Quiver Scaling Regimes & Black Holes

PILJIN YI
KOREA INSTITUTE for ADVANCED STUDY

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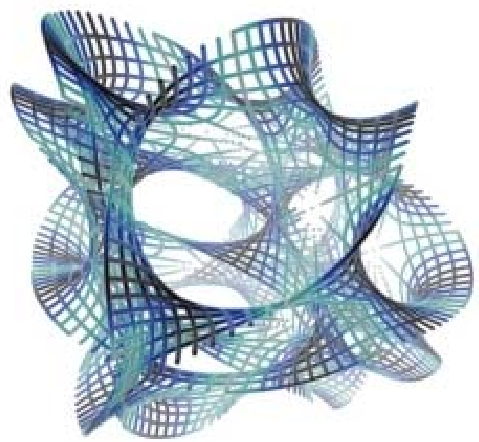
start with Denef's quiver quantum mechanics

Type IIB on CY3 → 4d N=2 theories

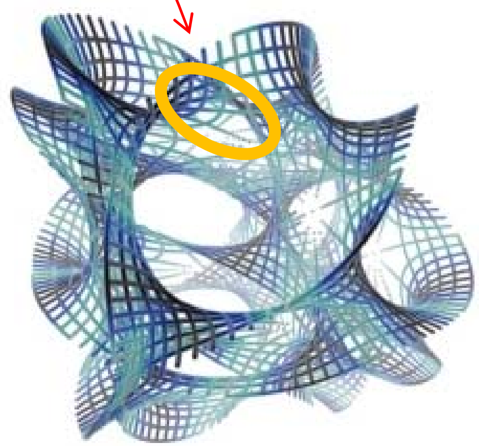
$$R^{1+3} \quad \times$$



D3 wrapped on a SL cycle in CY3 → 4d BPS particle

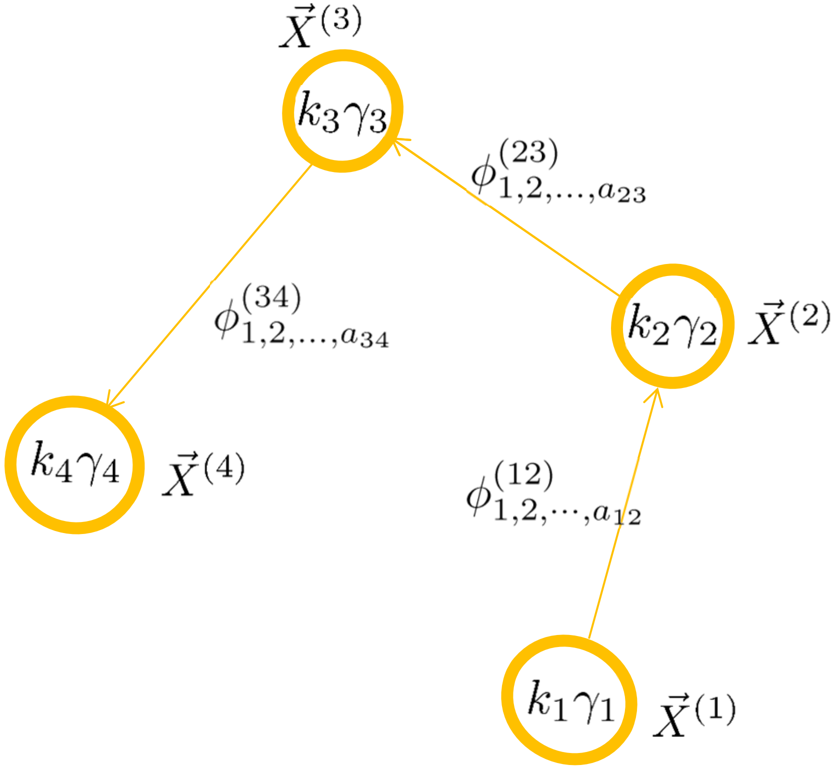
charged particle-like
BPS state in 4d

$$R^{1+3} \quad \times$$



Denef 2002

D3's wrapped on SL 3-cycles in CY3
→ quiver quantum mechanics for particle-like BPS states in 4d

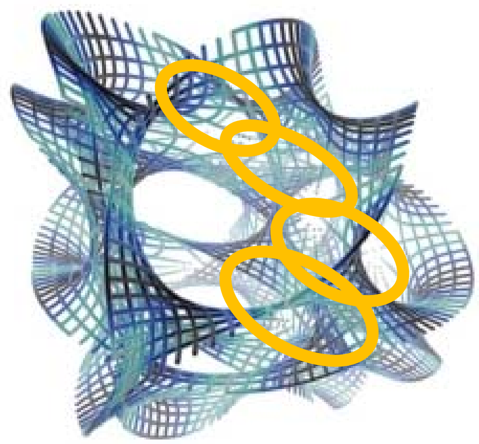


$$\vec{X}^{(1)} \qquad \vec{X}^{(2)} \qquad \vec{X}^{(3)} \qquad \vec{X}^{(4)}$$

$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

$$\phi^{(12)}_{1,2,\cdots,a_{12}} \qquad \phi^{(23)}_{1,2,\ldots,a_{23}} \qquad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$



where the main object of interest is the equivariant index

$$\Omega = -\frac{1}{2}\mathrm{tr}\left[(-1)^{2J_3}(2J_3)^2 y^{2(J_3+I)}\right]$$

$$= \mathrm{tr}'_{\text{without overall U(1)}}\left[(-1)^{2J_3} y^{2(J_3+I)}\right]$$

$SU(2)_R \times U(1)_R$

$J_{1,2,3}$ I

which counts BPS states with 4 supercharged preserved

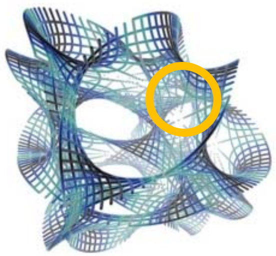
marginal stability wall

γ_1

γ_2

γ_3

$\vdots$



γ_1 $\quad$ $\gamma_1 + \gamma_2$

γ_2 $\quad$ $\gamma_2 + \gamma_3$

γ_3 $\quad$ $\gamma_1 + \gamma_3$

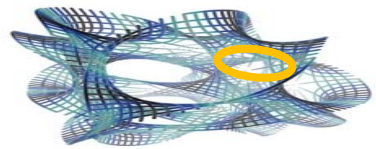
$\vdots$ $\quad$ $\vdots$

$\gamma_1 + \gamma_2 + \gamma_3$

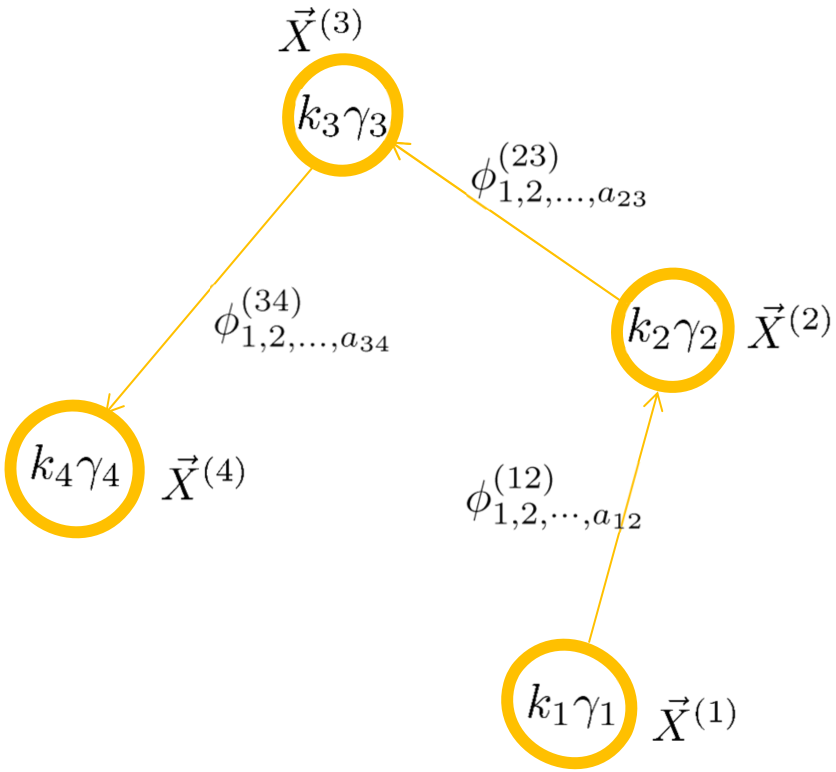
$\vdots$

$\sum k_i \gamma_i$

$\vdots$



Coulomb versus Higgs



$$\vec{X}^{(1)} \qquad \vec{X}^{(2)} \qquad \vec{X}^{(3)} \qquad \vec{X}^{(4)}$$

$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

$$\phi^{(12)}_{1,2,\cdots,a_{12}} \qquad \phi^{(23)}_{1,2,\ldots,a_{23}} \qquad \phi^{(34)}_{1,2,\ldots,a_{34}}$$

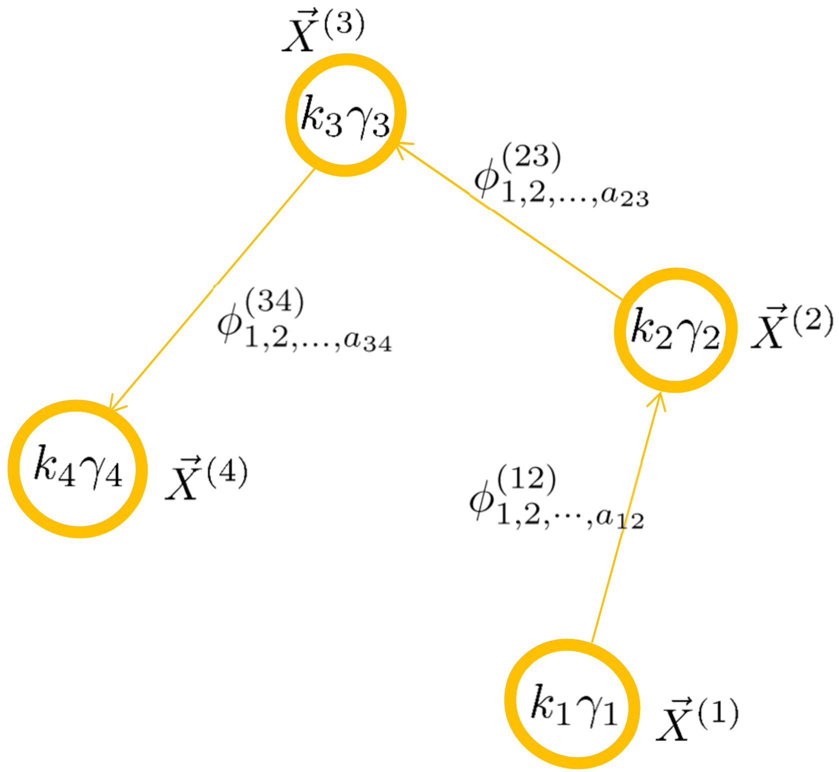
$$V \sim \sum_{ij} |(X^{(i)} - X^{(i)})\phi^{(ij)}|^2$$

$$+ \sum_{ij} |\partial W / \partial \phi^{(ij)}|^2$$

$$+ \sum_j \left(-D_j^2 + D_j \left(\sum_i \phi^{(ij)} \phi^{(ij)\dagger} - \zeta_j \right) \right)$$

Higgs : wrapped D-branes are fused into a single object

large & "positive"
FI constants



$$U(k_1) \times U(k_2) \times U(k_3) \times U(k_4)$$

$\phi^{(12)}_{1,2,\cdots,a_{12}}$ $\phi^{(23)}_{1,2,\ldots,a_{23}}$ $\phi^{(34)}_{1,2,\ldots,a_{34}}$

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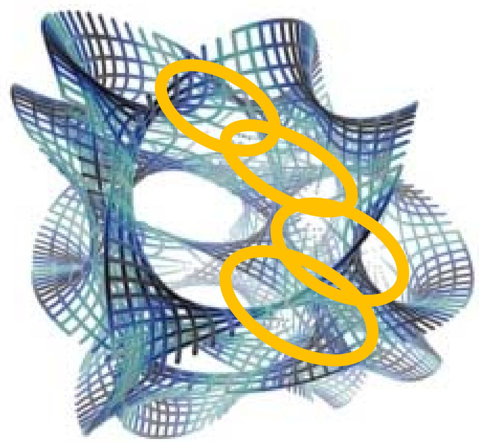


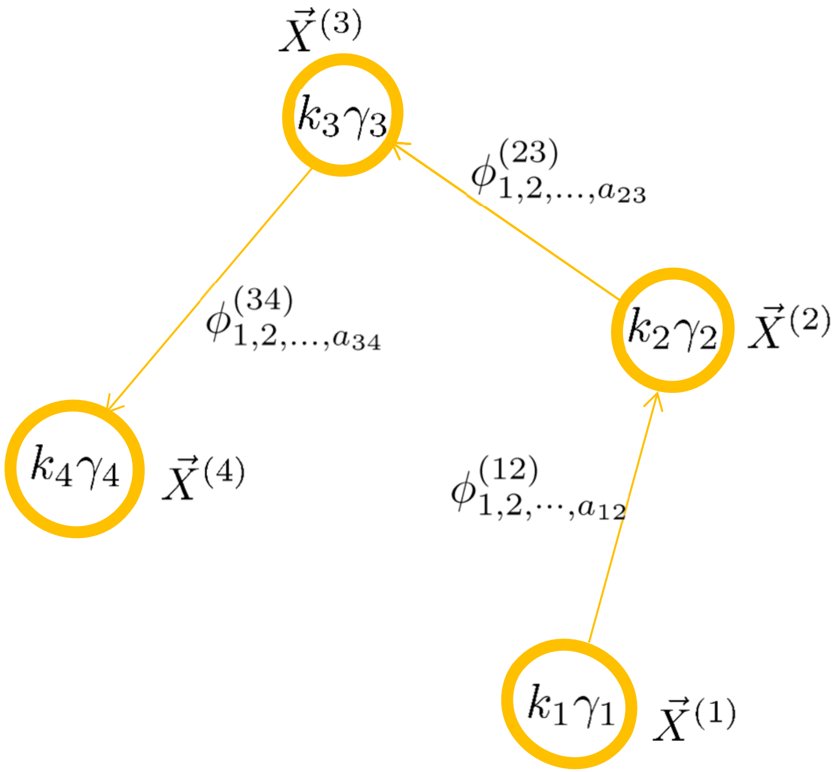
Higgs "phase"

= assume large values of chiral multiplets

and ignore heavy vector multiplets

$$V \sim \sum_{ij} |\cancel{(X^{(i)} - X^{(i)})\phi^{(ij)}}|^2 + \sum_j \left(-D_j^2 + D_j \left(\sum_i \phi^{(ij)}\phi^{(ij)\dagger} - \zeta_j \right) \right) + \sum_{ij} |\partial W / \partial \phi^{(ij)}|^2$$

