does not work. In 4D other method (appealing to topological censor) applies

Topology and Entropy bound:

(Gibbons - Woolgar - Cai & Galloway)

Under appropriate circumstances

If $\Lambda \geqslant 0$, then $\sigma(\Sigma) > 0$

If $\sigma(\Sigma) < 0$ and $\Lambda < 0$, $\quad \mathrm{Area}(\Sigma) \geqslant \left| \frac{\sigma(\Sigma)}{2\Lambda} \right|^{(D-2)/2}$

Let $[q]$ be conformal class of q of $(D-2)$-dim. (Σ, q):

Yamabe constant: $\quad Y(\Sigma : [q]) \equiv \inf_{q \in [q]} \frac{\int \mathrm{d}\Sigma_q \mathcal{R}_q}{\left(\int \mathrm{d}\Sigma_q\right)^{(D-4)/2}}$

Yamabe invariant: $\quad \sigma(\Sigma) \equiv \sup_{[q]} Y(\Sigma : [q])$

- **SYMMETRY IN ADS**

Symmetry (Rigidity) in AdS

Topology theorem does not hold as it stands

$\Longrightarrow$ Hawking's proof of the rigidity relies on spherical topology, hence does not work

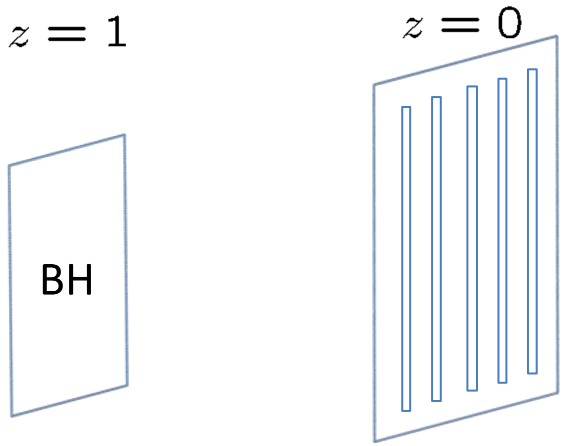
Appeal to a method used ergodic theory

Hollands - AI - Wald 07

-- can apply to any compact horizon in $D \geq 4$

Symmetry in AdS/CMP

Charged AdS BH (w/ scalar hair) and DC-conductivity



Planner horizon　　Ionized lattice

Iizuka - Maeda 2012
Horowitz-Santos-Tong 2012

No translational symmetry on bulk & boundary

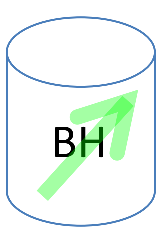
Nevertheless, for superfluid component,

Current is conserved → No dissipation

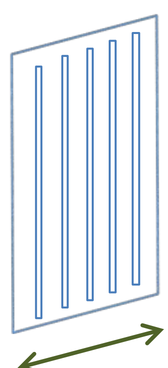
Horowitz-Santos 2013

$$A_t = \mu(1 + A_0 \cos(k_0 x)) - \rho z + O(z^2)$$

What if charged AdS BH w/ compact horizon is rotating?



Compact rotating horizon
should have symmetry



Periodic lattice on the boundary:
No translational symmetry

Apparent discrepancy with BH rigidity

→ Need dissipation effects

Iizuka - Maeda - AI work in progress

STABILITY IN ADS

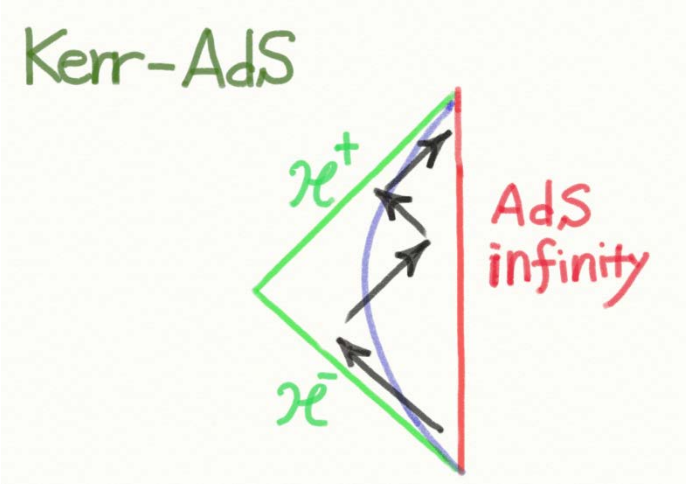
Stability in AdS

- Static (Schwarzschild-)AdS BHs → linear stable

  -- depend on the choice of boundary conditions

  Kodama - AI 04

- Rotating AdS BHs → Superradiant instability



Hawking-Reall 99, Cardoso-Dias 04
Cardoso-Dias-Yoshida 06,
Kodama 07 Murata-Soda 08
Uchikata-Yoshida-Futamase 09

