# 3D Chern-Simons Theory from M5 Branes

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based on M.Yamazaki, **S.L.**, arXiv:1305.2429
C.Cordova, D.Jafferis, arXiv:1305.2891
J.Yagi, arXiv:1305.0291

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# What I Want to Show

**3d/3d Correspondence**
[Terashima,Yamazaki][Dimofte,Gaiotto,Gukov] …

**"This is what I want to study"**

N M5-branes on $M_3$ x X

Top'l twist along $M_3$

X = $S^1$ x $S^2$ or squashed $S^3$

3d N=2 SUSY theory $T_{\mathfrak{g}}^{N}[M_3]$ on X

3d $\mathbf{g_c}$ Chern-Simons Theory on $M_3$

# 3d N=2 SUSY Theory, $T_g\,[M_3]$

## Geometric engineering of 3d N=2 Theories

N M5-branes on $M_3$ x X

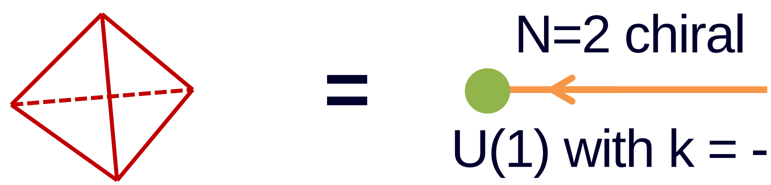
Top'l twist along $M_3$

3d N=2 SUSY theory

$T_{\mathfrak{g}}^{N}[M_3]$ on X

[Dimofte,Gaiotto,Gukov]

- 3d N=2 SUSY theories, characterized by hyperbolic space $M_3$
  - Tetrahedral decomposition of $M_3$
  - Building block (tetrahedron)




  - Use the "gluing rule" to construct $T_g\,[M_3]$

- The rule can't be applied to generic three-folds,

# Complexified Chern-Simons Theories

**Complexified CS theory** [Witten] [Gukov] …

$$\mathcal{L}_{\rm CS} = \frac{t}{8\pi}\Big(\mathcal{A}\wedge d\mathcal{A} - \frac{2i}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\Big) + \frac{\bar{t}}{8\pi}\Big(\bar{\mathcal{A}}\wedge d\bar{\mathcal{A}} - \frac{2i}{3}\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\wedge\bar{\mathcal{A}}\Big)$$

with CS levels $t = k + i\sigma$ and $\bar{t} = k - i\sigma$

- k is quantized (large gauge invariance) but $\sigma$ is a continuous parameter

- Unitary only when $\sigma$ is either real or pure imaginary

  **NB** Theory of 3d Euclidean gravity (at least classically) : ($\sigma = is$ )

$$\mathcal{L}_{\rm CS} = isI_{\rm grav}\ (\mathcal{A} = w + ie) \ \text{ with } \ I_{\rm grav} = \frac{1}{4\pi}\int_{M_3}\sqrt{g}\Big[R+2\Big]$$

- Lots of ambiguities though …

# What I Want to Show

**3d/3d Correspondence** [Terashima,Yamazaki][Dimofte,Gaiotto,Gukov]

| 3d N=2 SUSY theory $T^N_{\mathfrak{g}}[M_3]$ on X | | 3d complexified Chern-Simons on $M_3$ |
| --- | --- | --- |
| SUSY Vacua | = | Flat Connection on $M_3$ $d\mathcal{A} - i\mathcal{A}\wedge\mathcal{A} = 0$ |
| SUSY Partition Function on X (N=2) | | Partition Function of (SL(2,C)) CS with level t determined by X |

**e.g.**

$X = S^1 \times S^2$ $\qquad t \propto i\ R_{S^2}/R_{S^1}$

$X = S^3_b$ $\qquad t = 1 + if(b)$

# What I Want to Show

[Dimofte,Gaiott,Gukov] have speculated that

6d $\mathfrak{g}$ N=(2,0) on $M_3$ x X

$$\Big\downarrow \; X$$

3d G = $\mathfrak{g}_{\mathbf{C}}$ Chern-Simons Theory on $M_3$

# 6d N=(2,0) SCFT

## 6d N=(2,0) Theories

- Maximally supesymmetric conformal **FIELD** theories in 6D

- ADE classification

- Theory of tensor supermultiplet : [B$\mu\nu$, $\phi$, $\lambda$]

- Strongly interaction system: no tunable coupling constant.