Higgs "phase" ground states ~ cohomology → Euler index

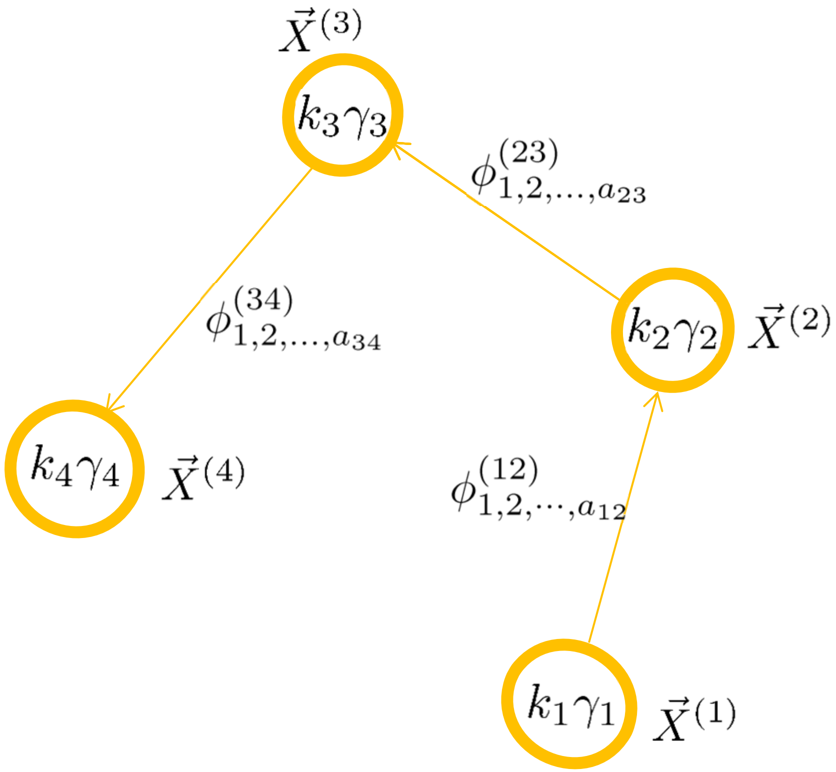


$$\Omega_{\mathrm{Higgs}}\left(\sum_i k_i\gamma_i;\xi^{(i)}\right)$$

$$\sim \chi\left(\mathcal{M}_{\mathrm{H}}\right)$$

$$= \sum_l (-1)^l \dim\left[H^l\left(\mathcal{M}_{\mathrm{H}}\right)\right]$$

or the Hirzebruch characteristic



$$\Omega_{\mathrm{Higgs}}[y]\left(\sum_i k_i\gamma_i;\xi^{(i)}\right)$$

$$= \mathrm{tr}(-1)^{p+q-d}y^{2p-d}$$

$$= \sum_{p,q}(-1)^{p+q-d}y^{2p-d}h^{(p,q)}$$

Higgs "phases" have branches with different vacuum geometry

marginal stability wall

$$\xi^{(1)} > \xi^{(2)} > \xi^{(3)} > \xi^{(4)}$$

$$\chi\left(\mathcal{M}_{\mathrm{H}}\right) = 0$$

$$\chi\left(\mathcal{M}_{\mathrm{H}}\right) = a_{12} \times a_{23} \times a_{34}$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$

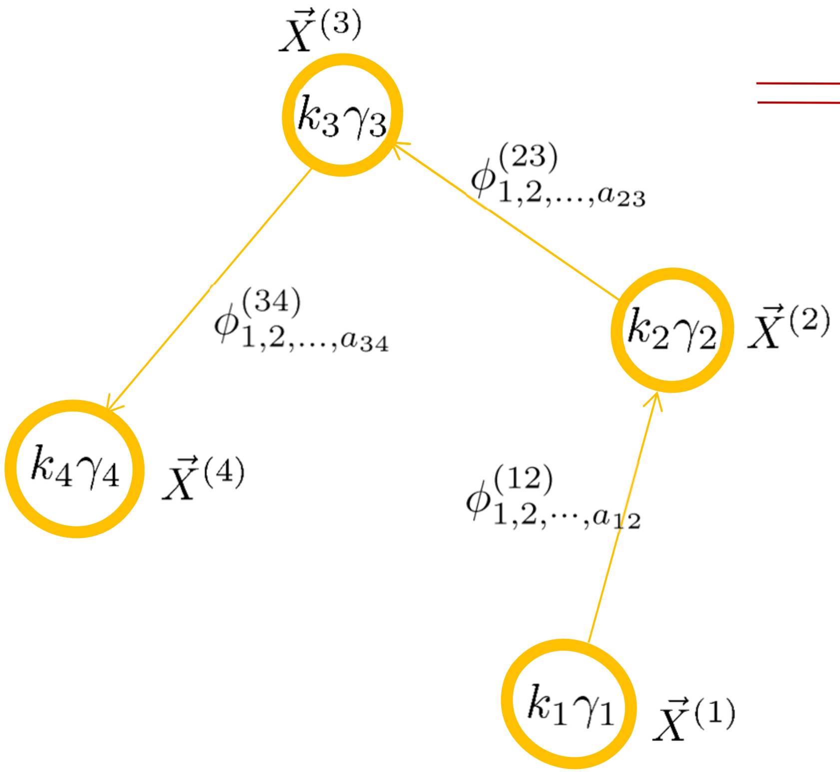
$$\mathcal{M}_{\mathrm{H}}\left(\sum_i \gamma_i; \xi^{(i)}\right) = 0$$

$$\mathcal{M}_{\mathrm{H}}\left(\sum_i \gamma_i; \xi^{(i)}\right)$$

$$= CP^{a_{12}-1} \times CP^{a_{32}-1} \times CP^{a_{34}-1}$$

Coulomb : wrapped D-branes are separated along real space

small & “positive”
FI constants



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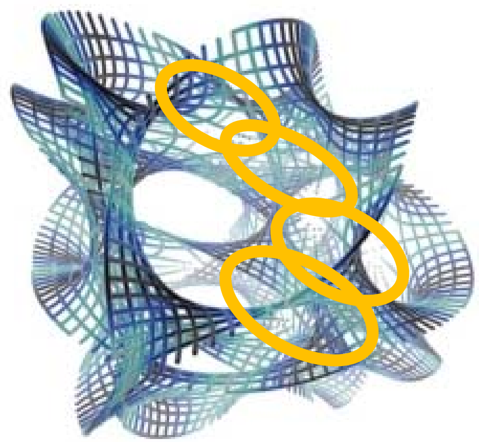


Coulomb "phase"

= assume large values of vector multiplets

and integrate out heavy chiral multiplets

$$V \sim \sum_{ij} |(X^{(i)} - X^{(i)})\phi^{(ij)}|^2 \; + \sum_j \left(-D_j^2 + D_j \left(\sum_i \phi^{(ij)} \phi^{(ij)\dagger} - \zeta_j \right) \right) + \cancel{\sum_{ij} |\partial W / \partial \phi^{(ij)}|^2}$$

Coulomb "phase"

= assume large values of vector multiplets

and integrate out heavy chiral multiplets

$$\Rightarrow \qquad V_{\text{Coulomb}}^{0+1} \sim \sum_j \left(-D_j^2 + D_j \left(\sum_i \frac{a_{ij}/2}{|X^{(i)} - X^{(j)}|} - \zeta_j \right) \right)$$

Coulomb "phase"

= assume large values of vector multiplets

and integrate out heavy chiral multiplets

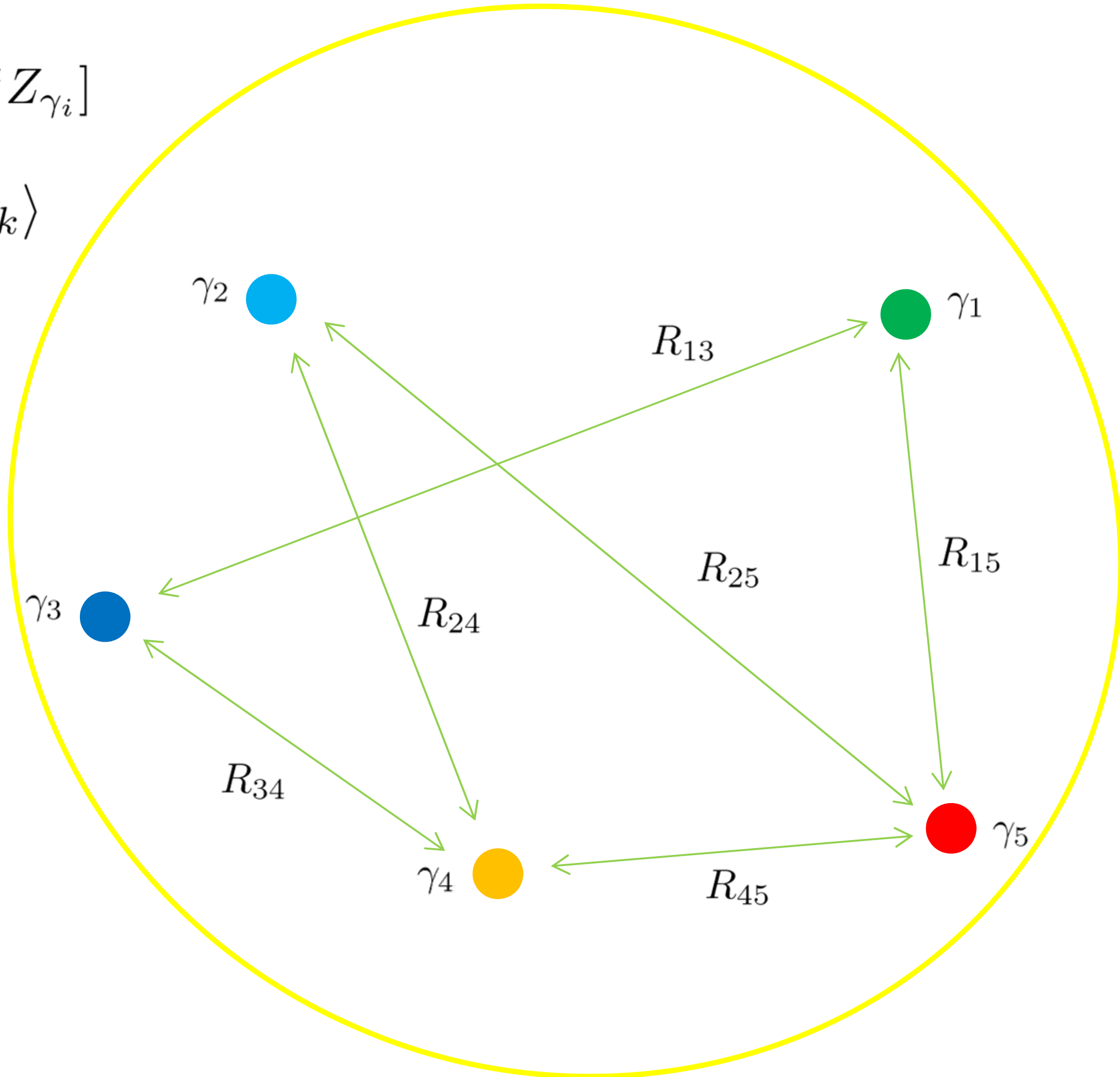
$$\Rightarrow \quad V^{0+1}_{\text{Coulomb}} \sim \sum_j \left(\sum_i \frac{a_{ij}/2}{|X^{(i)} - X^{(j)}|} - \zeta_j \right)^2$$

→ multi-center picture of BPS states

$$\zeta^i \sim \mathrm{Im}[e^{-i\alpha} Z_{\gamma_i}]$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$

$$R^3 = \{\vec{X}\}$$



1998 Lee + P.Y.
N=4 SU(n) ¼ BPS states via semiclassical multi-center dyon solitons

1999 Bak + Lee + Lee + P.Y.
N=4 SU(n) ¼ BPS states via semi-classical multi-center monopole dynamics

1999-2000 Gauntlett + Kim + Park + P.Y. / Gauntlett + Kim + Lee + P.Y. / Stern + P.Y.
N=2 SU(n) BPS state counting via semi-classical multi-center monopole dynamics

2000 Denef
N=2 supergravity via classical multi-center black holes attractor solutions

2001 Argyres + Narayan / Ritz + Shifman + Vainshtein + Voloshin
UV-incomplete string-web picture for N=2 BPS dyons

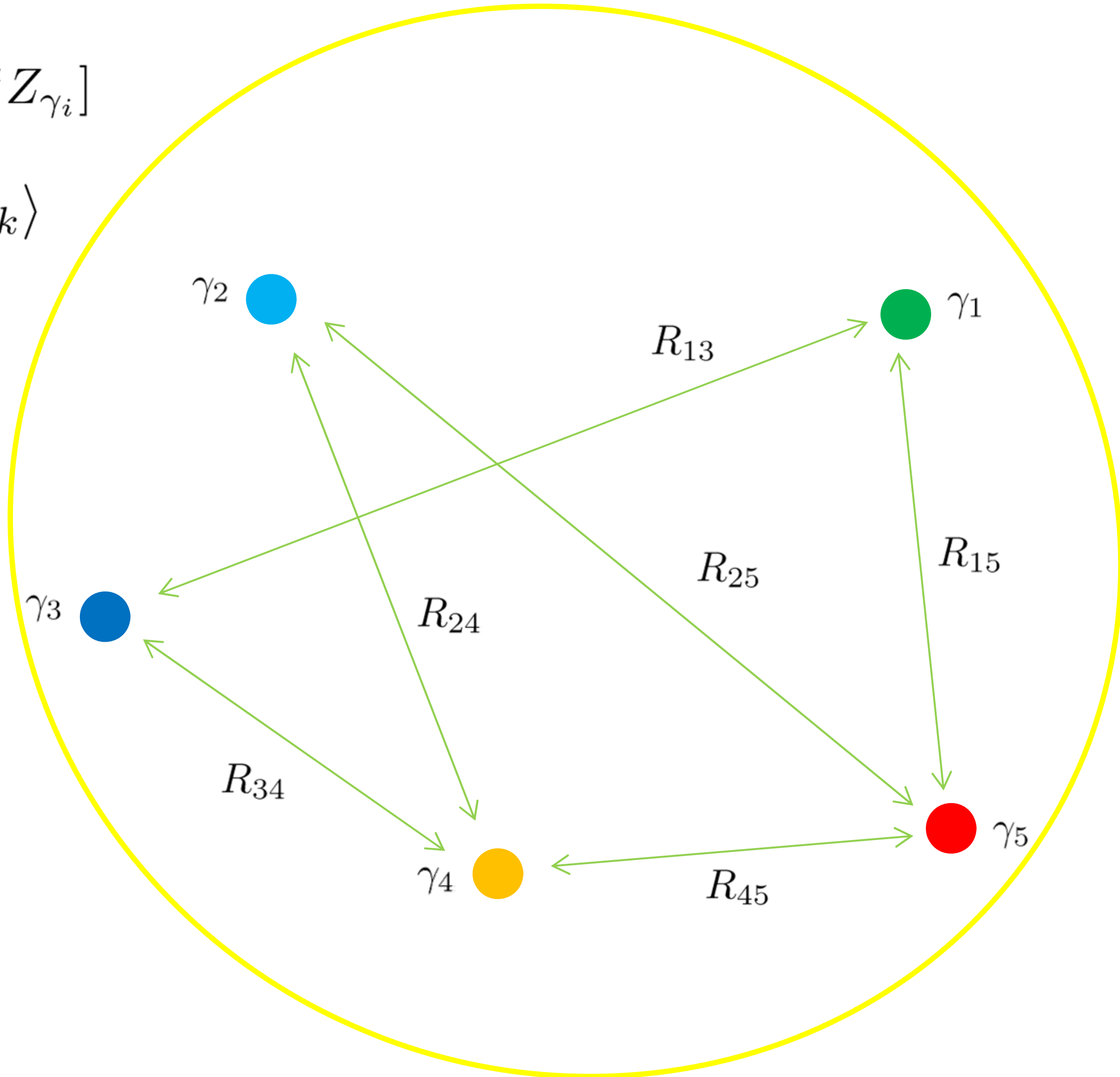
2002 Denef
quiver dynamics of BPS states / primitive wall-crossing formula

wall-crossing ~ supersymmetric Schroedinger problem

$$\zeta^i \sim \mathrm{Im}[e^{-i\alpha} Z_{\gamma_i}]$$

$$a_{ik} = \langle \gamma_i, \gamma_k \rangle$$

$$R^3 = \{\vec{X}\}$$



N=4 many body quantum mechanics, to be orbifolded by the Weyl symmetry

Denef 2002
Sungjay Lee+P.Y. 2011
Heeyeon Kim+Jaemo Park+Zhao-Long Wang+P.Y, 2011

$$\int dt\, \mathcal{L}_{kinetic} = \int dt \int d\theta^2 d\bar{\theta}^2 F(\hat{\Phi}^{Aa})$$

$$\int dt\, \mathcal{L}_{potential} = \int dt \int d\theta\ \left(i\mathcal{K}(\Phi)_A \Lambda^A - iW(\Phi)_{Aa} D\Phi^{Aa}\right)$$

$$\mathcal{K}_A = \mathrm{Im}[e^{-i\alpha} Z_{\gamma_A}] - \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle /2}{|\vec{x}_A - \vec{x}_B|}$$

$$\vec{W}_A = \sum_{B\neq A} \frac{\langle \gamma_A, \gamma_B \rangle}{2}\ \vec{W}^{4\pi}_{Dirac}(\vec{x}_A - \vec{x}_B)$$

$$F(\vec{x}_A) \simeq \sum_A |Z_A||\vec{x}_A|^2 = \sum_A m_A |\vec{x}_A|^2 \qquad \text{asymptotically}$$

each charge-center feels the long-range tails due to the rest

Sungjay Lee+P.Y. 2011

$$\mathcal{L}_\gamma = -|\mathcal{Z}_\gamma|\sqrt{1-\dot{\vec{x}}^2} + \mathrm{Re}[e^{-i\alpha}\mathcal{Z}_\gamma] - \dot{\vec{x}}\cdot\vec{W}$$