$$E := -\int_S dS n^a \chi^b T_{ab}$$ χ^a: co-rotate Killing vector

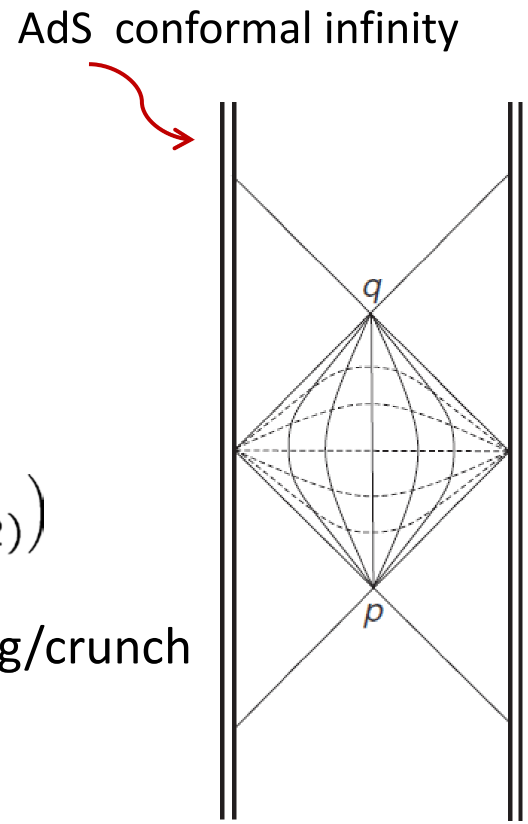
Note: χ^a can be non-spacelike if $a^2 \leqslant r_H^4/\ell^2 \Longrightarrow E \geqslant 0$

How about the stability of asymptotically AdS spacetimes without horizons ?

Global AdS

- AdS is maximally symmetric, geodesically complete, but *non-globally hyperbolic*.

- AdS conformal infinity plays a role of a confining box: Waves bounce off at AdS-infinity



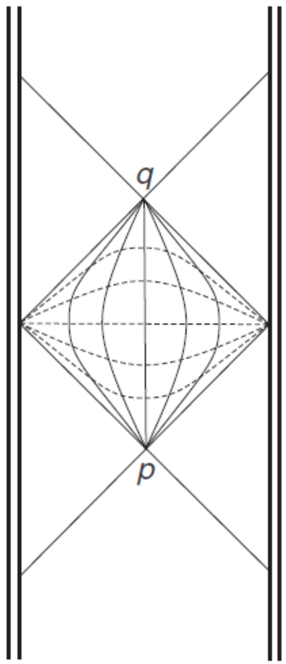
e.g. cosmological chart:

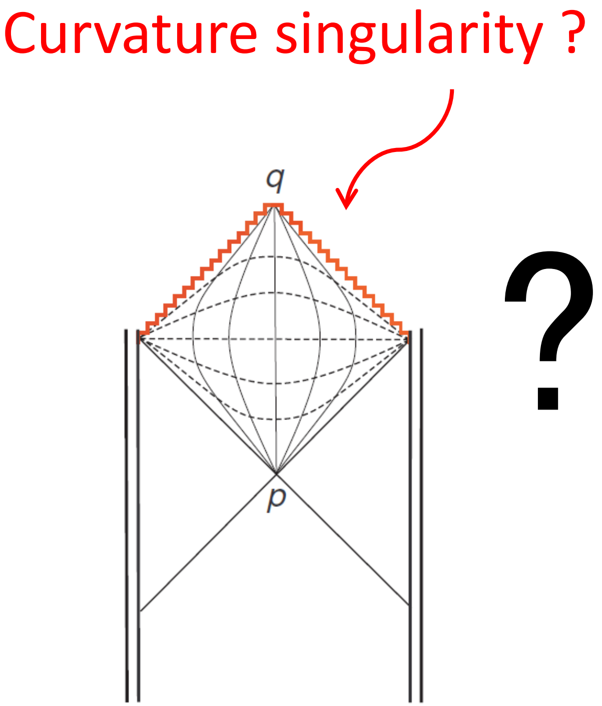
$$ds^2 = -d\tau^2 + \cos^2\tau\left(d\chi^2 + \sinh^2\chi d\Omega^2_{(D-2)}\right)$$

Points p, q, are conjugate, corresponding to Big-bang/crunch

All timelike geodesics emanating from p re-converge to q

Does a perturbation make the conjugate points true curvature singularities?