- Mysterious scaling law: **$N^3$ - N** (AdS$_7$/CFT$_6$ & Anomaly-Inflow)

However nothing much is known (not even Lagrangian)

# What I Want to Show

**Direct Derivation of CS from M5s:**

$$
\begin{tikzcd}
\text{6d } A_{N-1} \text{ (2,0) on } M_3 \times S^1 \times_q S^2 \arrow[d, "S^1"] \arrow[dd, bend left, "???"] \\
\text{5d N=2 SYM on } M_3 \times S^2 \arrow[d, "S^2"] \\
\begin{array}{c}\text{3d G=SL(N,C)} \\ \text{Chern-Simons on } M_3\end{array}
\end{tikzcd}
$$

# 5D N=2 SYM

**Compactify (2,0) theory on $S^1$ x Y (=$S^2$ x $M_3$)**

- When a circle is small, low-energy theory can be described as 5D N=2 SYM on Y (=$S^2$ x $M_3$) with

$$g_{\rm YM}^2 = 8\pi^2 R_{\rm M}$$

**-** Non-renormalizable theory

**-** Valid in the limit $\mathcal{E} \ll \frac{1}{g_{\rm YM}^2}$

**e.g.** Y = $S^2$ x $M_3$ : (radius of $S^2$: r) $\frac{1}{r} \ll \frac{1}{g_{\rm YM}^2}$

**-** Corrections are controlled by a parameter $\beta \propto \mathcal{E} g_{\rm YM}^2 \propto \mathcal{E} R_{\rm M}$

# 5D N=2 SYM

However, we believe that some observables are still relevant even for finite $\beta$ !!

- Massive mode in KK tower = YM instanton-soliton

$$M_{\rm inst} = \frac{8\pi^2}{g_{\rm YM}^2} = \frac{1}{R_{\rm M}} = M_{\rm KK}$$

- Irrelevant operators to describe the UV fixed point are Q-exact

**e.g.** X = $S^5$ (radius of $S^5$: r) [H.Kim,S.Kim]

$$Z_{\rm 5d\ SYM}[S^5] = Z_{6d\ (2,0)}[S^1 \times_q S^5]$$

where q = $e^{-\beta}$, **finite**, is a fugacity of the 6d index

(**c.f.** see also the work of Kawano et al.)

# SUSY on $S^2$ x $M_3$

**SUSY on $S^2$ x $M_3$ :** SU(2|1)

- Two requirements

  [1] Topological twisting along $M_3$

  [2] 3d N=2 superconformal index (**$S^1$ x$_q$ $S^2$**) : SU(2|1)

Sp(4)$_R$

$$SO(2)_{S^1} \times SU(2)_{S^2} \times SO(3)_{M_3} \times SO(3)_{\mathrm{R}} \times SO(2)_{\mathrm{R}} \subset OSp(2,6|4)$$

broken to diag. SO(3)$_{twist}$

- Parametrized by Killing spinors (m = 1, 2 and $\mu$ = 3, 4, 5)

$$\nabla_m \varepsilon_I = -\frac{i}{2r}\Gamma_m \Gamma_{12}\varepsilon_I \ , \qquad \nabla_\mu \varepsilon_I = 0$$

# Field Decomposition

**Under SO(3)$_{twist}$ x U(1)$_R$**

- Supercharge

| | | 16 | | 4 | | 12 |
|---|---|---|---|---|---|---|
| | | $\varepsilon_I$ | : | $\mathbf{1}_{\pm 1}$ | $\oplus$ | $\mathbf{3}_{\pm 1}$ |
| | | | | $(\xi, \bar{\xi})$ | | |

- 5d N=2 vector multiplet fields ($A_M$, $\phi^A$, $\lambda_I$) can be decomposed into 1+3 supermultiplets

| $A_M$ | | | $\phi^A$ | | | $\lambda_I$ | | |
|---|---|---|---|---|---|---|---|---|
| $\mathbf{1}_0$ | $\oplus$ | $\mathbf{3}_0$ | $\mathbf{1}_{\pm 2}$ | $\oplus$ | $\mathbf{3}_0$ | $\mathbf{1}_{\pm 1}$ | $\oplus$ | $\mathbf{3}_{\pm 1}$ |
| $A_m$ | | $A_\mu$ | $\varphi_\pm$ | | $\phi^\mu$ | $(\lambda, \bar{\lambda})$ | | $(\psi^\mu, \bar{\psi}^\mu)$ |