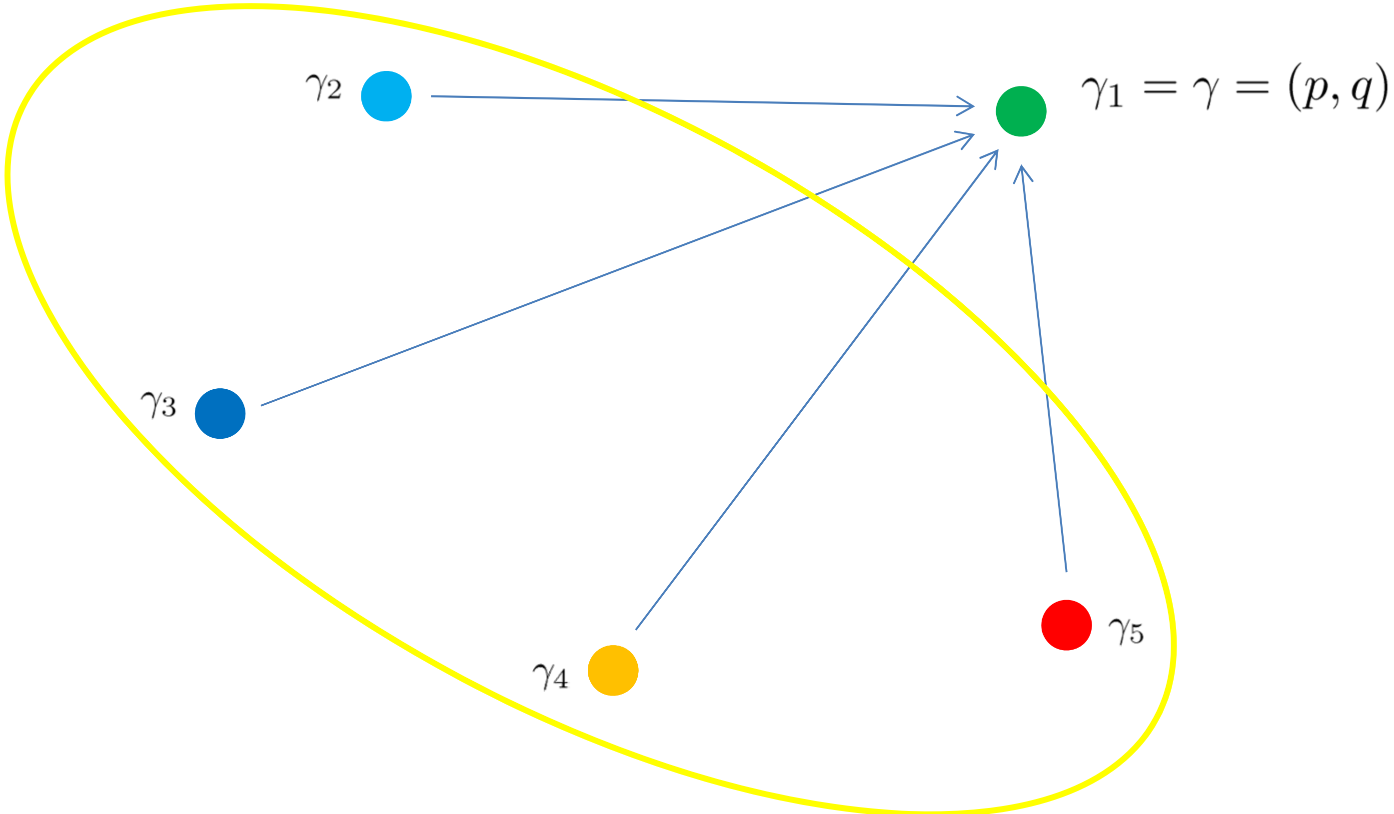
$$\simeq \frac{1}{2}|\mathcal{Z}_\gamma|\dot{\vec{x}}^2 - (|\mathcal{Z}_\gamma| - \mathrm{Re}[e^{-i\alpha}\mathcal{Z}_\gamma]) - \dot{\vec{x}}\cdot\vec{W}$$

$$\simeq \frac{1}{2}|\mathcal{Z}_\gamma|\dot{\vec{x}}^2 - \frac{(\mathrm{Im}[e^{-i\alpha}\mathcal{Z}_\gamma])^2}{2|\mathcal{Z}_\gamma|} - \dot{\vec{x}}\cdot\vec{W}$$

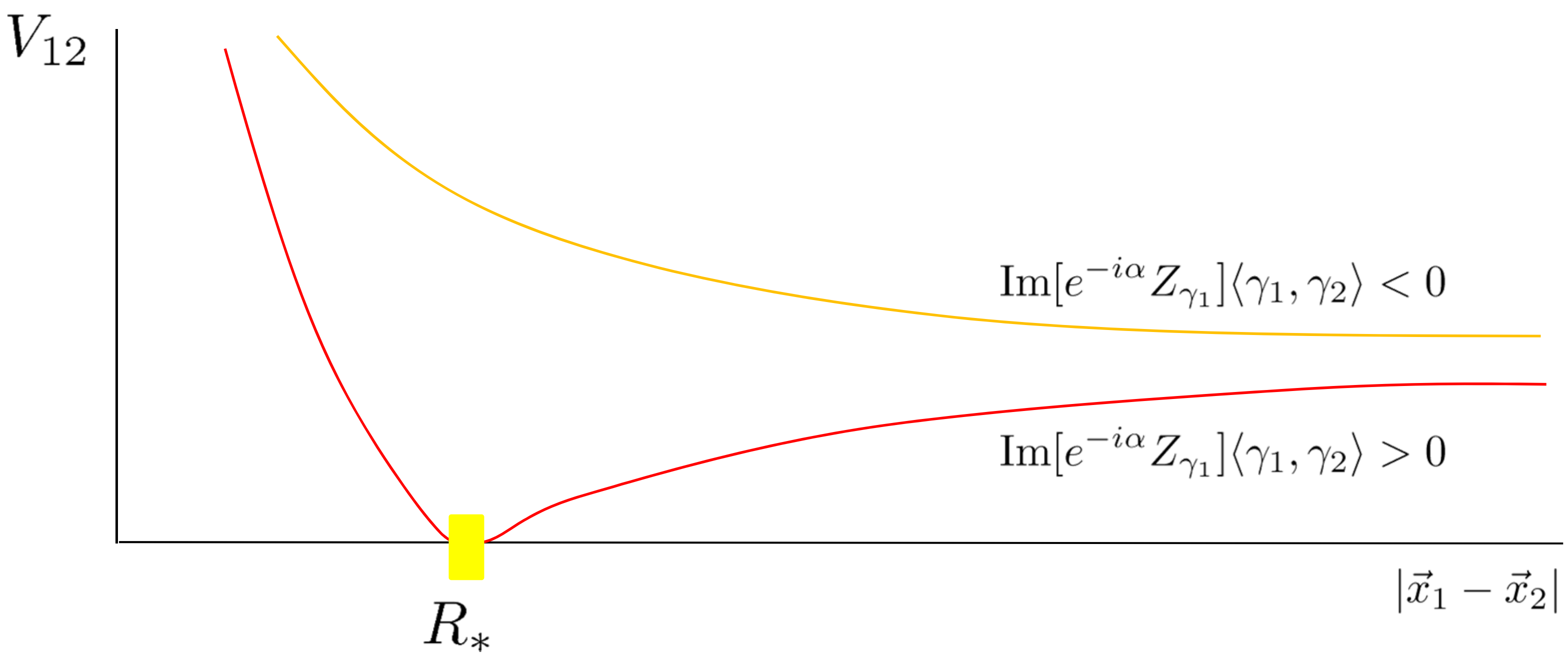
$$\vec{\partial}\times\vec{W} \equiv \vec{\partial}\,\mathrm{Im}\,[e^{-i\alpha}\mathcal{Z}_\gamma]$$

$$e^{-i\alpha}\mathcal{Z}_\gamma = |\mathcal{Z}_\gamma|e^{i\epsilon}, \quad |\epsilon| \ll 1$$

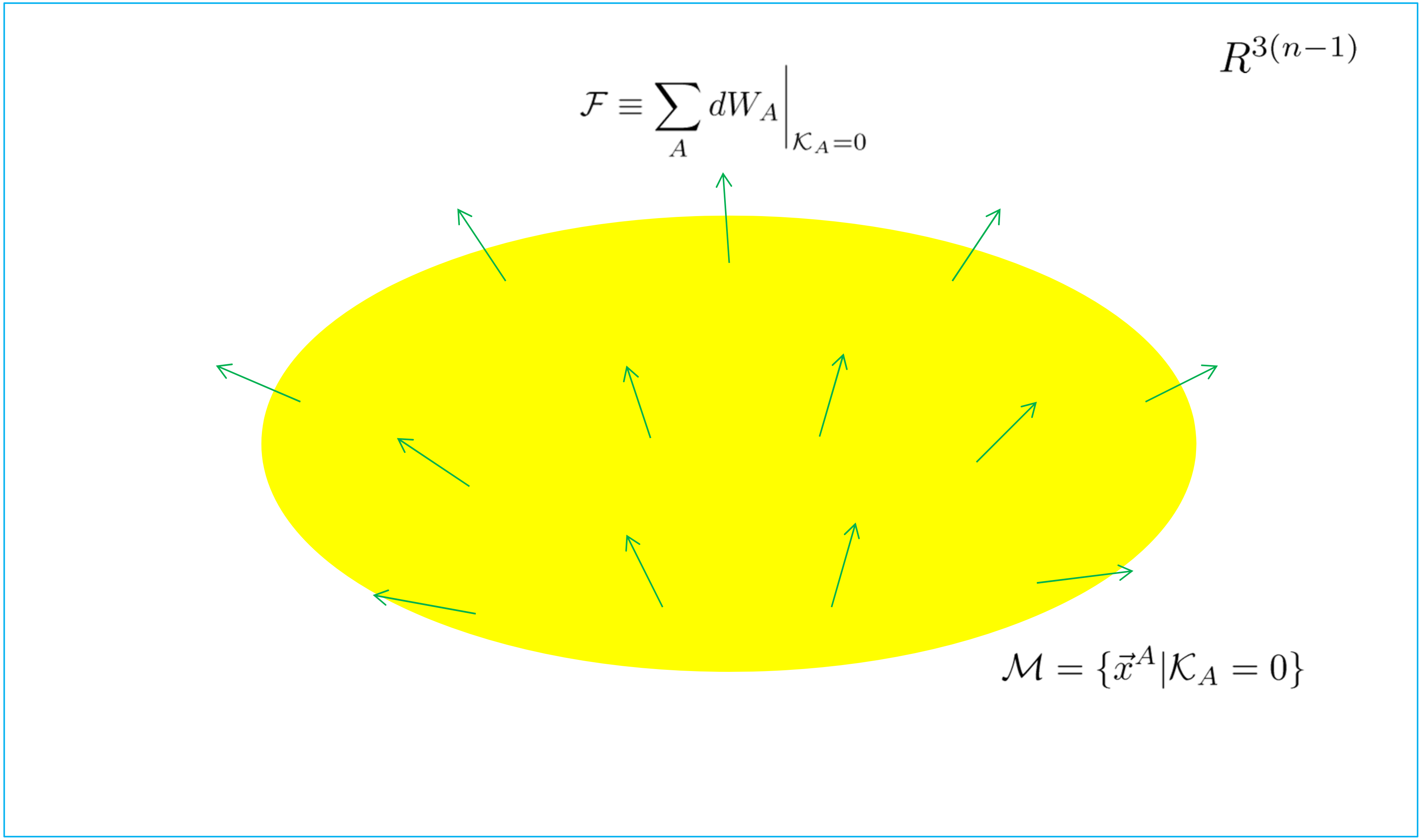
$$\mathcal{Z}_{\gamma=(p,q)} \equiv [p^i \phi^i_D + q^i \phi^i]\Big|_{\gamma_{A'=2,3,4,\ldots}} \qquad R^3 = \{\vec{X}\}$$



$$V(\{\vec{x}_{12}\}) \sim \left(\mathrm{Im}[e^{-i\alpha}Z_{\gamma_1}] - \frac{\langle\gamma_1,\gamma_2\rangle/2}{|\vec{x}_1-\vec{x}_2|}\right)^2$$



deform & localize N=4 3(n-1) dimensional dynamics
→ N=1 2(n-1) dim nonlinear sigma model with U(1) bundle



an index theorem before the Weyl division

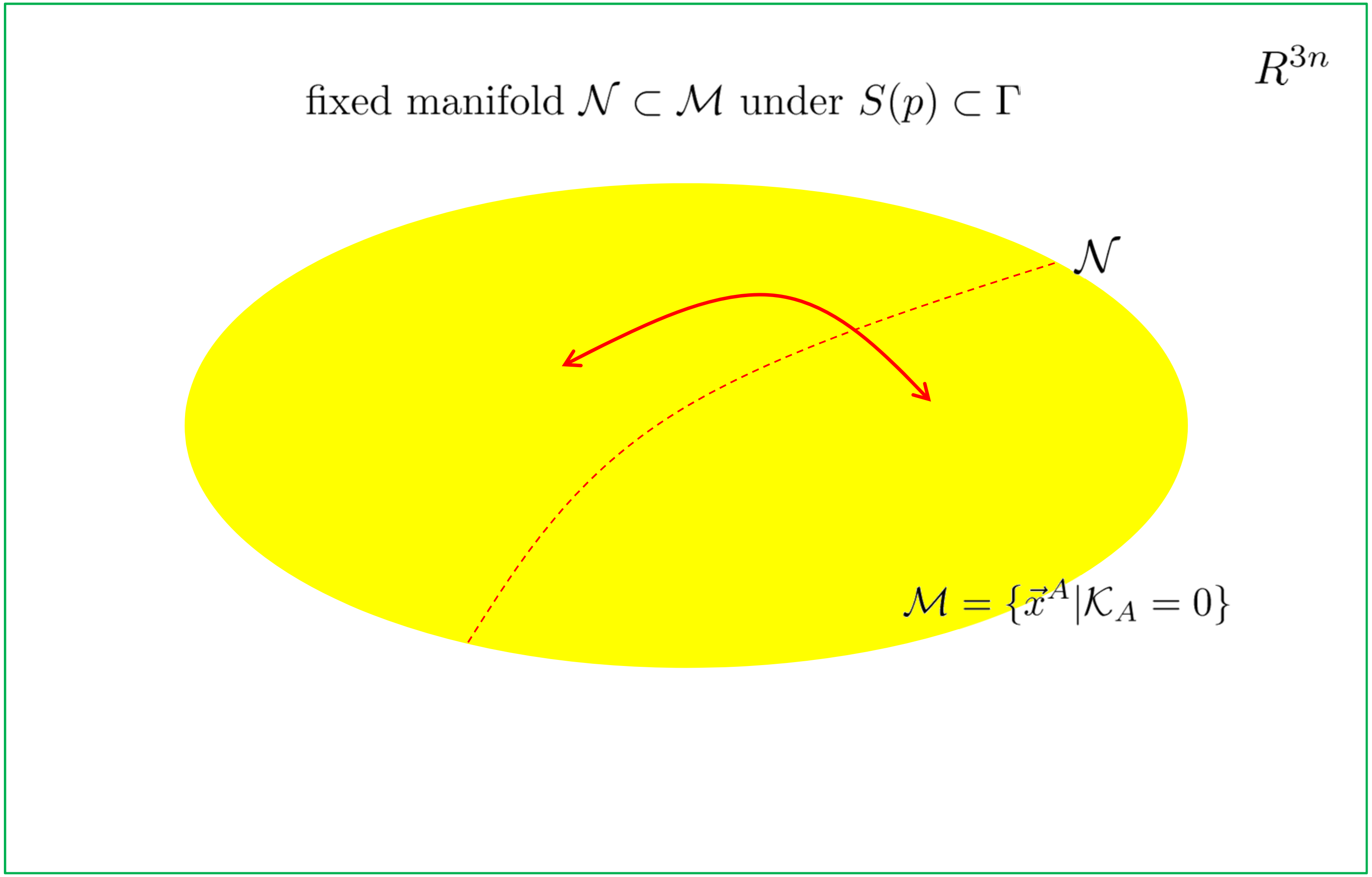
Manschot+Pioline+ Sen 2010/2011
Kim+Park+Wang+P.Y. 2011