Global AdS is stable at least for **linear** perturbations

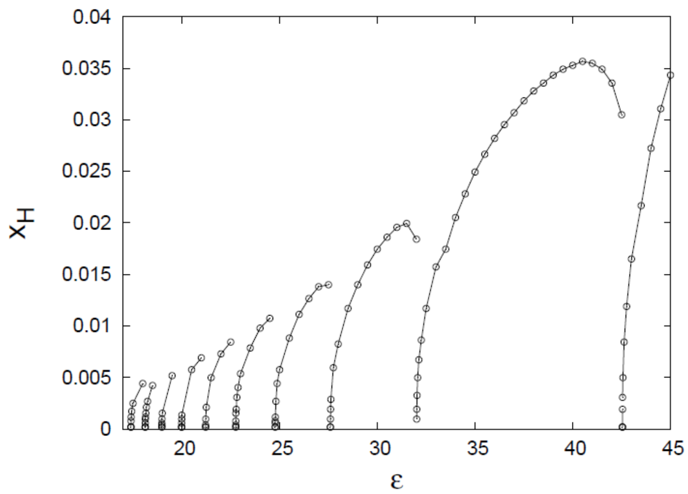
e.g. Al - Wald 04

--- but **nonlinearly** unstable

Bizon, Rostworowski 2011

Turbulent instabiliy

The energy cascades from low frequency to high frequency



Initial small perturbations grow by repeating bounce off by AdS infinity

Black hole forms even starting from arbitrarily small initial perturbations

Massless scalar Spherical symmetry

Bizon, Rostworowski 2011 Jalmuzna, Rostworowski, Bizon 2011

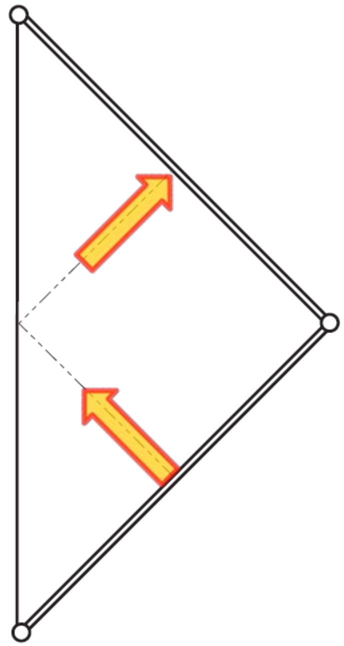
Vacuum, gravitational waves

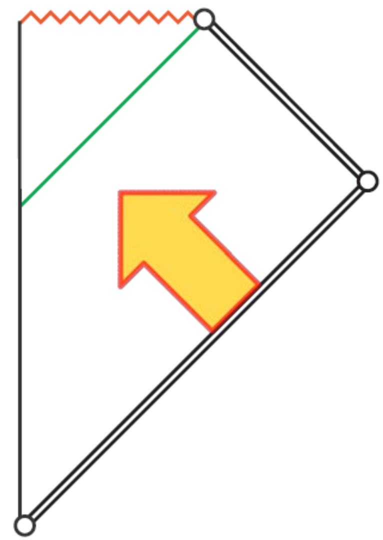
Dias, Horowitz, Santos 2011

c.f. Collapse in flat space

- Scalar field collapse in asymptotically Flat spactime

  Critical phenomena: Choptuik 93

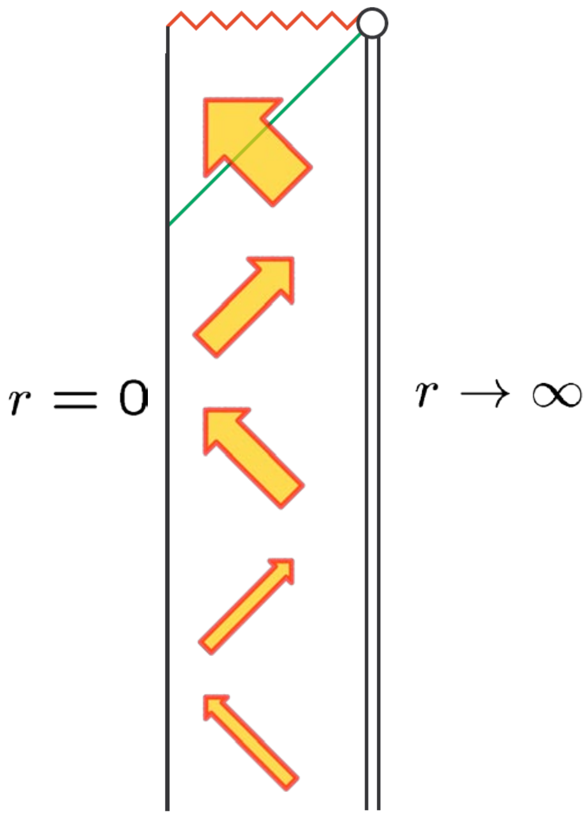




Subcritical initial data:
Wave packet bounces off the center and escapes to infinity

Supercritical initial data:
Wave packet collapses to form a black hole

Collapse in AdS spacetimes



Wave packet cannot disperse in AdS as it bounces off at AdS infinity

Subcritical initial data gets amplified and becomes supercritical after reflecting several times at AdS infinity, and finally collapses to form a black hole

A single Poincare patch is not enough

Boundary perspective:

If a black hole always forms

⇓

CFT on the boundary of *global* AdS always thermalizes…

(… even starting from a pure state _?)