# 5d N=2 SYM on $S^2$ x $M_3$

SU(2|1) Representation : 1/r corrections (**r** : radius of $S^2$)

$$\delta\lambda = \delta^{\mathrm{flat}}\lambda + \frac{1}{r}\bar{\xi}\varphi_+$$

SUSY Lagrangian on $S^2$ x $M_3$ : 1/r corrections

$$\mathcal{L}_{S^2} = \mathcal{L}_{\mathrm{flat}} + \frac{1}{4rg^2}\Big[\mathcal{L}_{\mathrm{CS}}(\mathcal{A}) - \mathcal{L}_{\mathrm{CS}}(\bar{\mathcal{A}})\Big] + \frac{i}{2rg^2}\mathrm{tr}\Big[\bar{\lambda}\gamma^3\lambda\Big]$$

where

$$\mathcal{L}_{\mathrm{CS}}(\mathcal{A}) = \epsilon^{\mu\nu\rho}\mathrm{tr}\Big[\mathcal{A}_\mu\partial_\nu\mathcal{A}_\rho - \frac{2i}{3}\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\rho\Big] \quad \mathcal{A}_\mu = A_\mu + i\phi_\mu$$

# 5d N=2 SYM on $S^2$ x $M_3$

Convenient to reorganize the (bosonic) Lagrangian as follows

$$\mathcal{L}_{\rm b} = \mathcal{L}_{\rm tv} + \mathcal{L}_{\rm tc} + \mathcal{L}_W + \mathcal{L}_{\bar{W}}$$

where

$$\mathcal{L}_{\rm tv} = {\rm tr}\Big[\frac{1}{2}(F_{12})^2 + \frac{1}{2}D_m\varphi_+ D_m\varphi_- + \frac{1}{8}\big[\varphi_+,\varphi_-\big]^2 + \frac{1}{2}D^2\Big]$$

$$\begin{aligned}\mathcal{L}_{\rm tc} = {\rm tr}\Big[ &+\frac{1}{2}(F_{m\mu})^2 + \frac{1}{2}(D_m\phi_\mu)^2 + \frac{1}{2}D_\mu\varphi_- D_\mu\varphi_+ - \frac{1}{2}[\phi_\mu,\varphi_+][\phi_\mu,\varphi_-] \\ &+ \frac{1}{2}\bar{G}_\mu G^\mu + iD(D^\mu\phi_\mu)\Big]\end{aligned}$$

$$\mathcal{L}_W = {\rm tr}\Big[-\frac{i}{2}\frac{\partial}{\partial\mathcal{A}^\mu}W(\mathcal{A}^\mu)G^\mu\Big] + i\frac{1}{2r}W(\mathcal{A}^\mu)$$

$$\mathcal{L}_{\bar{W}} = {\rm tr}\Big[-\frac{i}{2}\frac{\partial}{\partial\bar{\mathcal{A}}^\mu}\bar{W}(\bar{\mathcal{A}}^\mu)\bar{G}^\mu\Big] + i\frac{1}{2r}\bar{W}(\bar{\mathcal{A}}^\mu)$$

$$W = -\frac{i}{2}\epsilon^{\mu\nu\rho}{\rm tr}\Big[\mathcal{A}_\mu\partial_\nu\mathcal{A}_\rho - \frac{2i}{3}\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\rho\Big]$$

$$\frac{\partial}{\partial\mathcal{A}^\mu}W(\mathcal{A}) = -\frac{i}{2}\epsilon_{\mu\nu\rho}\mathcal{F}^{\nu\rho}$$