$$\Omega_{\text{before statistics}} = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1} \boxed{I_n(\{\gamma_A\})} \prod_A \Omega(\gamma_A)$$

$$I_n(\{\gamma_A\}) = \text{tr}\left[(-1)^F e^{-\beta H}\right] = \int_{\mathcal{M}=\{\vec{x}_A \mid \mathcal{K}_A=0\}/R^3} ch(\mathcal{F}) \wedge \mathbf{A}(\mathcal{M})$$

$$\mathcal{F} \equiv \sum_{A>B} \frac{\langle\gamma_A,\gamma_B\rangle}{2}\, \mathcal{F}^{4\pi}_{Dirac}(\vec{x}_A - \vec{x}_B)\Big|_{\mathcal{K}_A=0}$$

division by Weyl symmetries → an iterative sum over fixed submanifolds under permutation of identical particles



→ orbifolding of the index

fixed manifold $\mathcal{N} \subset \mathcal{M}$ under $S(p) \subset \Gamma$

$$\Gamma' = \Gamma / S(p) \qquad \mathcal{P}' = \sum_{\sigma \in \Gamma'} (\pm 1)^{|\sigma|} \sigma$$

$$\mathrm{tr}\,(-1)^F e^{-\beta H} \mathcal{P}$$

$$= \mathrm{tr}_{\mathcal{M}/\Gamma - \mathcal{N}}\,(-1)^F e^{-\beta H} \mathcal{P} + \boxed{\Delta_{\mathcal{N}}}\, \mathrm{tr}_{\mathcal{N}/\Gamma' - \mathcal{N}'}\,(-1)^F e^{-\beta H} \mathcal{P}' + \cdots$$

for p identical particles & with internal degeneracy

P.Y. 1997
Green + Gutperle 1997
Kim+Park+Wang+P.Y. 2011

$$\text{fixed manifold } \mathcal{N} \subset \mathcal{M} \text{ under } S(p) \subset \Gamma$$

$$\Delta_{\mathcal{N}}(p\gamma) = \mathrm{tr}_{\mathcal{N}^\perp} \left[(-1)^{F^\perp} e^{-\beta H^\perp} \mathcal{P}^{(\pm)}_{S(p)}\right] \quad = \quad \frac{\Omega(\gamma)}{p^2}$$

$$\Downarrow$$

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

cf) Manschot + Pioline + Sen 2010/2011

e.g., for an identical pair of unit degeneracy each

$$\mathcal{P}_2^{(\pm)} \; : \; x \to -x, \; \psi \to -\psi$$

P.Y. 1997

$$\Delta_{\mathcal{N}}^{(\pm)}\Big|_{p=2} \leftarrow \lim_{\beta\to 0} \mathrm{tr}_{R^d;n_f} \left[(-1)^{F^\perp} e^{\beta\partial^2/2}\mathcal{P}_2^{(\pm)}\right]$$

$$= \lim_{\beta\to 0} \int_{R^d} d^d x \, \langle -x|e^{\beta\partial^2/2}|x\rangle \times (\pm 2^{n_{fermion}/2-1})$$

$$= \lim_{\beta\to 0} \frac{\pm 2^{n_{fermion}/2-1}}{(2\pi\beta)^{d/2}} \int_{R^d} d^d x \; e^{-(x+x)^2/2\beta}$$

$$= \lim_{\beta\to 0} \frac{\pm 2^{n_{fermion}/2-1}}{2^d}$$

$$\boxed{\begin{array}{l} \to \quad \dfrac{\pm 1}{2^2} \\[2ex] \begin{array}{cccccc} n_f & = & 2 & 4 & 8 & 16 \\ d & = & 2 & 3 & 5 & 9 \end{array} \end{array}}$$

universal wall-crossing formula
from Coulomb 'phase' dynamics / real space dynamics

Manschot+Pioline+Sen 2011
Kim+Park+Wang+P.Y. 2011

$$\boxed{\bar{\Omega}(\gamma) = \sum_{p|\gamma} \Omega(\gamma/p)/p^2}$$

$$\bar{\Omega}^-\left(\sum \gamma_A\right) = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1} \frac{\prod_A \bar{\Omega}^+(\gamma_A)}{|\Gamma|} \int_{\mathcal{M}} ch(\mathcal{F}) \wedge \mathcal{A}(M)$$

$$\vdots$$

$$+(-1)^{\sum_{A'>B'}\langle\gamma'_{A'},\gamma'_{B'}\rangle+n'-1} \frac{\prod_{A'} \bar{\Omega}^+(\gamma'_{A'})}{|\Gamma'|} \int_{\mathcal{M}'} ch(\mathcal{F}') \wedge \mathcal{A}(M')$$

$$\vdots$$

$$+(-1)^{\sum_{A''>B''}\langle\gamma''_{A''},\gamma''_{B''}\rangle+n''-1} \frac{\prod_{A''} \bar{\Omega}^+(\gamma''_{A''})}{|\Gamma''|} \int_{\mathcal{M}''} ch(\mathcal{F}'') \wedge \mathcal{A}(M'')$$

$$\vdots$$

$$\sum_{A=1}^{n} \gamma_A = \sum_{A'=1}^{n'} \gamma'_{A'} = \sum_{A''=1}^{n''} \gamma''_{A''} = \cdots$$

← an Abelianization formula
via a sum over all partitions of charges with rational invariants

Manschot+Pioline+Sen 2011
Kim+Park+Wang+P.Y. 2011

$$\boxed{\bar{\Omega}(\gamma) = \sum_{p|\gamma} \Omega(\gamma/p)/p^2}$$

$$\bar{\Omega}^-\left(\sum \gamma_A\right) = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1} \frac{\prod_A \bar{\Omega}^+(\gamma_A)}{|\Gamma|} \int_{\mathcal{M}} ch(\mathcal{F}) \wedge \mathcal{A}(M)$$

$$\vdots$$

$$+(-1)^{\sum_{A'>B'}\langle\gamma'_{A'},\gamma'_{B'}\rangle+n'-1} \frac{\prod_{A'} \bar{\Omega}^+(\gamma'_{A'})}{|\Gamma'|} \int_{\mathcal{M}'} ch(\mathcal{F}') \wedge \mathcal{A}(M')$$

$$\vdots$$

$$+(-1)^{\sum_{A''>B''}\langle\gamma''_{A''},\gamma''_{B''}\rangle+n''-1} \frac{\prod_{A''} \bar{\Omega}^+(\gamma''_{A''})}{|\Gamma''|} \int_{\mathcal{M}''} ch(\mathcal{F}'') \wedge \mathcal{A}(M'')$$

$$\vdots$$

$$\sum_{A=1}^{n} \gamma_A = \sum_{A'=1}^{n'} \gamma'_{A'} = \sum_{A''=1}^{n''} \gamma''_{A''} = \cdots$$

this computes BPS bound state index $\bar{\Omega}^{-}(\Gamma)$'s
given input data $\bar{\Omega}^{+}(\gamma)$'s → wall-crossing formulae

marginal stability wall

$$\gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \vdots$$

$$\gamma_1 \quad \gamma_2 \quad \gamma_3 \quad \vdots$$

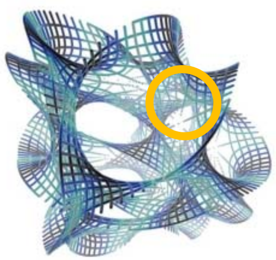
$$\gamma_1 + \gamma_2 \quad \gamma_2 + \gamma_3 \quad \gamma_1 + \gamma_3 \quad \vdots$$

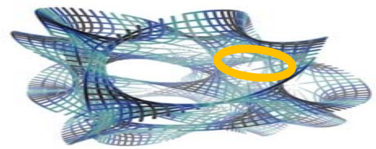
$$\gamma_1 + \gamma_2 + \gamma_3$$

$$\vdots$$

$$\sum k_i \gamma_i$$

$$\vdots$$





which is easily elevated to the equivariant index of the quiver as

Lee+Wang+P.Y. 2012

$$Q^A_\pm \qquad SU(2)_J \times U(1)^R_I \qquad \Omega = -\frac{1}{2}\mathrm{tr}\left[(-1)^{2J_3}(2J_3)^2 y^{2(J_3+I)}\right]$$

$$H = H_{\text{center of mass}} \otimes H_{\text{reduced}}$$

$$\mathcal{J}_3 = J_3 + I$$

$$\Omega = \mathrm{tr}_{H_{\text{reduced}}}\left[(-1)^{2L_3+2(S_3-I)}(-1)^{2I_3}y^{2(J_3+I)}\right]$$

reduction to $\mathcal{M}$

$$Q^1_- + Q^2_+ \qquad SO(3)_{\mathcal{J}} \qquad \Omega = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\mathrm{tr}((-1)^F y^{2\mathcal{J}_3})$$

and can be easily evaluated via localization

Lee+Wang+P.Y. 2012

$ch(\mathcal{F})$

$$\mathrm{tr}((-1)^F y^{2\mathcal{J}_3}) = \frac{G_n(y, y^{-1})}{(y - y^{-1})^n} \quad = a_d(y^d + y^{-d}) + a_{d-1}(y^{d-1} + y^{-d+1}) + \cdots + a_0$$

$\mathcal{A}(M)$

with all charges γ_A on a single plane of charge lattice,
and in the absence of a scaling regime,
the resulting wall-crossing formula has been shown to be
equivalent to the Kontsevich-Soibelman proposal

(Ashoke Sen, December 2011)

Coulomb versus Higgs

large FI constants

small FI constants

$$\Omega_{\text{Higgs}} = \Omega_{\text{Coulomb}}$$

Denef 2002

why?

in quantum mechanics, the word "phase" is very misleading
since vacuum expectation values do not imply
superselection sectors

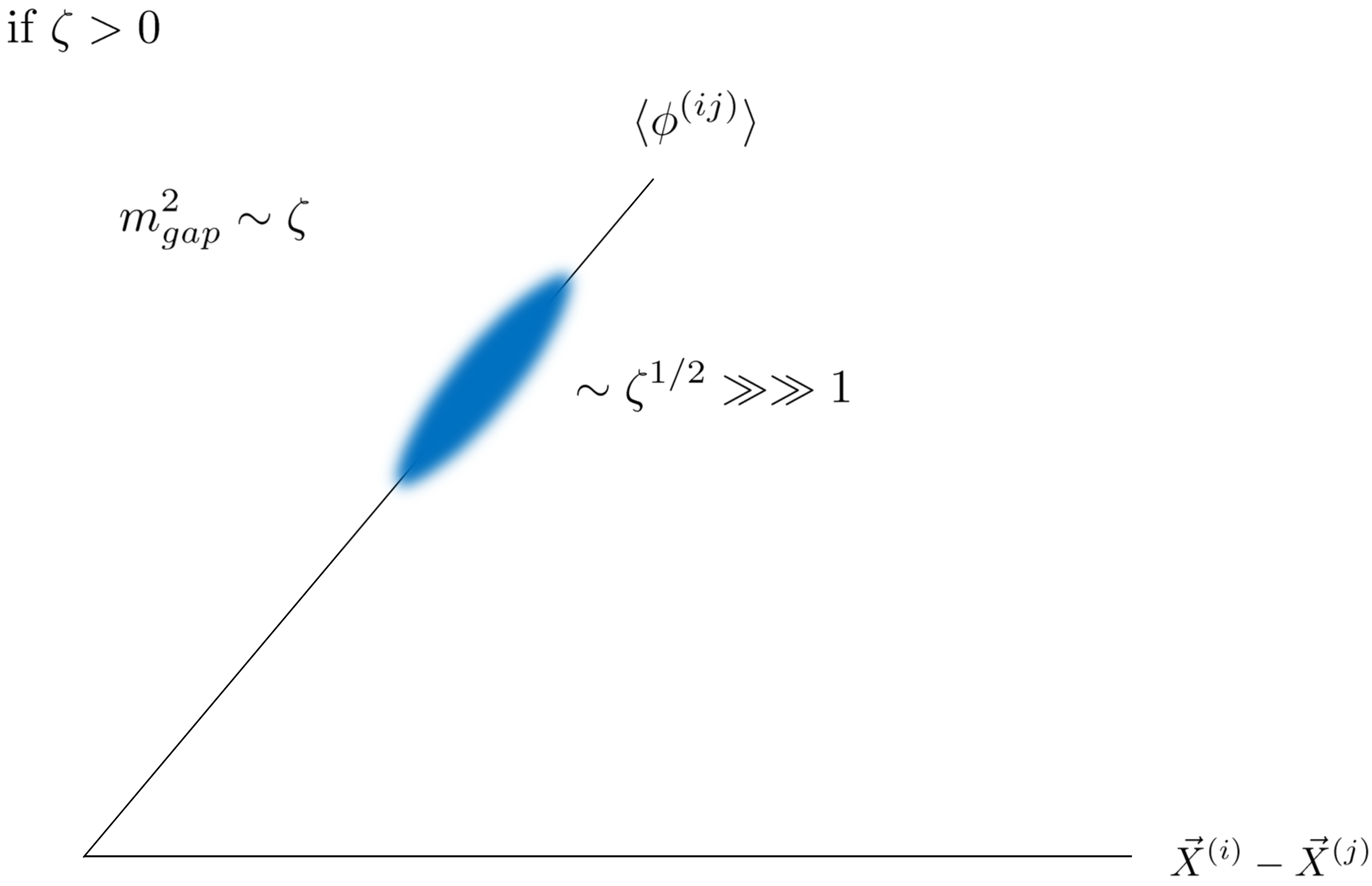
what one really means by this word is a truncation process
depending on where the ground state wavefunctions are localized;
at large values of chiral multiplets
or at large values of vector multiplets

$$V^0_{\text{Higgs}} \sim \sum_j \left(\sum_{i \neq j} \phi^{(ij)} \phi^{(ij)\dagger} - \zeta_j \right)^2$$