Is AdS generically singular ?

No!

Some asymptotically AdS are nonlinearly stable

Dias-Horowitz-Marolf-Santos 2012

Buchel-Liebling-Lehner 2013

Maliborski-Rostworowski 2013

e.g. geons, boson stars in AdS

-- indicating there exist generic initial data
in strongly coupled CFT that never thermalize

Under what kind of circumstances can asymptotically global AdS spacetimes be singular ?

3. SINGULARITIES IN ADS

Some cases that guarantee instability/singularity formation

A singularity theorem for spherically symmetric perfect fluid in AdS

Maeda - AI 2012

If the following averaged convergence holds

$$\int ds \left(4\pi(\mu + 3P) + |\Lambda| + \sigma^2 + \frac{1}{n}\theta^2 - \nabla_a \dot{V}^a\right) \geq c > 0$$

then a singularity must form.

μ : Energy density $\quad P$: Pressure

$P = (\gamma - 1)\mu$: $1 < \gamma < 2$

Gravitational contraction > Repulsive force by Pressure

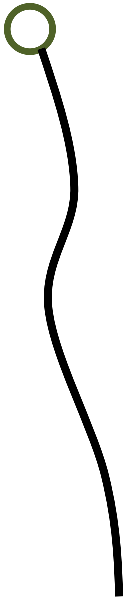
Energy density at the origin grows indefinitely!

Attempts toward a more generic singularity theorem

Hawking-Penrose singularity theorems

- **Singularity (incomplete causal geodesic) must form under the conditions of**

  1. **Convergence (generic & energy conditions)**
  2. **Global structure (causality or Cauchy surface)**
  3. **Strong-gravity (trapped set)**



World line of a particle

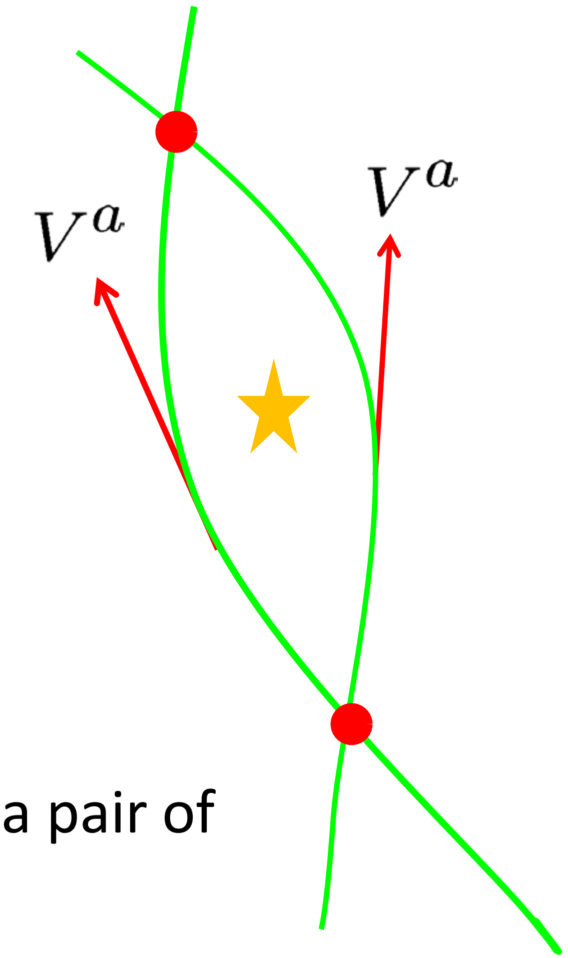
1. Convergence (generic & energy conditions)

Generic condition: $V_{[a}R_{b]cd[e}V_{f]}V^cV^d \neq 0$

Raychauduri equation:

