Entanglement, geometry and the Ryu Takayanagi formula

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Kyoto, 2013

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Tom Faulkner

Lewkowycz, JM ArXiv:1304.4926 → Previously argued by Fursaev

&

Faulkner, Lewkowycz, JM, to appear

Entanglement entropy

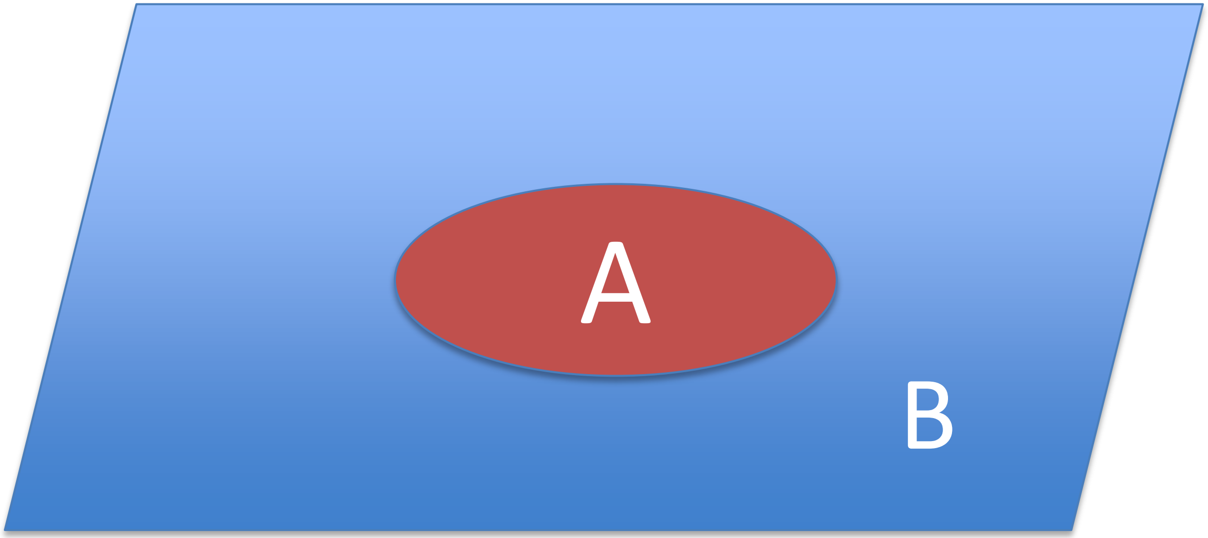
$$H = H_A \times H_B$$

$$\rho_A = Tr_{H_B} |\Psi\rangle\langle\Psi| = Tr_{H_B}[\rho]$$

$$S_A = -Tr[\rho_A \log \rho_A]$$

(Not a proper observable in the sense of Dirac, not a linear function of the density matrix)

Entanglement for subregions in QFTs



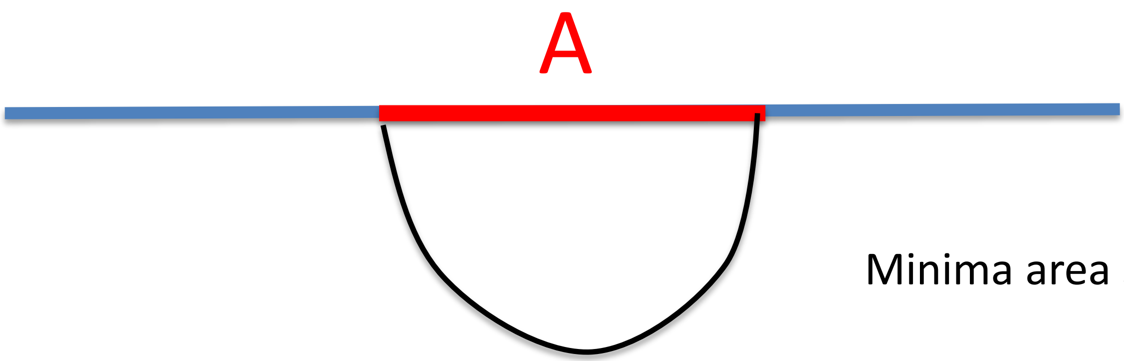
Bombelli, Koul, Lee, Sorkin
Srednicki…..

It is divergent, but divergencies are well understood. Divergencies are universal. We will often talk about finite quantities.

Usually difficult to compute.