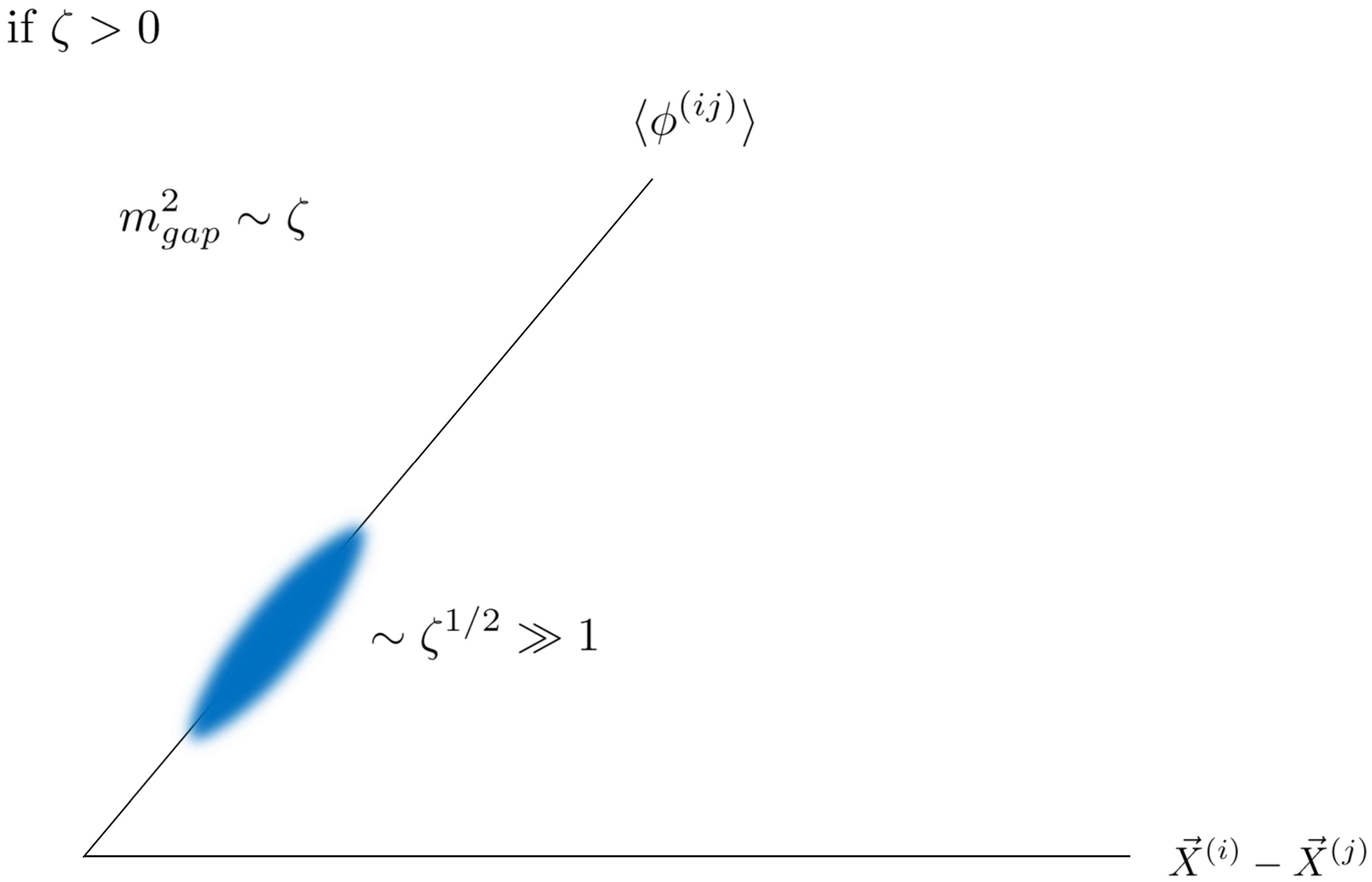
$$+ \sum_{ij} |\partial W / \partial \phi^{(ij)}|^2$$

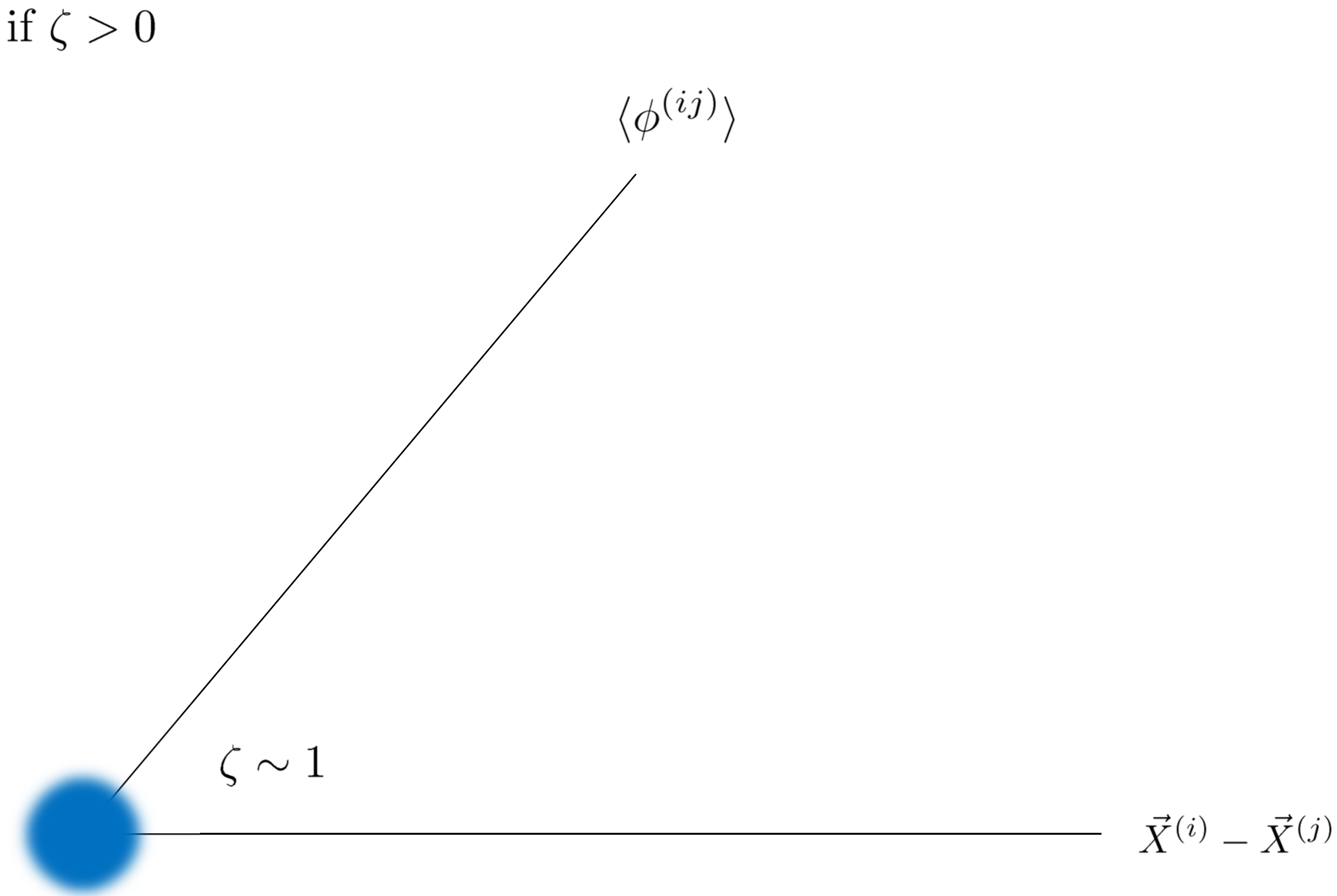
versus

$$V^{0+1}_{\text{Coulomb}} \sim \sum_j \left(\sum_i \frac{a_{ij}/2}{|X^{(i)} - X^{(j)}|} - \zeta_j \right)^2$$

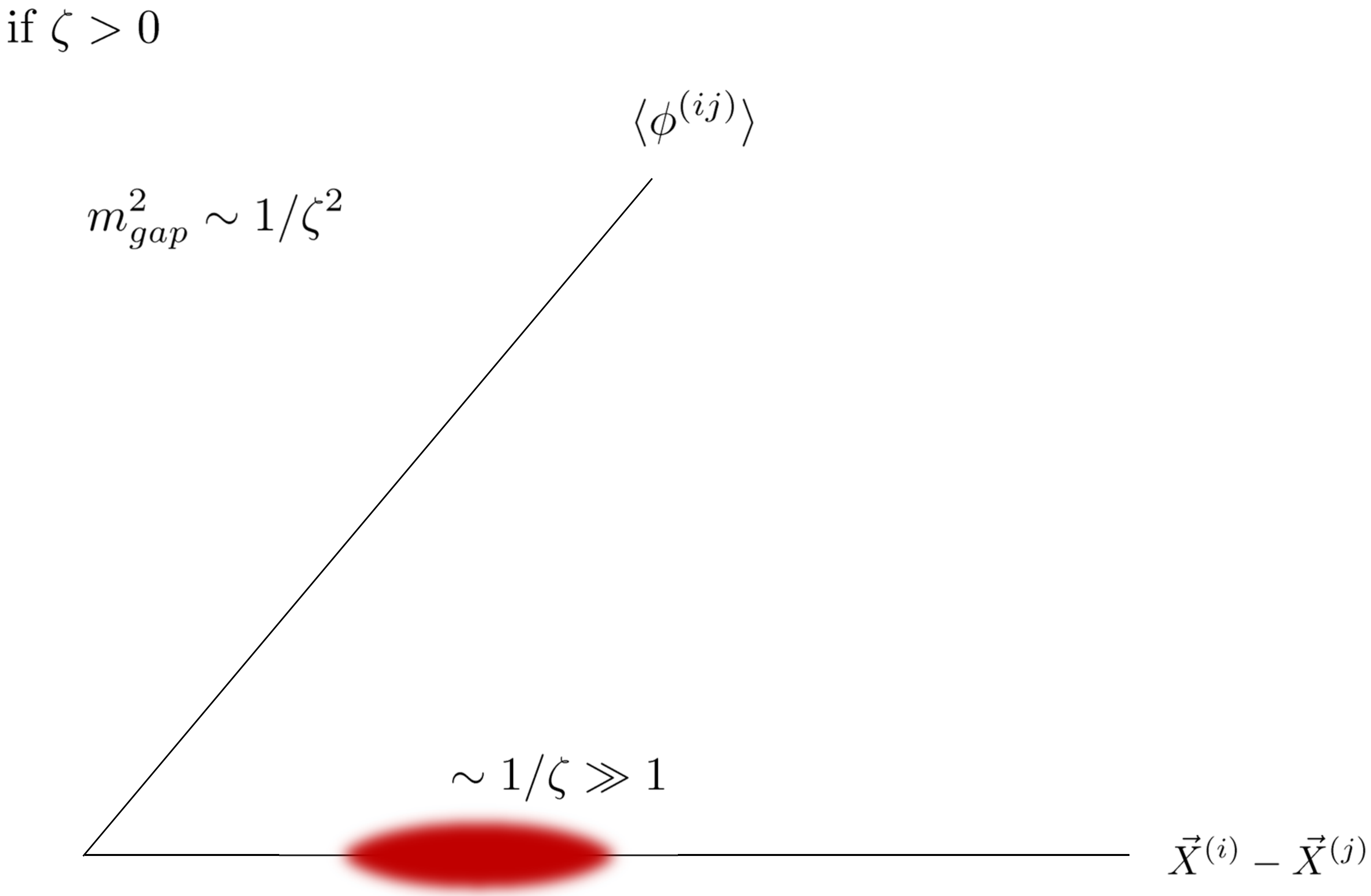


if $\zeta > 0$
$\langle \phi^{(ij)} \rangle$
$m_{gap}^2 \sim \zeta$
$\sim \zeta^{1/2} \ggg 1$
$\vec{X}^{(i)} - \vec{X}^{(j)}$

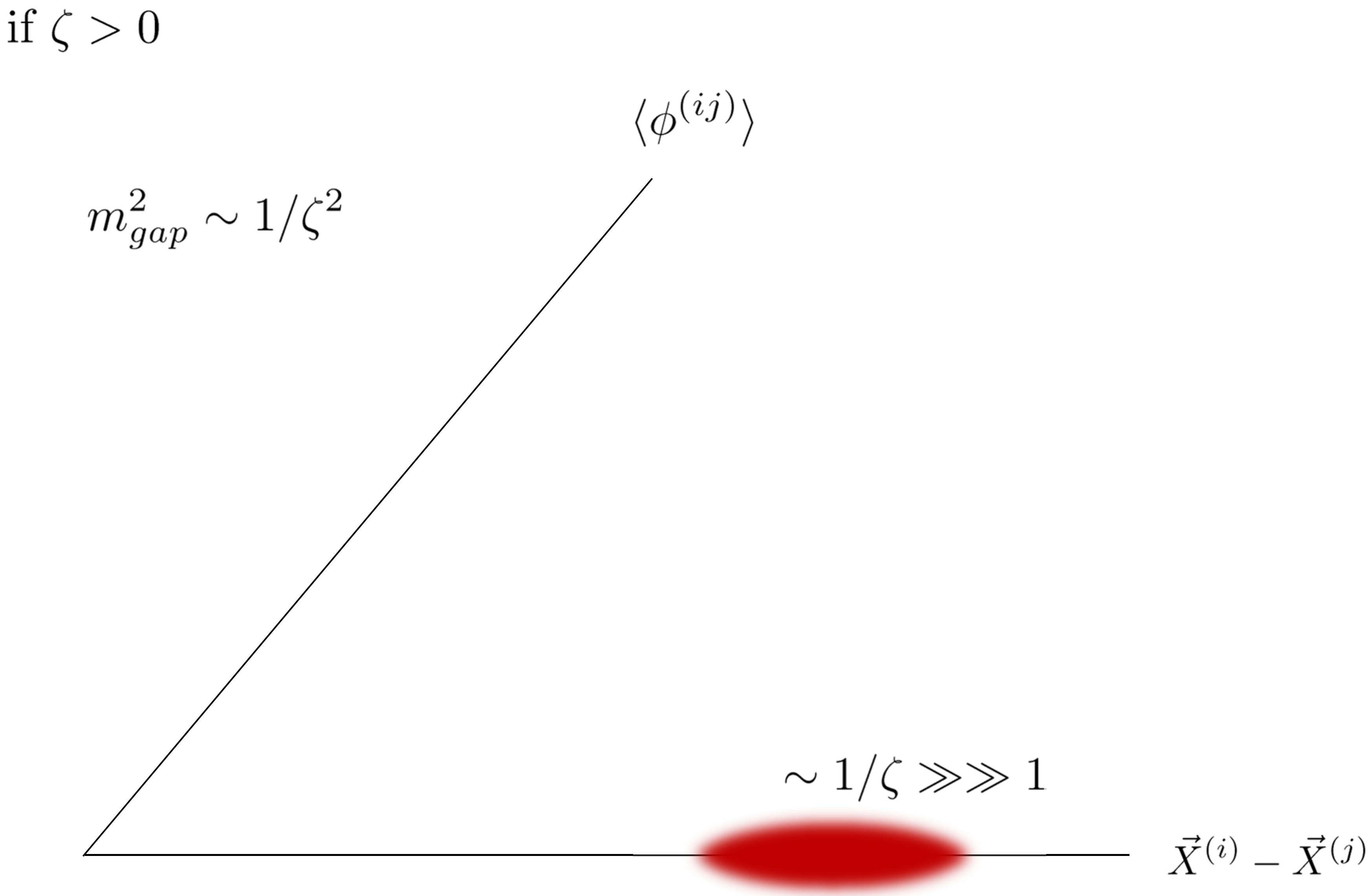

if $\zeta > 0$
$\langle \phi^{(ij)} \rangle$
$m^2_{gap} \sim \zeta$
$\sim \zeta^{1/2} \gg 1$
$\vec{X}^{(i)} - \vec{X}^{(j)}$



if $\zeta > 0$
$\langle \phi^{(ij)} \rangle$
$\zeta \sim 1$
$\vec{X}^{(i)} - \vec{X}^{(j)}$



if $\zeta > 0$
$\langle \phi^{(ij)} \rangle$
$m_{gap}^2 \sim 1/\zeta^2$
$\sim 1/\zeta \gg 1$
$\vec{X}^{(i)} - \vec{X}^{(j)}$