$$\frac{d}{d\lambda}\theta = -\frac{1}{n}\theta^2 - \sigma^2 - R_{ab}V^aV^b$$

$$\theta := \frac{1}{A}\frac{dA}{d\lambda}$$



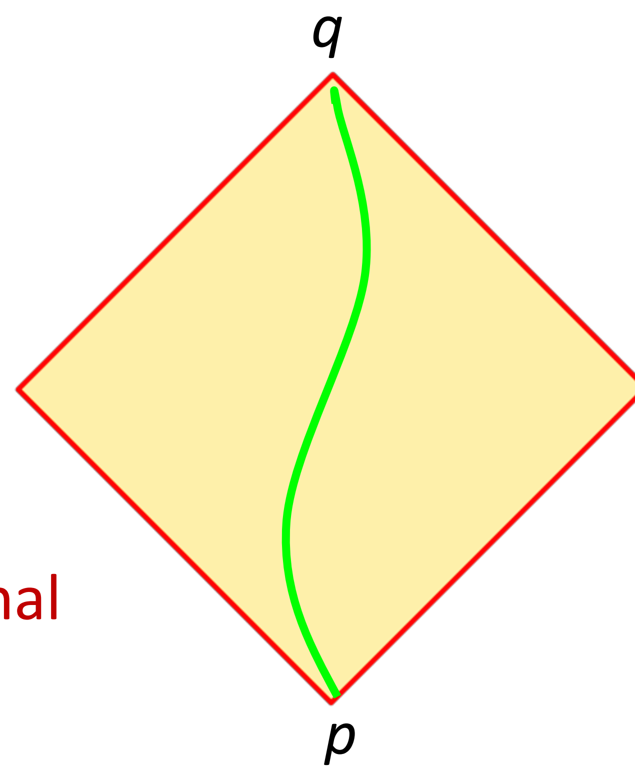
Every (endless) timelike / null geodesic contains a pair of conjugate points due to gravitational pull

Negative $\Lambda < 0$ enhances the timelike convergence

— does **not** affect null convergence

2. Global structure (causality or Cauchy surface)

- **Globally hyperbolic sub-region**
  i.e., covered by diamond shape (causally convex) subsets



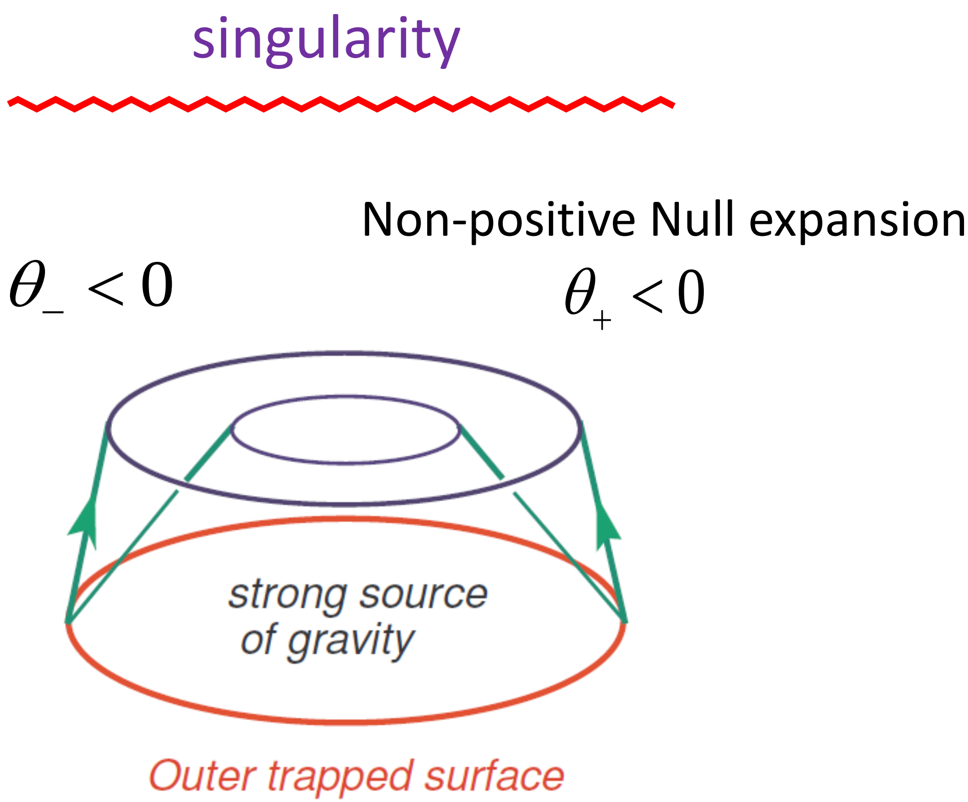
- **Importance in singularity theorem:**

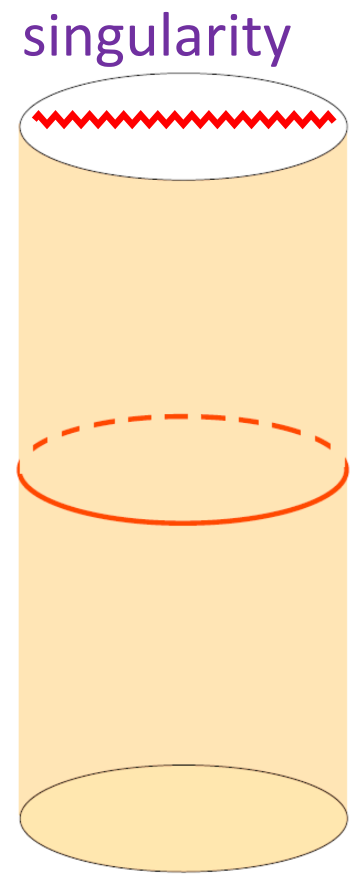
  guarantee the existence of the maximal length curve among all causal curves from $p \rightarrow q$