Encodes interesting dynamical information → c, f, a theorems

Entanglement in theories with gravity duals



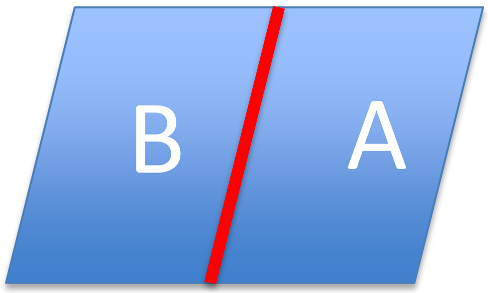
Minima area surface in the bulk

Ryu-Takayanagi

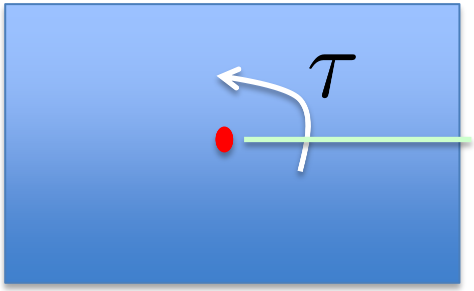
$$S_A = \frac{A_{\mathrm{minimal}}}{4G_N}$$

Leading order in G_N expansion

Half space



Half of space



Euclidean time

Circle: $\tau = \tau + 2\pi$

By a Weyl transformation on the metric we can map this to $H_3 \times S_1$

Casini, Huerta, Myers

The entaglement entropy is equal to the thermodynamic entropy

Replica trick

$$Z_n \propto Tr[\rho^n]$$

Callan Wilczek

$$\tau = \tau + 2\pi n$$

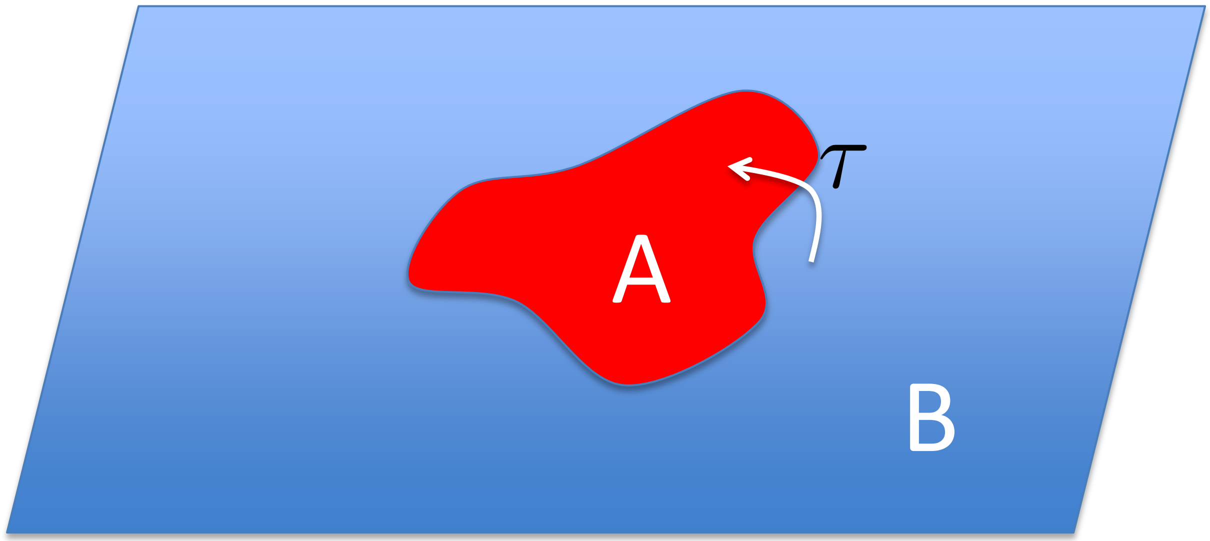
$$S = -\,\partial_n \left(\log Z_n - n \log Z_1\right)\Big|_{n=1}$$

$$H_3 \times S^1 \to H_3 \times S^1_n$$

Usual thermodynamic computation

Bulk dual = Black brane with a hyperbolic horizon

More general regions



We still have a tau circle. Choose a Weyl factor so that the circle is not contractible in the boundary metric. The metric is tau dependent.