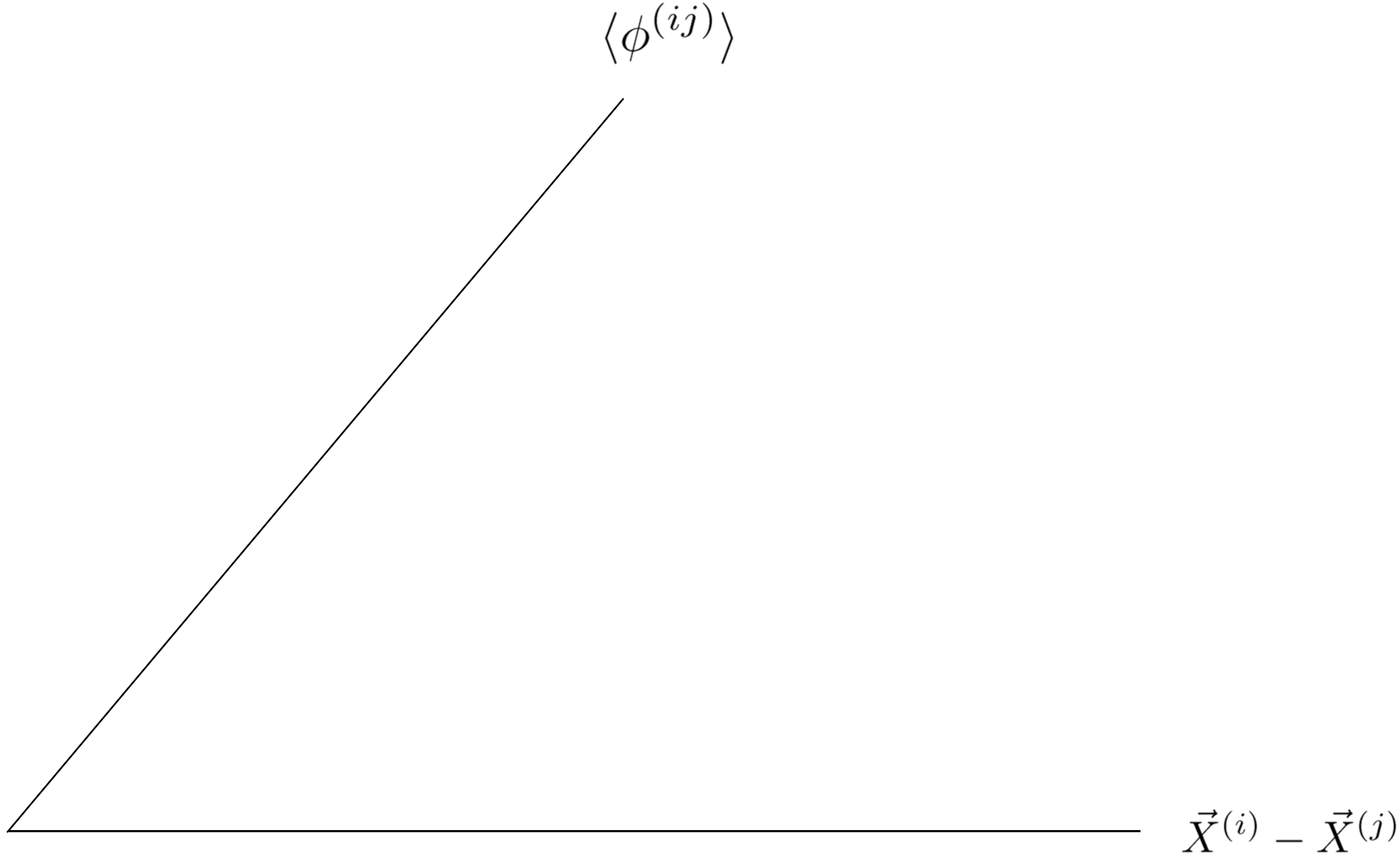

if $\zeta > 0$
$\langle \phi^{(ij)} \rangle$
$m^2_{gap} \sim 1/\zeta^2$
$\sim 1/\zeta \ggg 1$
$\vec{X}^{(i)} - \vec{X}^{(j)}$

in quantum mechanics, the word "phase" is very misleading
since vacuum expectation values do not imply
superselection sectors

what one really means by this word is a truncation process
depending on where the ground state wavefunctions are localized;
at large values of chiral multiplets
or at large values of vector multiplets

as long as wavefunctions do not move away to infinite,
and as long as the truncation process is reliable,
the supersymmetric index seems to be preserved

if $\zeta < 0$



large FI constants

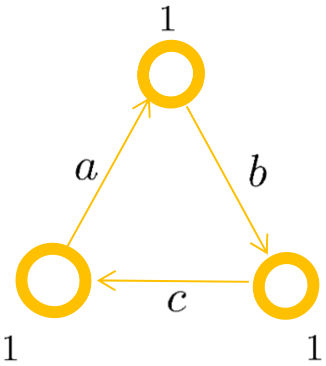
small FI constants

$$\Omega_{\text{Higgs}} = \Omega_{\text{Coulomb}}$$

F. Denef 2002 + A. Sen 2011

however, a simple 3-body problem says otherwise

Denef + Moore 2007



$$\Omega_{\text{Higgs}} = \underbrace{\boxed{\begin{pmatrix} a\cdot(c-b)\\ b\cdot(a-c)\\ c\cdot(b-a)\end{pmatrix}}}_{=\,\Omega_{\text{Coulomb}}} + \boxed{\# \cdot 2^{(a+b+c)/2} + \cdots}$$

large FI constants

small FI constants

$$\Omega_{\rm Higgs} \neq \Omega_{\rm Coulomb}$$

why not ?

practically, however, what one also means by “phase” is
certain truncation processes where we integrate out
either the chiral multiplets or the vector multiplets

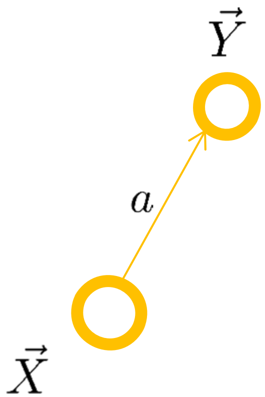
however, this process can sometimes fail spectacularly,
if the “heavy” multiplet in question become light
somewhere in the classical vacuum moduli space

precisely this happens in the Coulomb “phase” scaling regime

Coulomb "phase"

$$\sum_i \frac{a_{ij}}{|X^{(i)} - X^{(j)}|} = \zeta_j$$

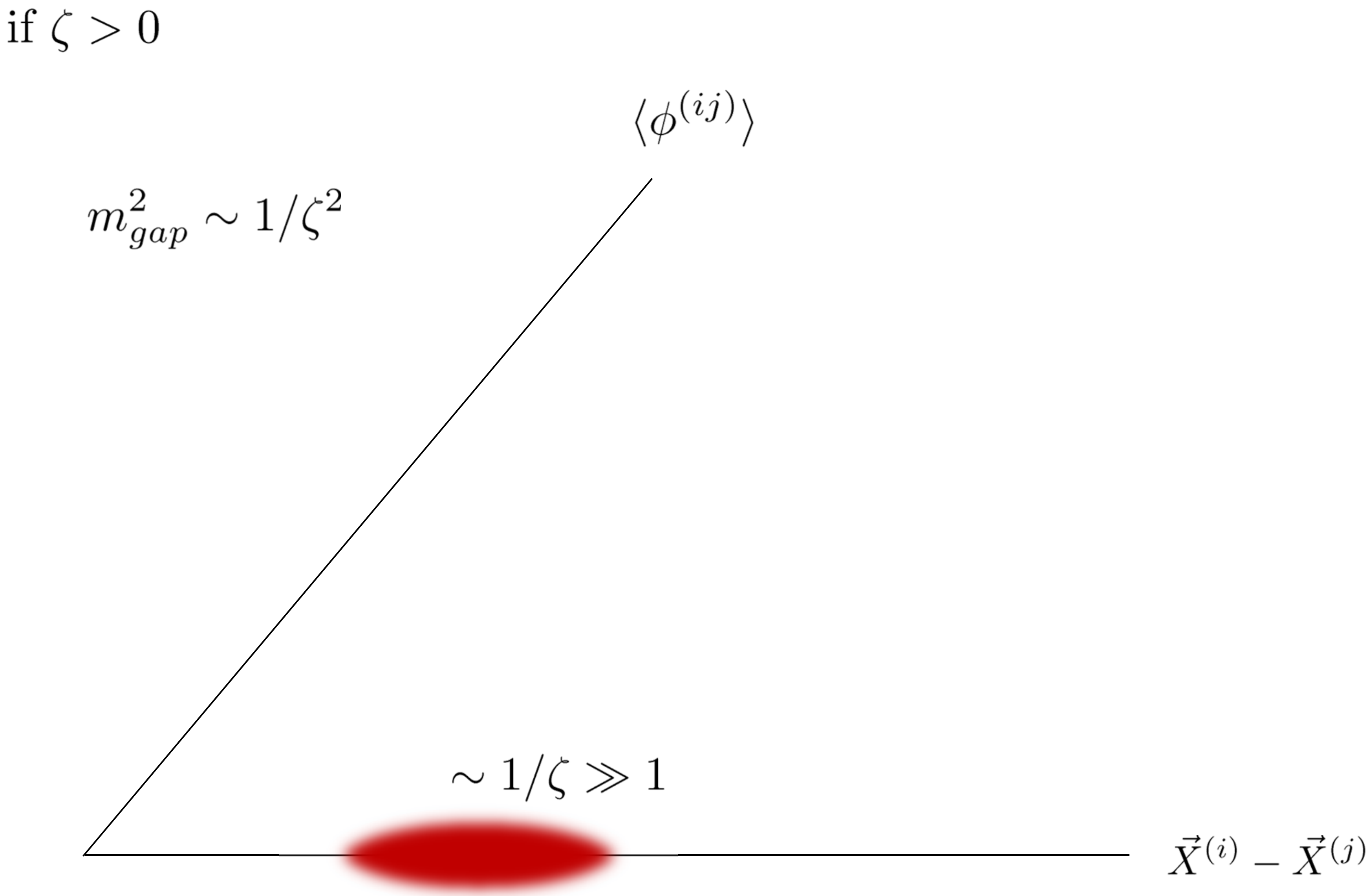
Coulomb "phase"



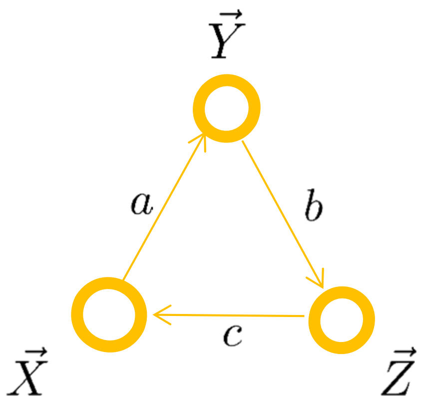
$$\frac{a}{|\vec{X}-\vec{Y}|} = \zeta$$

$$|\vec{X}-\vec{Y}| \sim a/\zeta$$

$$m_\phi^2 \sim |\vec{X}-\vec{Y}|^2 \sim (a/\zeta)^2$$


if $\zeta > 0$
$\langle \phi^{(ij)} \rangle$
$m^2_{gap} \sim 1/\zeta^2$
$\sim 1/\zeta \gg 1$
$\vec{X}^{(i)} - \vec{X}^{(j)}$

Coulomb "phase" scaling regime

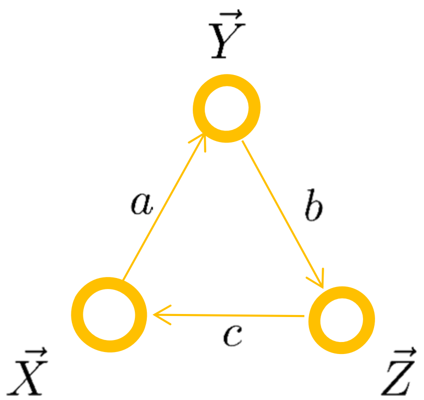


$$\frac{c}{|\vec{Z}-\vec{X}|} - \frac{a}{|\vec{X}-\vec{Y}|} = \zeta_1$$

$$\frac{a}{|\vec{X}-\vec{Y}|} - \frac{b}{|\vec{Y}-\vec{Z}|} = \zeta_2$$

$$\frac{b}{|\vec{Y}-\vec{Z}|} - \frac{c}{|\vec{Z}-\vec{X}|} = \zeta_3 = -(\zeta_1 + \zeta_2)$$