The maximal length is attained by a timelike geodesic curve

3. Strong-gravity (trapped set)

If no singularity → light cone from a trapped set would be closed





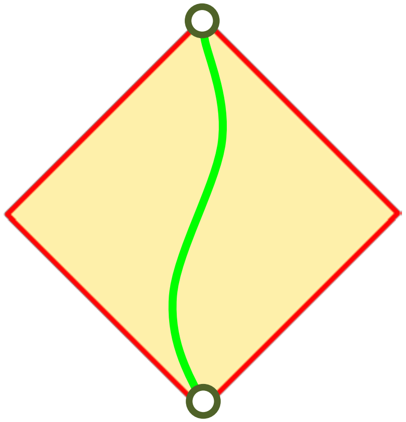
Ex. 1 Closed outer trapped surface

Ex. 2 Compact surface without edge

Sketch of proof

(S) Suppose all causal geodesics would be complete

(A) Conditions 1. 2. 3. imply
existence of an **endless** timelike geodesic
in a globally hyperbolic subset

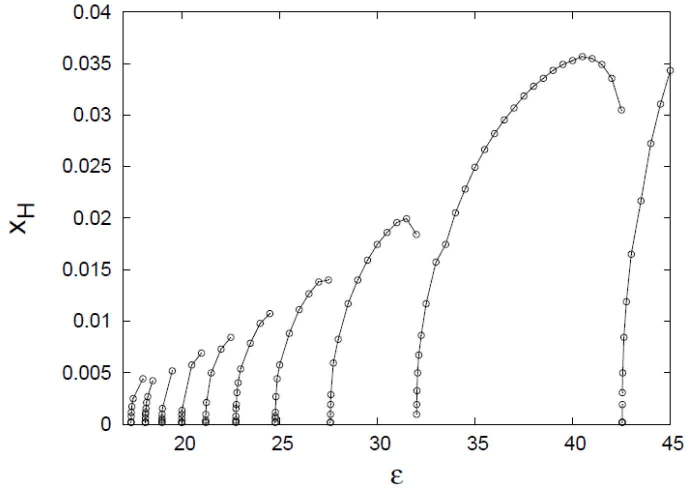


(B) The maximal length timelike geodesic does not allow a pair of conjugate points on it

(A) & (B) lead a contradiction under **(S)**

⇒ **(S)** must be false

However …



According to the result of turbulent instability, a black hole -- hence singularity -- forms even starting from arbitrarily small initial perturbations

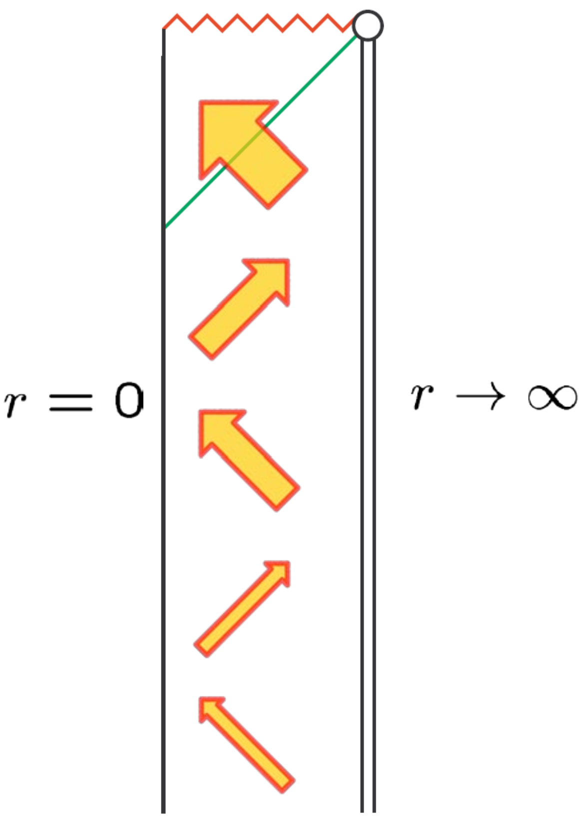
-- wish to remove the strong gravity condition
(i.e. a closed trapped surface)

Collapse in AdS

Remember:

For turbulent instability, the process of bouncing-off at AdS infinity many times has to be taken into account: Need to deal with a non-globally hyperbolic region