Bulk problem

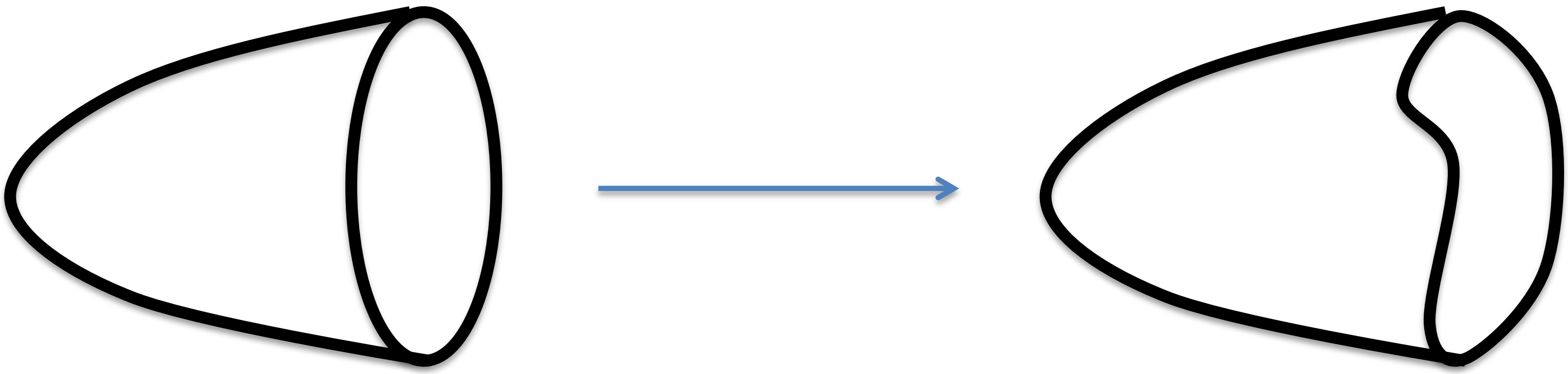
Finding a sequence of bulk geometries where the boundary has the previous geometry but with

$$\tau = \tau + 2\pi n$$

AdS$_3$ → Faulkner
(from CFT → Hartman)