Coulomb "phase" scaling regime

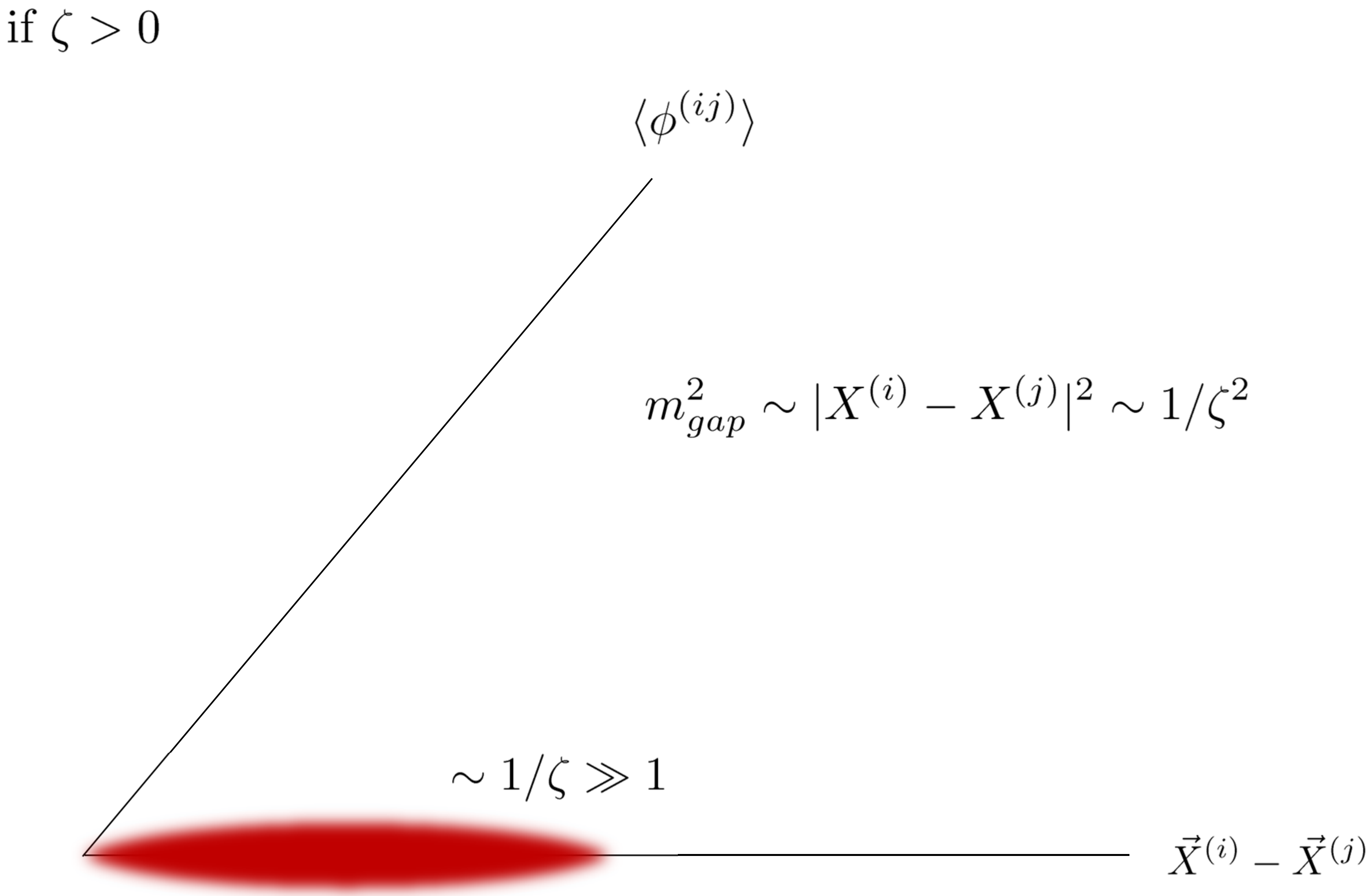


$$|\vec{X} - \vec{Y}| = \epsilon \times a + O(\epsilon^2/\zeta)$$

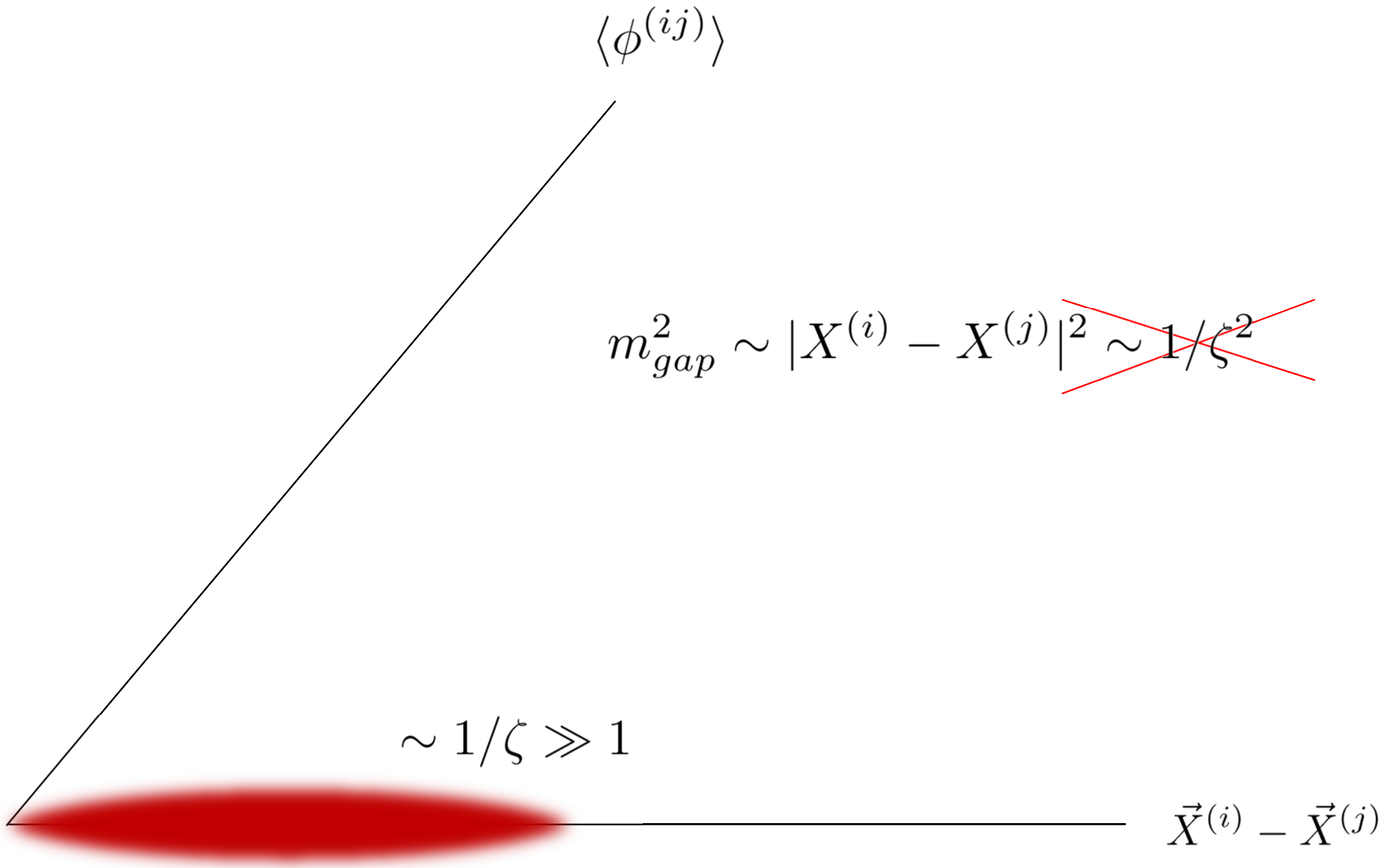
$$|\vec{Y} - \vec{Z}| = \epsilon \times b + O(\epsilon^2/\zeta)$$

$$|\vec{Y} - \vec{Z}| = \epsilon \times c + O(\epsilon^2/\zeta)$$

if (a, b, c) are lengths of edges of a single triangle


if $\zeta > 0$
$\langle \phi^{(ij)} \rangle$
$m^2_{gap} \sim |X^{(i)} - X^{(j)}|^2 \sim 1/\zeta^2$
$\sim 1/\zeta \gg 1$
$\vec{X}^{(i)} - \vec{X}^{(j)}$

if $\zeta > 0$

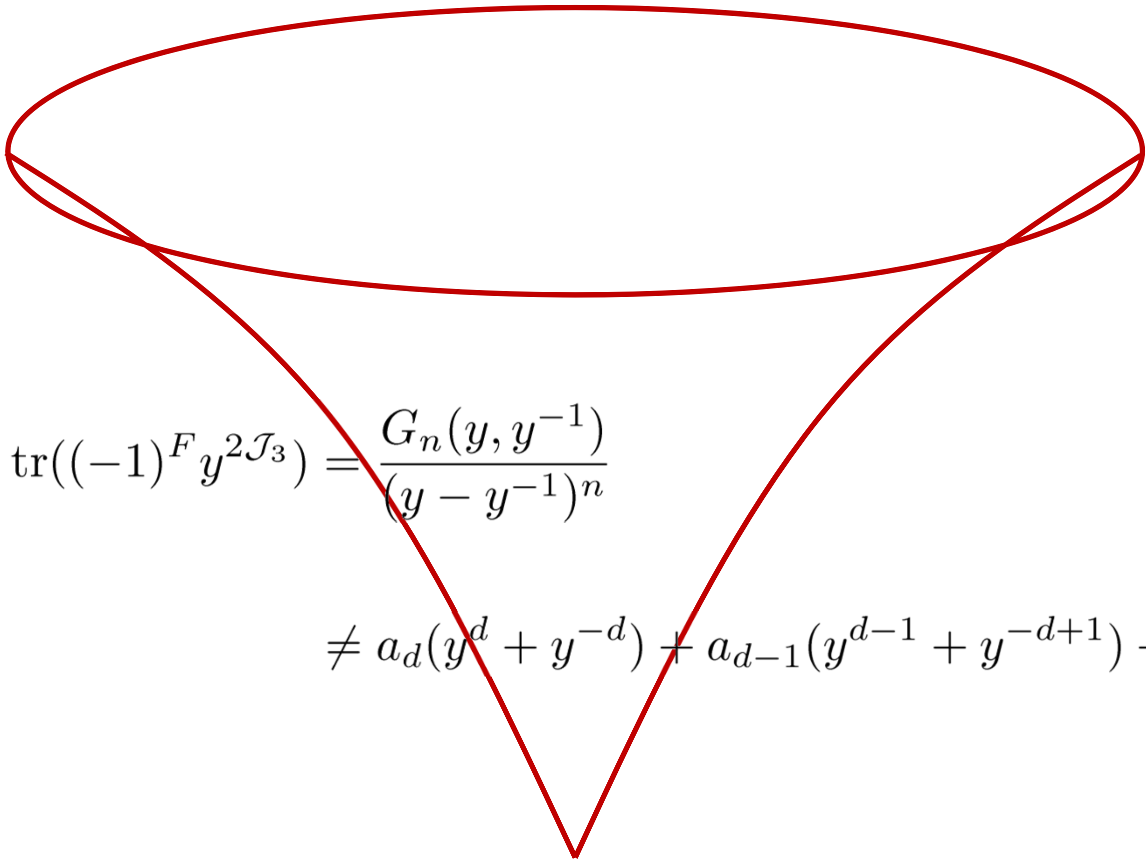

$\langle \phi^{(ij)} \rangle$
$m^2_{gap} \sim |X^{(i)} - X^{(j)}|^2 \cancel{\sim 1/\zeta^2}$
$\sim 1/\zeta \gg 1$
$\vec{X}^{(i)} - \vec{X}^{(j)}$

$$\Omega = -\frac{1}{2}\mathrm{tr}\left[(-1)^{2J_3}(2J_3)^2 y^{2(J_3+I)}\right]$$

$$\mathrm{tr}((-1)^F y^{2\mathcal{J}_3}) = \frac{G_n(y, y^{-1})}{(y - y^{-1})^n}$$

$$= a_d(y^d + y^{-d}) + a_{d-1}(y^{d-1} + y^{-d+1}) + \cdots + a_0$$

Coulomb "phase" scaling regime

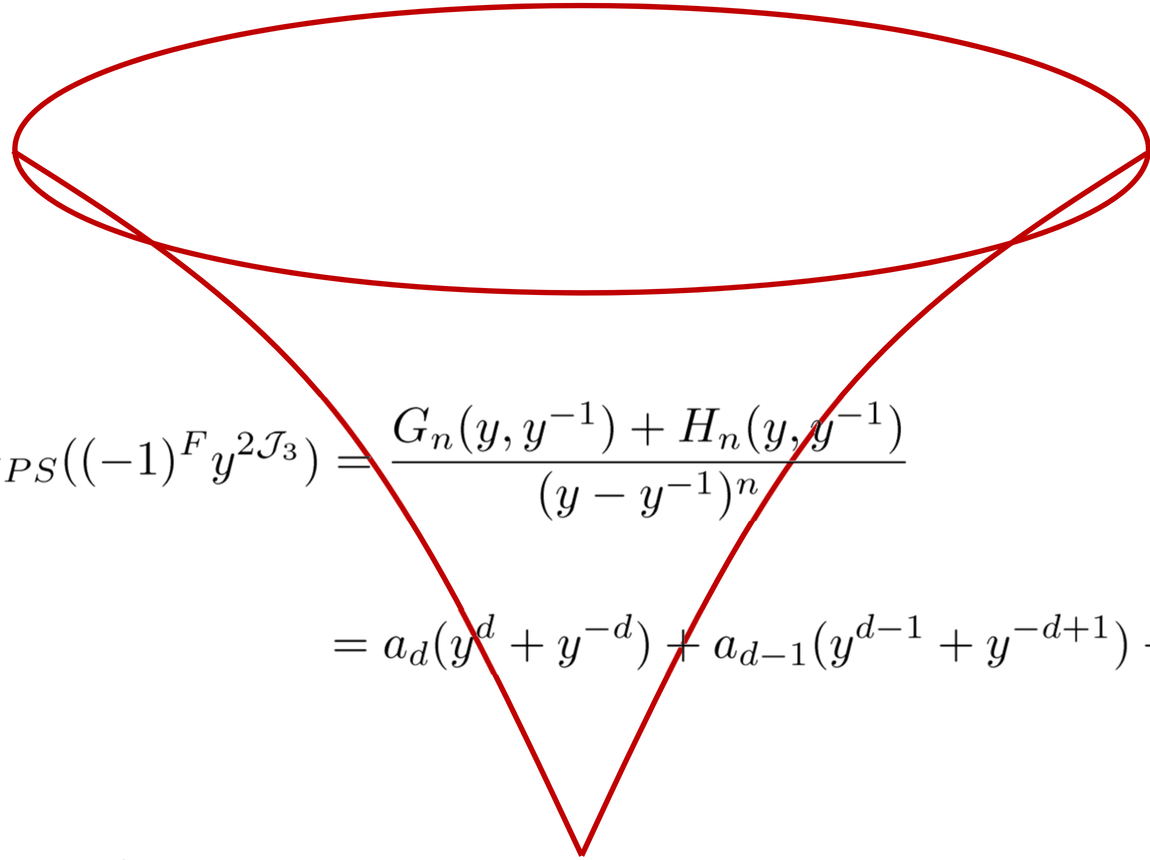


$$\mathrm{tr}((-1)^F y^{2\mathcal{J}_3}) = \frac{G_n(y, y^{-1})}{(y - y^{-1})^n}$$

$$\neq a_d(y^d + y^{-d}) + a_{d-1}(y^{d-1} + y^{-d+1}) + \cdots + a_0$$

Coulomb “phase” scaling regime

Manschot+Pioline+ Sen 2011



$$\mathrm{tr}_{MPS}((-1)^F y^{2\mathcal{J}_3}) = \frac{G_n(y,y^{-1}) + H_n(y,y^{-1})}{(y-y^{-1})^n}$$

$$= a_d(y^d + y^{-d}) + a_{d-1}(y^{d-1} + y^{-d+1}) + \cdots + a_0$$

$H_n(y,y^{-1})$ → an ad hoc, canceling Laurent polynomial of degree less than n & of the same parity as $G_n(y,y^{-1})$

Bena + Berkooz + de Boer +
El-Showk + d. Bleeken, 2012
Lee+Wang+P.Y. 2012

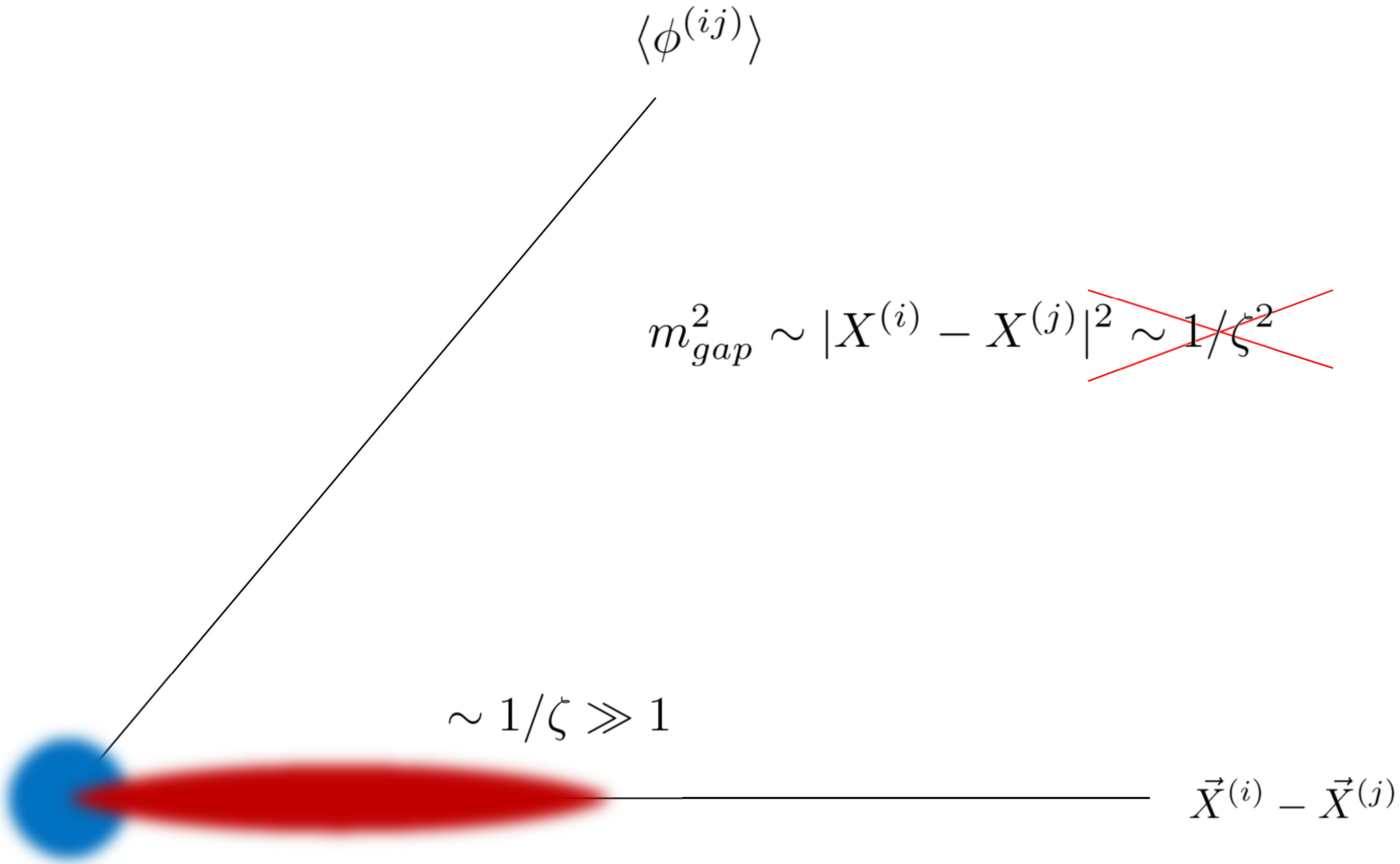
$$\Omega = -\frac{1}{2}\mathrm{tr}\left[(-1)^{2J_3}(2J_3)^2 y^{2(J_3+I)}\right]$$