# Comments I

Dimensional reduction down to $S^2$

- Supermultiplets : In the language of $S^2$ SUSY, 5d N=2 vector multiplet can be decomposed into

| $A_M$ | $\phi^A$ | $\lambda_I$ |
|---|---|---|
| $\mathbf{1}_0 \oplus \mathbf{3}_0$ | $\mathbf{1}_{\pm 2} \oplus \mathbf{3}_0$ | $\mathbf{1}_{\pm 1} \oplus \mathbf{3}_{\pm 1}$ |
| $A_m \quad A_\mu$ | $\varphi_\pm \quad \phi^\mu$ | $(\lambda, \bar{\lambda}) \quad (\psi^\mu, \bar{\psi}^\mu)$ |

**[1] 1** Twisted vector multiplet

**[2] 3** Twisted chiral multiplets

# Comments I

Dimensional reduction down to $S^2$

- SUSY Lagrangian : a theory of twisted vector + twisted chiral multiplets

  - Kinetic Lagrangian terms

$$\mathcal{L}_{\rm tv} = \mathrm{tr}\Big[\frac{1}{2}(F_{12})^2 + \frac{1}{2}D_m\varphi_+ D_m\varphi_- + \frac{1}{8}\big[\varphi_+,\varphi_-\big]^2 + \frac{1}{2}D^2\Big]$$

for twisted vector

$$\begin{aligned}\mathcal{L}_{\rm tc} =& \mathrm{tr}\Big[+\frac{1}{2}(F_{m\mu})^2 + \frac{1}{2}(D_m\phi_\mu)^2 + \frac{1}{2}D_\mu\varphi_- D_\mu\varphi_+ - \frac{1}{2}[\phi_\mu,\varphi_+][\phi_\mu,\varphi_-] \\ &+ \frac{1}{2}\bar{G}_\mu G^\mu + iD(D^\mu\phi_\mu)\Big]\end{aligned}$$

for twisted chiral

  - Twisted superpotential terms

$$\mathcal{L}_W = \mathrm{tr}\Big[-\frac{i}{2}\frac{\partial}{\partial\mathcal{A}^\mu}W(\mathcal{A}^\mu)G^\mu\Big] + i\frac{1}{2r}W(\mathcal{A}^\mu)$$

NB On $S^2$, it is **different** to a theory of vector + chiral multiplets

# Comments II

A different 5d SYM that reduces a theory of vector + chiral on $S^2$ ?

Yes, but this model is nothing to do with 3d/3d correspondence

**e.g. On $S^2$ x $R^3$** ($SU(2)_R$ indices a,b=1,2)

$$\mathcal{L} = \frac{1}{g^2}\mathrm{tr}\Big[\frac{1}{4}F_{MN}^2 + \frac{1}{2}\big(D_M\sigma\big)^2 + \frac{i}{2}\epsilon^{\mathfrak{ab}}\lambda_{\mathfrak{a}}\Gamma^M D_M\lambda_{\mathfrak{b}} - \frac{i}{2}\epsilon^{\mathfrak{ab}}\lambda_{\mathfrak{a}}[\sigma,\lambda_{\mathfrak{b}}] - \frac{1}{2}D_{\mathfrak{ab}}D^{\mathfrak{ab}} + \frac{\sigma^2}{2r^2} - \frac{\sigma}{2r}\epsilon^{mn}F_{mn}\Big] - \boxed{\frac{i}{2rg^2}\epsilon^{\mu\nu\rho}\mathrm{tr}\Big[A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho\Big]}$$

Standard CS coupling

- Physical theory preserving 8 supercharges with $SU(2)_R$ symmetry
- $U(1)_R$ of $SU(2|1) \subset SU(2)_R$ : $M_3$ is impossible while preserving SU(2|1)
- "$r/g^2$" is quantized

# Evaluate the Path-Integral of 5d N=2 SYM on $S^2 \times M_3$

# Localization

Start with a following path-integral

$$Z[t] = \int \mathcal{D}\Phi \; e^{-S[\Phi] - tQ.V[\Phi]}$$

$$Q.S[\phi] = 0$$
$$Q^2 = J$$

- $S[\Phi]$ **:** action of a theory we want to study
- The term V is invariant under J, $J.V[\Phi] = 0$

**Supersymmetry tells us**

| $Z[0]$ | **=** | $Z[\infty]$ |
|---|---|---|
| $S[\Phi]$ | | $S_{\text{def}} = Q.V[\Phi]$ |
| Quantum | | **Semi-Classical** |
| Hard to evaluate | | Easy to evaluate (**Gaussian Integral**) |