Typically the circle shrinks smoothly in the interior

Generalized gravitational entropy

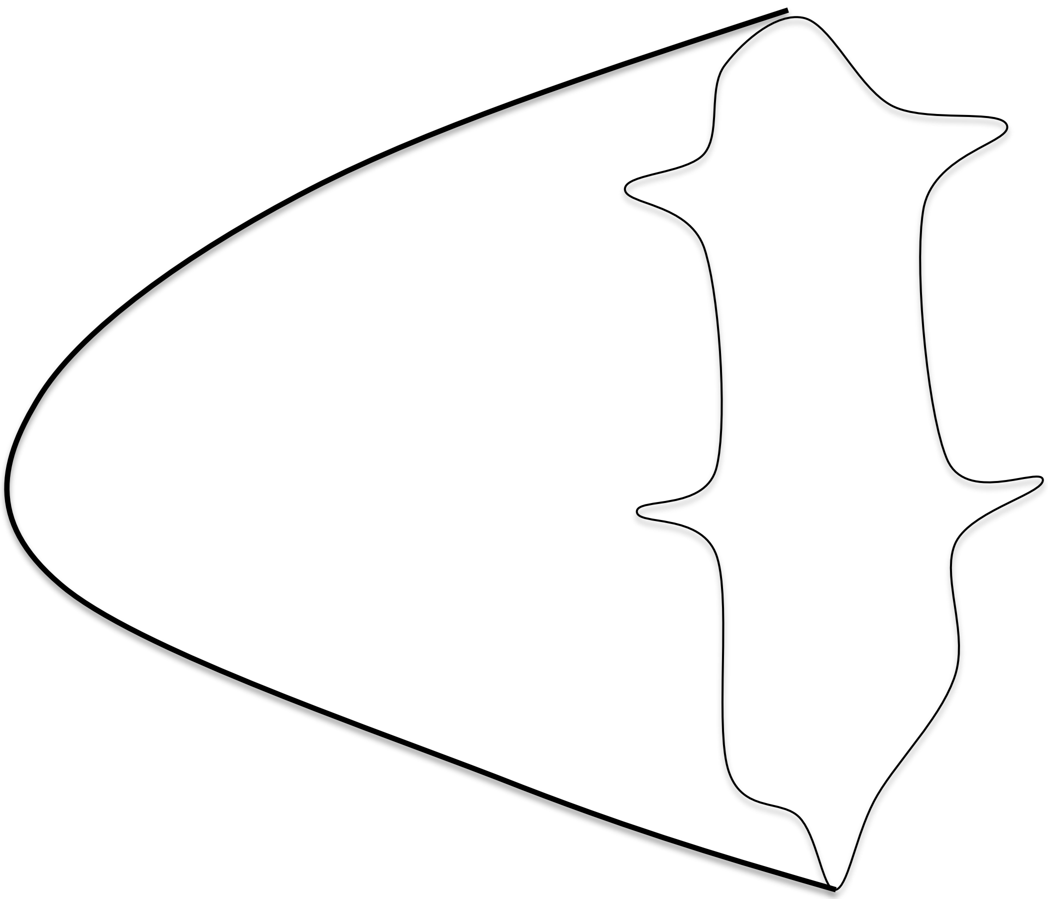


Gibbons
Hawking

U(1) invariant

Not U(1) invariant

$$\rho \propto P e^{-\int_0^{2\pi} d\tau H(\tau)}$$

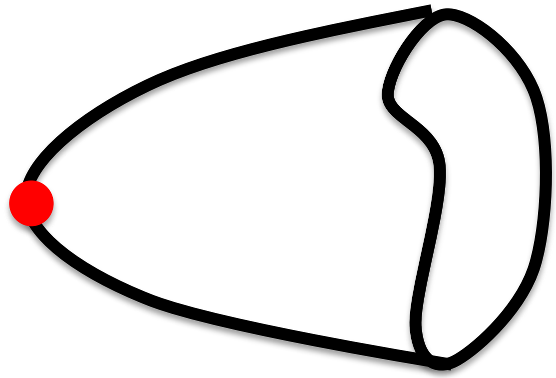


n=5
Z_5 symmetry in the bulk

Evaluating the gravitational action

$$-I_n \sim \log Z_n \sim \log Tr[\rho^n]$$

we can compute the entropy using the replica trick



$$S = \partial_n (I_n - n I_1)|_{n=1} = \frac{A_{\rm min}}{4G_N}$$

The Ryu Takayanagi conjecture reduces to proving the following classical statement:

The above combination of derivatives of the analytic continuation in n of the gravitational actions gives the area of the minimal surface.
This is a minimal codimension two surface such that the tau circle shrinks at its location

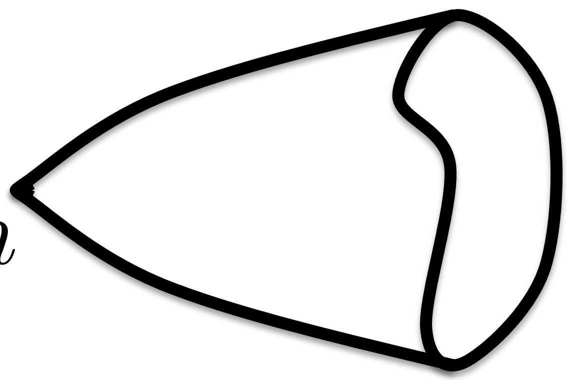
We will argue for this statement by giving the analytic continuation of the geometries.

The analytic continuation of the geometries

$$I_n = I[g_n] = nI[g_n/Z_n]$$

With no contribution from the singularity !!

$2\pi/n$



$$\tau = \tau + 2\pi$$

Analytic continuation = Same for all values of n

For integer n → non singular in covering space. Here no clear covering space…
We just evaluate the action, with no contribution from the singularity and multiply by n

Equation for the surface

As n → 1 we have the spacetime geometry produced by a cosmic string.

Equations of motion → from expanding Einstein's equations near the singularity and demanding that the geometry of the singularity is not modified.