However, in non-globally hyperbolic region, there is no guarantee for the existence of maximal length causal curves, which is the key point of the proof



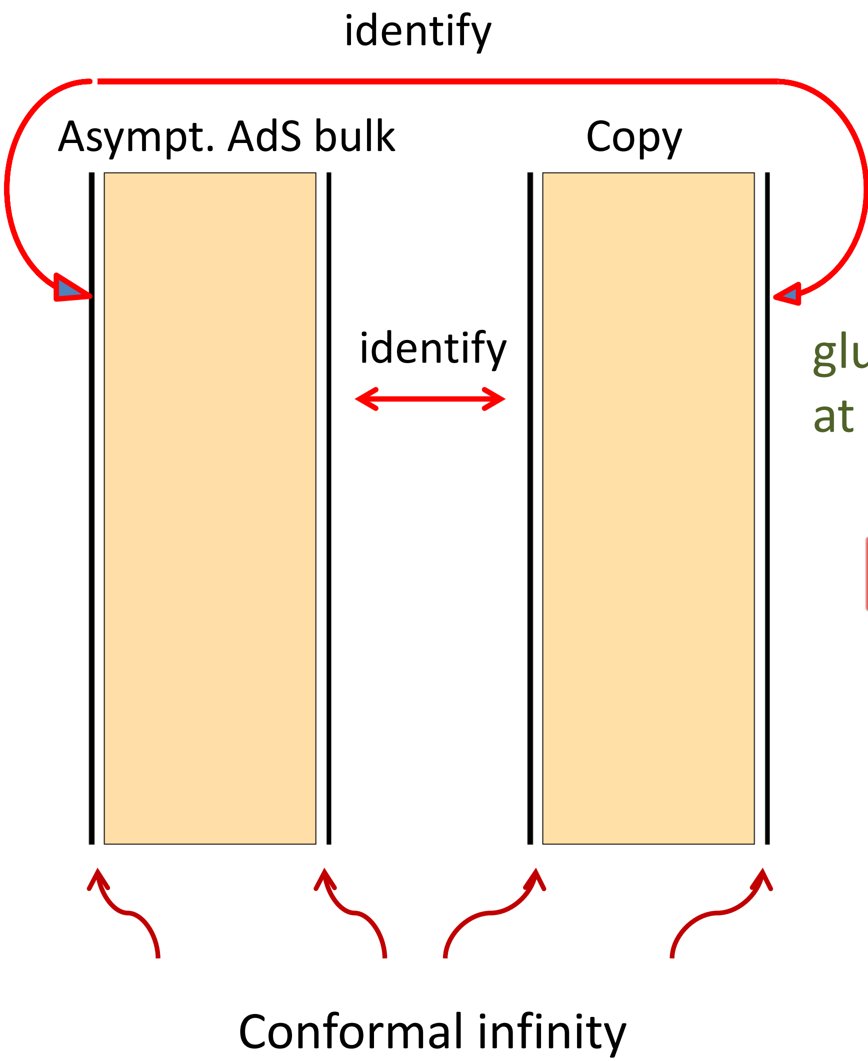
To get a desired maximal length curve, one may think of

double covering

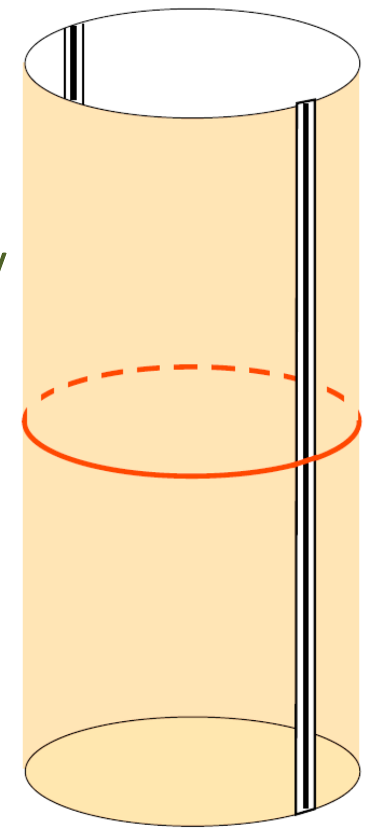
of the physical, asymptotically AdS spacetime to construct a *globally hyperbolic* unphysical spacetime w/ compact Cauchy surface.

Attempt to show a singularity theorem in the unphysical spacetime rather than in the physical spacetime.

Globally hyperbolic spacetime
w/ compact Cauchy surface



glue them together
at conformal boundary



-- can apply the argument of maximum
length causal curve?