# Localization

**RESULT**

$$Z[0] = \sum_{\phi_*} e^{-S[\phi_*]} \left( \frac{\det \Delta_f^{\mathrm{def}}}{\det \Delta_b^{\mathrm{def}}} \right) \Bigg|_{\phi_*}$$

where $\boldsymbol{\phi}_*$ satisfy (1) equation of motion of the deformed theory $\left. \frac{\delta S_{\mathrm{def}}}{\delta \phi} \right|_{\phi=\phi_*} = 0$

(2) supersymmetric condition

For BPS operators $Q.\mathcal{O}_{\mathrm{BPS}} = 0$,

$$\langle \mathcal{O}_{\mathrm{BPS}} \rangle = \sum_{\phi_*} e^{-S[\phi_*]} \left( \frac{\det \Delta_f^{\mathrm{def}}}{\det \Delta_b^{\mathrm{def}}} \right) \Bigg|_{\phi_*} \mathcal{O}_{\mathrm{BPS}}(\phi_*)$$

# Localization Scheme I

**[1]** **Choice of supercharge**

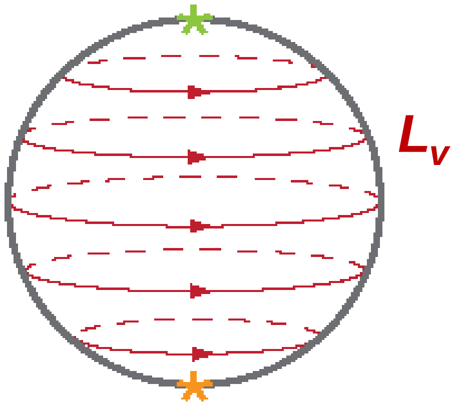
$$\mathcal{Q}^2 = -i\mathcal{L}_v + \mathrm{Gauge}(\gamma) + \mathrm{Lorentz}(\Theta) + U(1)_R(\alpha)$$

where $(\boldsymbol{\theta}, \boldsymbol{\phi})$ is the polar coordinate of S$^2$ and

$$\mathcal{L}_v = -i\partial_\phi$$

$$\gamma = A_\phi + e^{-i\phi}\sin\theta\varphi_+ - e^{+i\phi}\sin\theta\varphi_-$$

$$\Theta = -\frac{1}{r}\cos\theta \qquad \alpha = \frac{1}{2r}$$



# Gauge-Fixing

Introduce ghost fields $(\ c,\ \ \bar{c},\ \ B\ )$ and a background field $a_0$

- BRST charge $Q_B$ : $Q_B^2 = \mathrm{Gauge}(a_0)$

$$Q_B c = \frac{i}{2}[c, c] + a_0$$

- Taking into account for gauge-fixing, $\hat{\mathcal{Q}} \equiv \mathcal{Q} + Q_B$ satisfies

$$\boxed{(\hat{\mathcal{Q}})^2 = -i\mathcal{L}_v + \mathrm{Gauge}(a_0) + \mathrm{Lorentz}(\Theta) + U(1)_R(\alpha)}$$

**NB** SUSY variation of ghost field should be determined by

$$\mathcal{Q}c = -\gamma = -\left(A_\phi + e^{-i\phi}\sin\theta\varphi_+ - e^{-i\phi}\sin\theta\varphi_-\right)$$

# Localization Scheme II

**[2] Choice of Q-exact deformation**

$$\mathcal{L}_{\rm def} \equiv \hat{\mathcal{Q}}\mathcal{V} = \hat{Q}\Big[\mathrm{tr}\big[(\hat{Q}\lambda)^\dagger\lambda + \bar{\lambda}(\hat{Q}\bar{\lambda})^\dagger + (\hat{Q}\psi_\mu)^\dagger\psi_\mu + \bar{\psi}_\mu(\hat{Q}\bar{\psi}_\mu)^\dagger\big]$$

$$+ \mathrm{tr}\big[\bar{c}\underbrace{f}_{\text{gauge-fixing condition}} + \bar{c}B_0 + c\bar{a}_0\big]\Big]$$