Unruh, Hayward, Israel, Mcmanus
Boisseau, Charmousis, Linet

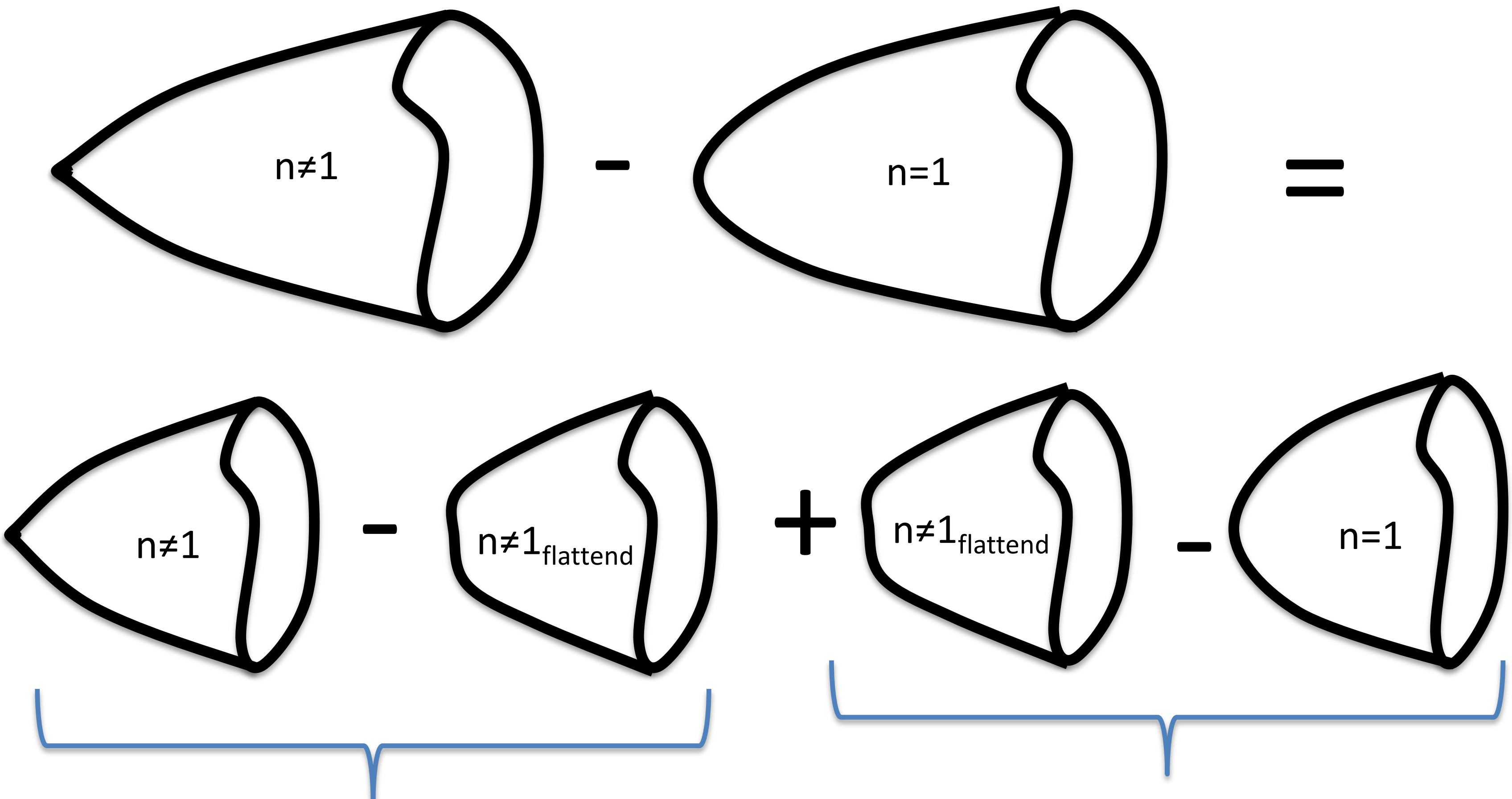
$$K^i_{aa} = 0$$

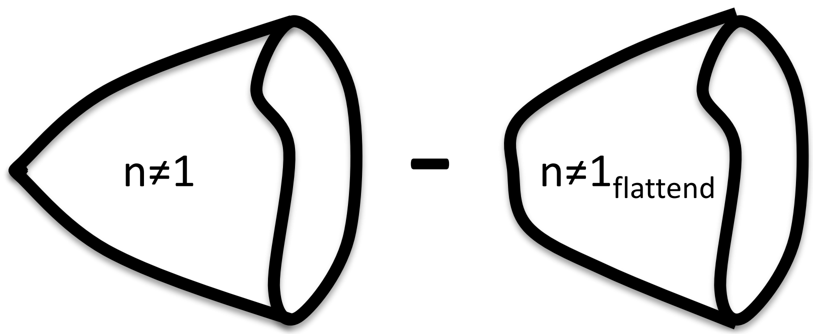
Equations for a minimal surface

Evaluating the action

Hawking, Fursaev, Solodukhin

We only want to evaluate it for n close to one.