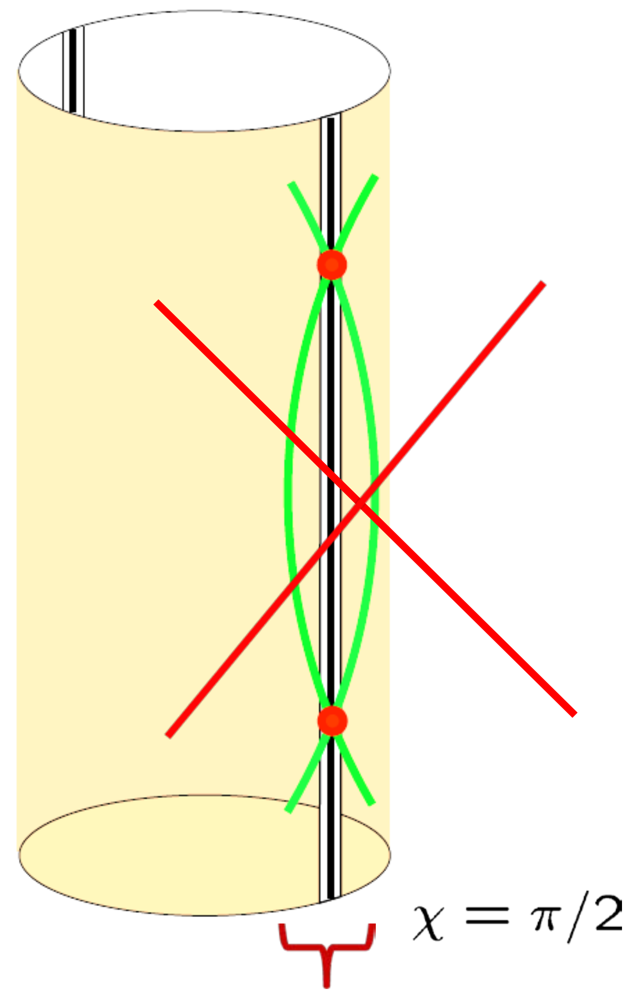
However :

Under the standard boundary conditions
(e.g. Dirichlet conditions)

The convergence (generic) condition is
NOT satisfied for timelike geodesics
at $\chi = \pi/2$, corresponding to AdS infinity

Geodesic congruence
at $\chi = \pi/2$ does NOT converge

-- cannot lead a contradiction!



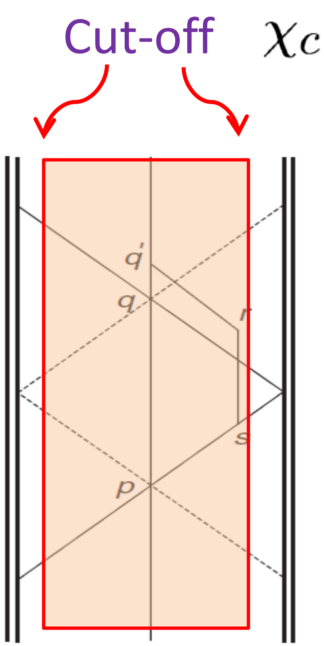
isometric to an open set of
the Einstein-Static Universe

This indicates that for a generic singularity theorem (without any strong gravity) , one may need to impose **dynamical boundary conditions** (that guarantee generic convergence for boundary timelike geodesics).

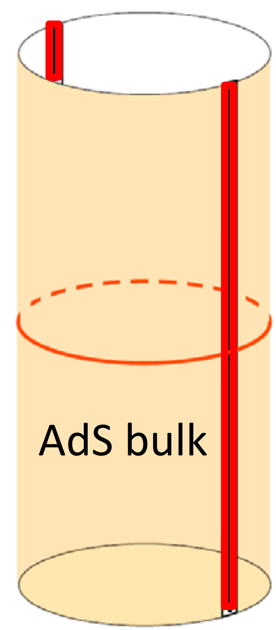
To get a desired maximal length curve, one may introduce a cut-off to make *compact* the space of all causal curves from $p \rightarrow q'$

-- physically implies that any causal, non-null curve has a *bounded acceleration* and *cannot reach AdS infinity*

-- with cut-off, turblent instability can still occur Buchel-Lehner-Liebling 12



Copy and glue them together at Cut-off



Convergence condition on the braneworld helps to show bulk AdS singularity theorem

Einstein-static universe as Randall-Sundrum type brane-world

Summary

- Timelike nature of AdS-conformal infinity makes AdS spacetimes a theoretical laboratory, modifies some of the standard results about basic properties of black holes and global structure.

- Some asymptotically AdS solutions are nonlinearly unstable.

- A singularity theorem holds for a specific case: Spherically symmetric perfect fluid -- needed to impose the averaged convergence conditions which seem too restrictive.

- Attempted to show a generic singularity theorem for asymptotically AdS without imposing strong gravity conditions. → Not successful

- Seems possible if imposing dynamical boundary conditions at AdS-infinity or cut-off, which however break AdS symmetry exploited in AdS/CFT context.