**[3] SUSY saddle point configurations**

$$\mathcal{A}_\mu(x^M) = \mathcal{A}_\mu(x^\mu)\ , \qquad \varphi_\pm = F_{12} = \boxed{a_0 = 0}$$

and all other fields vanish

$$\hat{\mathcal{Q}}c = 0$$

# Result

Collecting all the results, one can reduce the path-integral of 5d SYM to the path-integral of a 3d theory

$$Z_{\rm 5d} = \int \mathcal{D}\mathcal{A}_\mu(x^\nu)\mathcal{D}\bar{\mathcal{A}}_\mu(x^\nu)\, e^{-\frac{\pi r}{g^2}\int_{M_3}\left(\mathcal{L}_{\rm CS}(\mathcal{A})-\mathcal{L}_{\rm CS}(\bar{\mathcal{A}})\right)}\, Z_{\text{one-loop}}(\mathcal{A},\bar{\mathcal{A}})$$

complexified CS coupling

[1] Purely bosonic theory

[2] Purely imaginary CS level : $it = \dfrac{4\pi^2 r}{g^2} \propto \beta^{-1}$ (can take a finite $\boldsymbol{\beta}$ **!**)

[3] Nontrivial measure ?

We will show that **$Z_{\text{one-loop}}$ is trivial**, leading to the proof of 3d/3d relation

# One-Loop Determinant

Cohomological Basis : 18 boson + 18 fermion variables

$X$ **: 9 B** $\Psi$ **: 9 F**

$\hat{Q}\Psi$ **: 9 B** $\hat{Q}X$ **: 9 F**

In the above basis, the deformed Lagrangian can be written as follows :

$$\mathcal{L}_{\text{def}} = \hat{Q}\mathcal{V} = \mathcal{L}_b + \mathcal{L}_f \qquad \hat{Q}^2 \equiv H$$

where

$$\mathcal{V} = \begin{pmatrix} \hat{Q}X & \Psi \end{pmatrix} \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X \\ \hat{Q}\Psi \end{pmatrix} ,$$

$$\mathcal{L}_{\text{b}} = \begin{pmatrix} X & \hat{Q}\Psi \end{pmatrix} \begin{pmatrix} H & \\ & 1 \end{pmatrix} \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} X \\ \hat{Q}\Psi \end{pmatrix}$$

$$\mathcal{L}_{\text{f}} = \begin{pmatrix} \hat{Q}X & \Psi \end{pmatrix} \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} 1 & \\ & H \end{pmatrix} \begin{pmatrix} \hat{Q}X \\ \Psi \end{pmatrix}$$

# One-Loop Determinant

**One-loop determinant :**

$$\left(\frac{\det K_f}{\det' K_b}\right)^2 = \frac{\det_\Psi H}{\det'_X H} = \frac{\det_{\mathrm{Coker}D_{10}} H}{\det'_{\mathrm{Ker}D_{10}} H}$$

Can compute the one-loop determinant from the index

$$\mathrm{ind}\, D_{10} = \mathrm{Tr}_{\mathrm{Ker}D_{10}}\left[e^{-Ht}\right] - \mathrm{Tr}_{\mathrm{Coker}D_{10}}\left[e^{-Ht}\right]$$

with

“0”

$$H = (\hat{\mathcal{Q}})^2 = -i\mathcal{L}_v + \mathrm{Gauge}(a_0) + \mathrm{Lorentz}(\Theta) + U(1)_R(\alpha)$$

BUT, H is independent of the saddle points, **so is the one-loop determinant !**

# Summary and Outlook

Therefore, one can show that

$$\boxed{Z_{5\mathrm{d}} = \int \mathcal{D}\mathcal{A}_\mu(x^\nu)\mathcal{D}\bar{\mathcal{A}}_\mu(x^\nu)\ e^{-\frac{\pi r}{g^2}\int_{M_3}\ \left(\mathcal{L}_{\mathrm{CS}}(\mathcal{A})-\mathcal{L}_{\mathrm{CS}}(\bar{\mathcal{A}})\right)}}$$

End of Proof

**M5 on $S^3_b$ x $M_3$** : [Cordova,Jafferis] have shown that one obtains complex CS theory with Chern-Simons level **$t = 1 + i(1-b)^{1/2}$**

Using a similar idea, can we prove the AGT correspondence ?

Can we learn something new about 3d quantum gravity or higher spin theory from M5-brane picture ?