Curvature of the flattened solution near the tip → $\int_{\Sigma_2} \sqrt{g} R = 4\pi(1-n)$

This is multiplied by the area of the rest of the dimensions →
Got the usual area formula

Comments

- This works if there is a moment of time reflection symmetry. (e.g. spatial regions in static spacetimes).
- We did not prove the more general HRT formula, valid for dynamical situations.
- Higher curvature action ? Should work, details ?

Hubeny,
Rangamani
Takayanagi

Hung, Myers, Smolkin

Fursaev, Patrushev, Solodukhin
Chen, Zhang,
Bhattacharyya, Kaviraj, Sinha

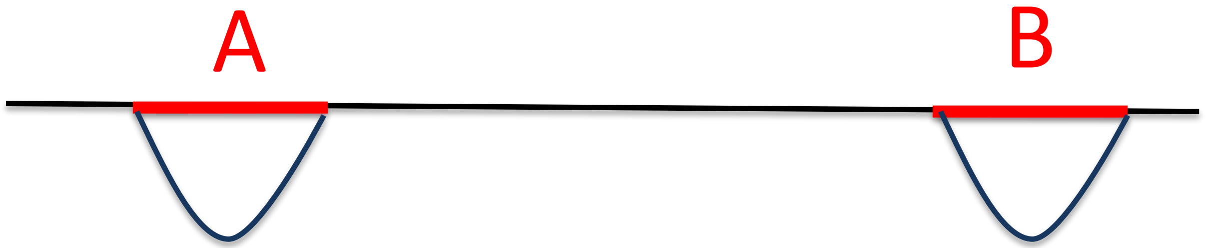
Quantum corrections

- So far we have discussed the terms of order $1/G_N$
- What about the first quantum correction in the bulk.

- We will motivate it with a puzzle.

Mutual information

$$I(A,B) = S(A) + S(B) - S(A \cup B)$$



Large separation

Headrick Takayanagi

$$I(A,B) = 0$$

But

$$\frac{\langle \mathcal{O}_A \mathcal{O}_B \rangle^2}{2|\mathcal{O}_A|^2|\mathcal{O}_B|^2} \leq I(A,B)$$

Wolf, Verstraete, Hastings, Cirac

- No problem
- I(A,B)=0 only <u>to leading order </u>in 1/N or G_N .
- Then it should be non-zero at order N^0 or $G_N{}^0$

Direct computation

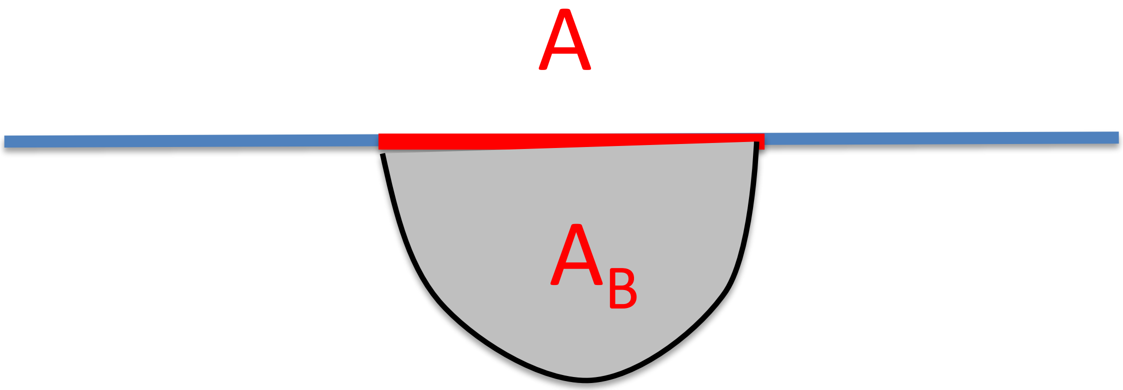
- Compute all one loop determinants around all the smooth integer n solutions.
- Continue in n →1

AdS_3 : Barrella, Dong,. Hartnoll, Martin Faulkner

- Is there a simpler formula for the final answer?

Quantum correction

$$S_q = S_{\text{Bulk entanglement}} + \cdots$$



Define a bulk region A_B , inside the minimal surface.

Compute the entanglement of the bulk quantum fields between A_B and the rest of the spacetime.

The …

$$S_q = S_{\text{Bulk entanglement}} + \frac{\delta A}{4G_N} + \frac{\langle \Delta A \rangle}{4G_N} + S_{\text{counterterms}}$$

Correction to the area
due to the one loop change in
the bulk metric. Area in the quantum
corrected metric.

Quantum expectation value of the
formal expression of the area.

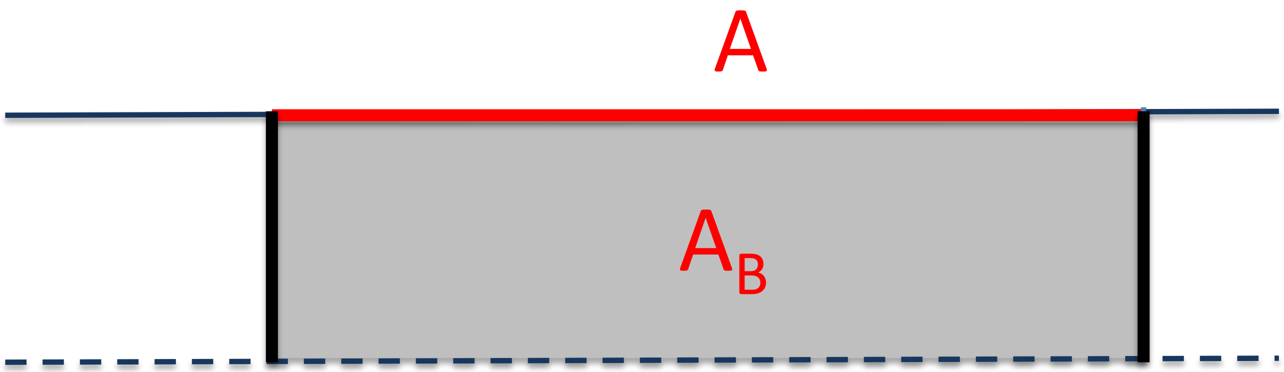
$$\int \phi^2 R \longrightarrow (\text{Area}) \langle \phi^2 \rangle$$

The counterterms that render the one loop bulk quantum theory finite
can lead to additional terms. They are of the usual Wald form.

Some interesting cases

KS theory

Theory with a mass gap for most degrees of freedom but with some remaining massless fields



Bulk computation → effectively four dimensional after KK reduction → only massless part contributes non-trivially at long distances.

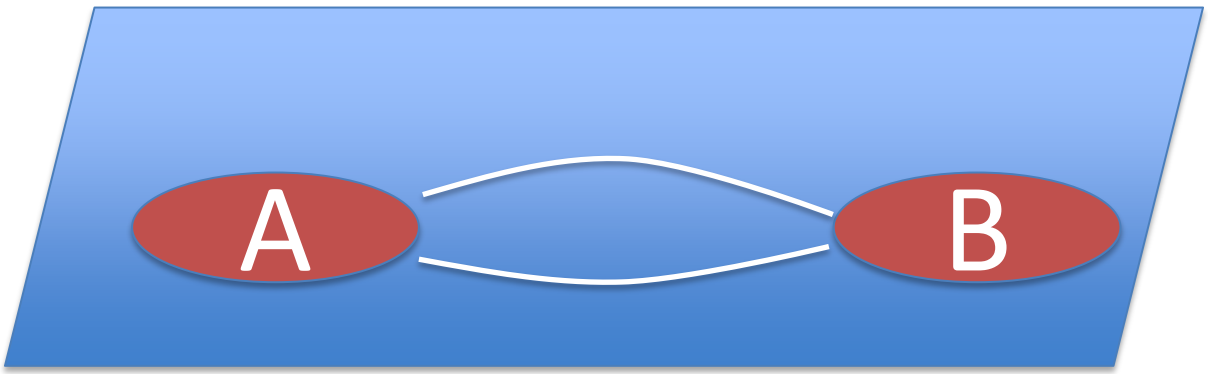
$$S = c_1 R^2 + \alpha \log R$$

Mutual information

Leading long distance entanglement = entanglement of the spins.

Mutual Information ``OPE''

General story in a CFT



Headrick
Calabrese,
Cardy Tonni

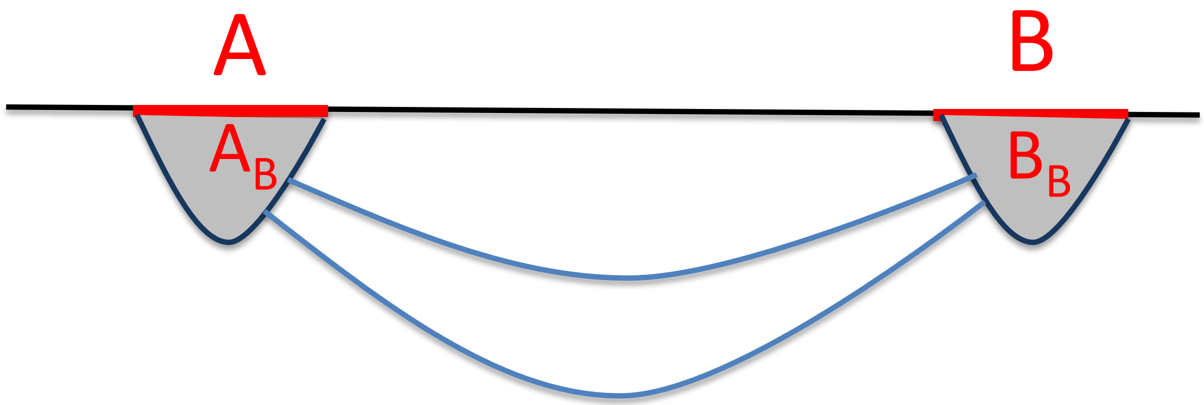
$$I(A,B) \sim \sum C \frac{1}{r^{4\Delta}} + \cdots$$

Analytic continuation of the OPE coefficients for the replicas.

Pairs of operators
(this comes from operators
on different replicas)

(not a true OPE in the original space)

In the holographic case



Large separation

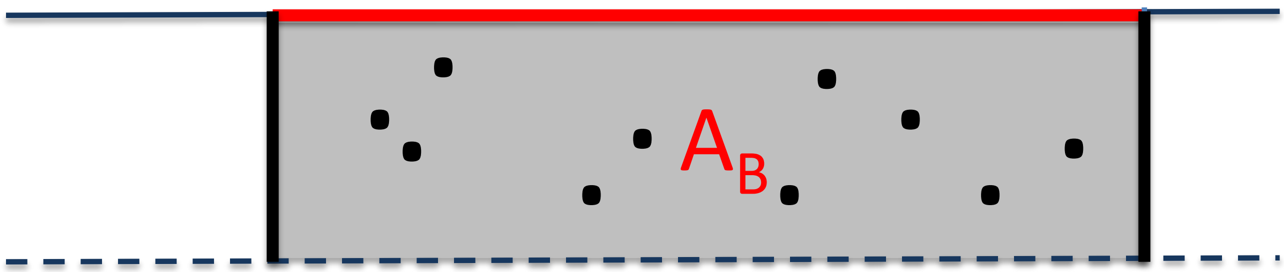
On the boundary we expect the previous OPE

In the bulk we also have an entanglement computation with two well separated regions. Bulk theory is not conformal and it is in curved space. However, we still expect some OPE where we exchange pairs of bulk particles.
(For conformally coupled bulk fields we can do it more precisely)

By GKPW → bulk particles → two point functions of operators.

Same structure for the OPE !

Thermal situations



Thermal gas in AdS → contribution from the entropy of the gas

Derivation

$$S_q = \partial_n (\log \det[g_n] - n \log \det[g_1])|_{n=1} =$$

$$S_q = \partial_n Tr\{(\rho[g_n/Z_n])^n\}|_{n=1}$$

$$S_q = \partial_n Tr\{(\rho[g_1])^n\}|_{n=1} + Tr\{\partial_n \rho[\tilde{g}_n]|_{n=1}\}$$

$$S_q = S_{\text{entanglement}} + S_{\text{corrections}}$$

Here ρ is the bulk density matrix, after propagation in tau for 2 π

Conclusions

- We proposed a simple formula for the quantum corrections of RT
- We checked it in a few cases where the quantum corrections are the dominant effect
- Further checks... ? More interesting checks …?

General comments

- There is an interesting connection between black hole entropy and entanglement entropy.

- Precise formulation of the Bekenstein formula

$$\Delta S \leq \langle \Delta H_{\rm Rindler} \rangle$$

Casini
arXiv:0804.2182
Inspired by
Marolf, Minic, Ross

- Proof of the generalized 2nd law.

Wall