$$\mathrm{tr}_{MPS}((-1)^F y^{2\mathcal{J}_3}) = \frac{G_n(y,y^{-1}) + H_n(y,y^{-1})}{(y-y^{-1})^n}$$

$$= a_d(y^d + y^{-d}) + a_{d-1}(y^{d-1} + y^{-d+1}) + \cdots + a_0$$

$H_n(y,y^{-1})$ → an ad hoc, canceling Laurent polynomial
of degree less than n & of the same parity as $G_n(y,y^{-1})$

if $\zeta > 0$



$\langle \phi^{(ij)} \rangle$
$m_{gap}^2 \sim |X^{(i)} - X^{(j)}|^2 \cancel{\sim 1/\zeta^2}$
$\sim 1/\zeta \gg 1$
$\vec{X}^{(i)} - \vec{X}^{(j)}$

large FI constants

small FI constants

$$\Omega_{\text{Higgs}} \neq \Omega_{\text{Coulomb}}$$

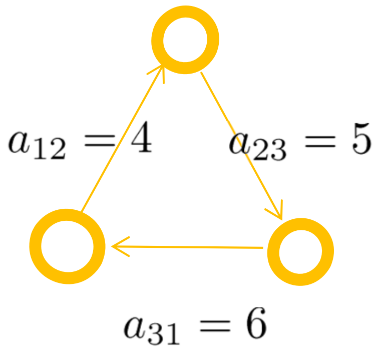
$$\Omega^{(k)}_{\text{Higgs}} = \boxed{\Omega_{\text{Invariant}}} + \boxed{\Omega^{(k)}_{\text{Coulomb}}}$$

1) loops in the quiver = superpotentials
2) geometric inequality for linking numbers
3) in all branches, Higgs “vacua” exist

S.J. Lee + Z.L. Wang + P.Y., 2012
Bena + Berkooz + de Boer + El-Showk + d. Bleeken, 2012

back to the simple 3-body example

$$\dim H^{(p,q)}(\mathcal{M}_H)$$



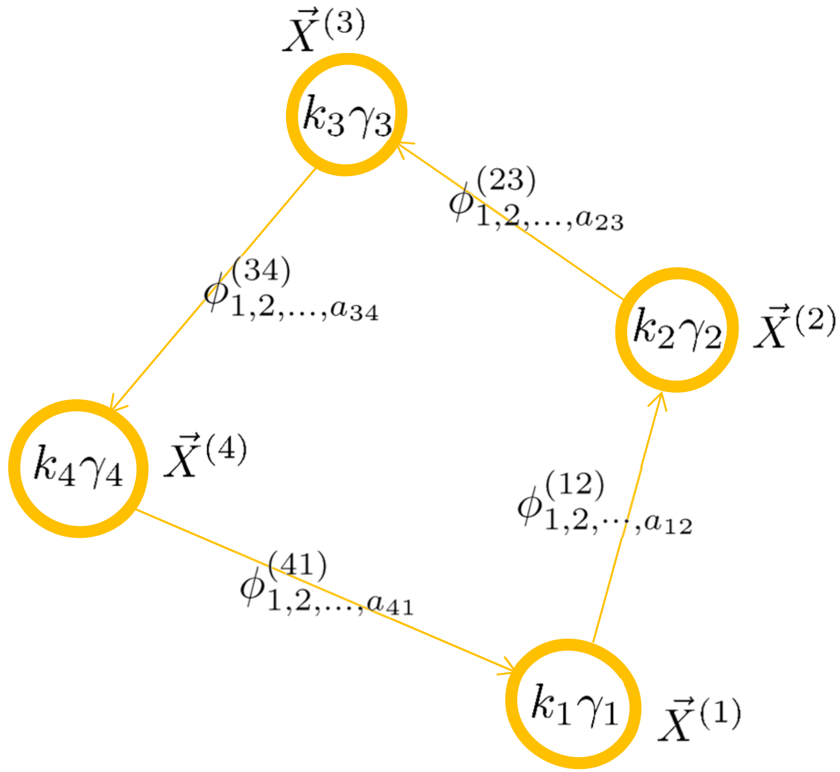
$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 2 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 3 & & 0 & & 0 & \\
0 & & 0 & & 26 & & 26 & & 0 & & 0 \\
 & 0 & & 0 & & 3 & & 0 & & 0 & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & & 0 & & 2 & & 0 & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & & & 1 & & & & &
\end{array}$$

$$\begin{array}{ccc}
 & 1 & \\
26 & & 26 \\
 & 1 &
\end{array}$$

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 2 & & 0 & \\
0 & & 26 & & 26 & & 0 \\
 & 0 & & 2 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

what physical & mathematical properties characterize these intrinsically Higgs, wall-crossing-safe BPS states ?

$$\Omega^{(k)}_{\rm Higg} = \Omega_{\rm Invariant} + \Omega^{(k)}_{\rm Coulomb}$$

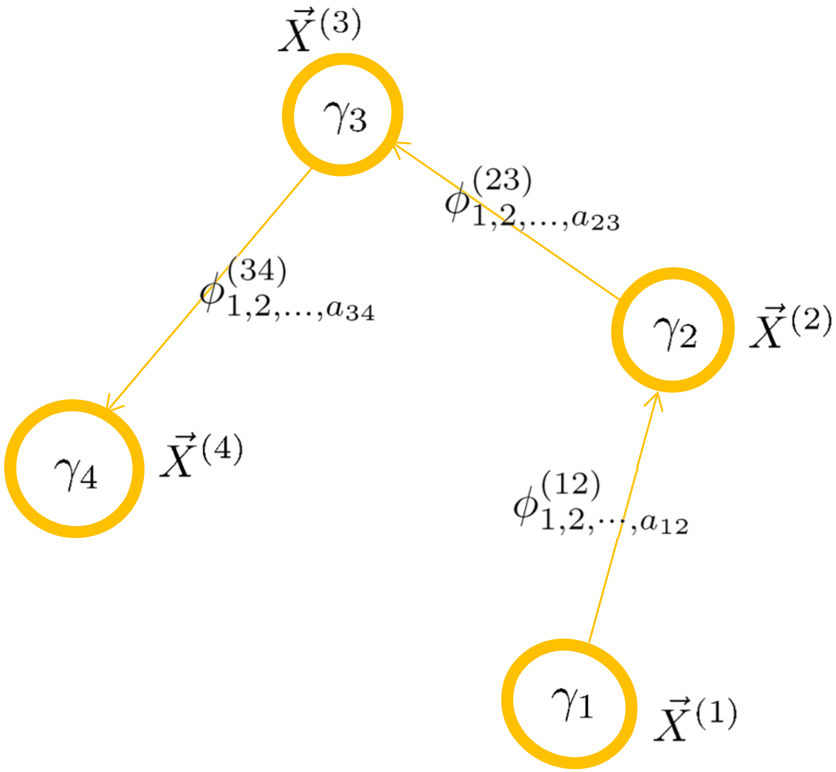


$$W(\phi) = \mathrm{tr}\left[\phi^{(12)}\phi^{(23)}\phi^{(34)}\phi^{(41)}\right]$$

quiver invariant

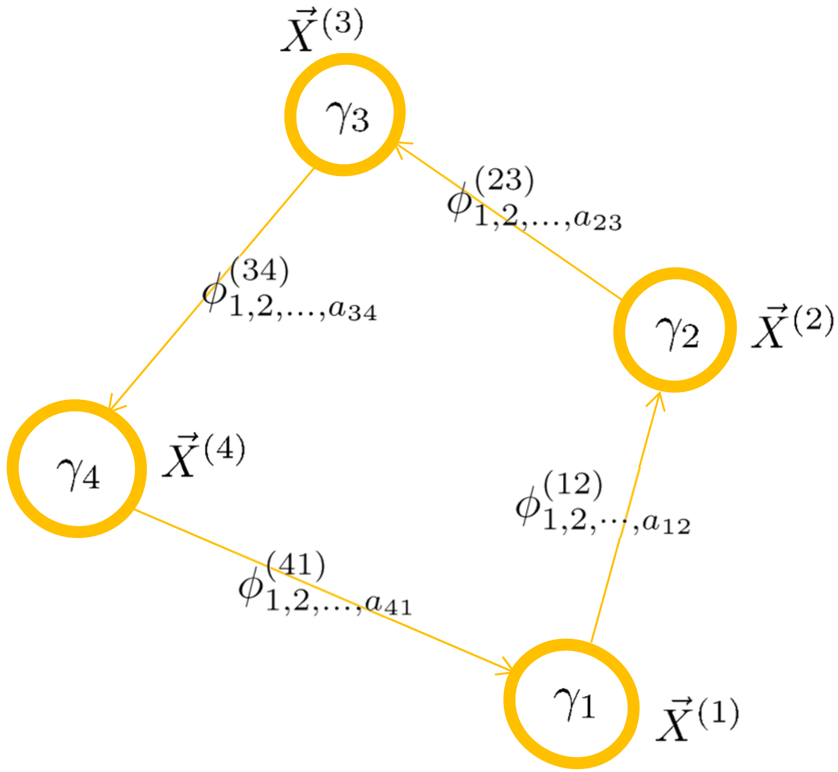
$$\Omega_{\text{Higgs}} = \Omega_{\text{Coulomb}}$$

$$\{\phi^{(12)}, \ldots\} \xrightarrow{D \;\sim\; // \prod_i U(1)_i} \mathcal{M}_{\text{H}}$$



$$\Omega_{\mathrm{Higgs}} \neq \Omega_{\mathrm{Coulomb}}$$

$$\begin{tikzcd}
\{\phi^{(12)},\ldots\} \arrow[d, "D \;\sim\; /\!/\prod_i U(1)_i"] \\
X_{\mathrm{H}} \\
\mathcal{M}_{\mathrm{H}} \arrow[u, "i \;\sim\; \partial_\phi W = 0"']
\end{tikzcd}$$



$$W(\phi) \sim \phi^{(12)}\phi^{(23)}\phi^{(34)}\phi^{(41)}$$

wall-crossing vs. wall-crossing-safe

$$\begin{tikzcd} \{\phi^{(12)},\ldots\} \arrow[d, "D"] \\ X_{\mathrm{H}} \\ \mathcal{M}_{\mathrm{H}} \arrow[u, "i \quad \sim \ \partial_\phi W = 0"'] \end{tikzcd}$$

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \quad \oplus \quad H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$

wall-crossing vs. wall-crossing-safe

$$\begin{tikzcd} \{\phi^{(12)},\ldots\} \arrow[d, "D"] \\ X_{\mathrm{H}} \\ \mathcal{M}_{\mathrm{H}} \arrow[u, "i"] \end{tikzcd}$$

$$H^*(\mathcal{M}_H) = \sum H^{(p,q)}(\mathcal{M}_H)$$

$$= \; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$

$$\mathrm{tr}_{i^*(H(X))}(-1)^{p+q-d}y^{2p-d} \qquad \mathrm{tr}_{\mathrm{Intrinsic}}(-1)^{p+q-d}y^{2p-d}$$

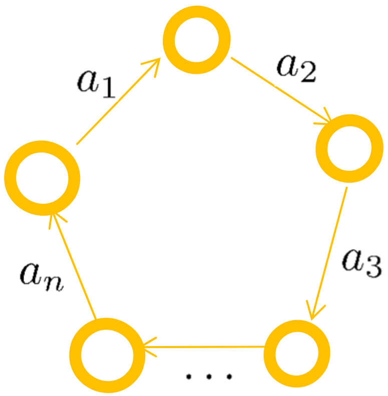
$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\Omega_{\mathrm{Coulomb}} \qquad \Omega_{\mathrm{Invariant}}$$

S.J. Lee + Z.L. Wang + P.Y., 2012

Bena + Berkooz + de Boer + El-Showk + d. Bleeken, 2012

general proof & explicit counting !



$$H^*(\mathcal{M}_H) \;=\; \sum H^{(p,q)}(\mathcal{M}_H)$$

$$= \; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\mathrm{tr}_{i^*(H(X))}(-1)^{p+q-d}y^{2p-d} \qquad \mathrm{tr}_{\text{Intrinsic}}(-1)^{p+q-d}y^{2p-d}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

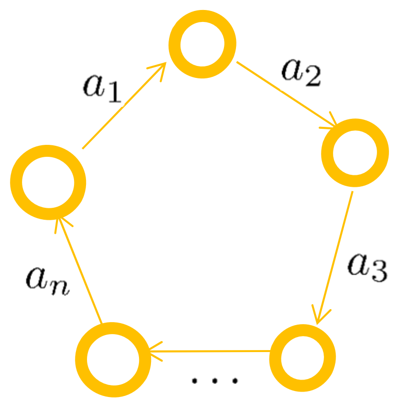
$$\Omega_{\text{Coulomb}} \qquad \Omega_{\text{Invariant}}$$

S.L. Lee + Z.L. Wang + P.Y., 2012
Manschot + Pioline + Sen, 2012

the total equivariant index ~ Hirzebruch character

$$\Omega^{(k)}_{\mathrm{Higgs}}(y) = \mathrm{tr}_{H^*(\mathcal{M}^{(k)}_H)}(-1)^{2J_3} y^{2J_3+2I} = \sum (-1)^{p+q-d} y^{2p-d} h^{(p,q)}(\mathcal{M}^{(k)}_H)$$

$$= (-y)^{-d_k} \chi_{t=-y^2}(\mathcal{M}^{(k)}_H)$$

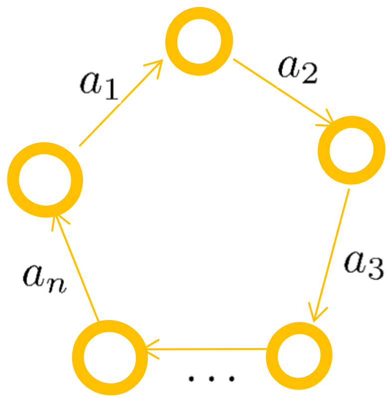


which is easily computable here, via Riemann-Roch theorem

$$\chi_t(\mathcal{M}_H^{(k)}) = \frac{1}{(1+t)^n} \int_{X_H^{(k)}} \left[\prod_{i \neq k} \left(J_i \frac{1+te^{-J_i}}{1-e^{-J_i}} \right)^{a_i} \right] \cdot \left(\frac{1-e^{-\sum_{i\neq k} J_i}}{1+te^{-\sum_{i\neq k} J_i}} \right)^{a_k}$$

$$X_{\mathrm{H}}^{(k)} = \prod_{i \neq k} CP^{a_i - 1}$$

$$\begin{array}{c} X_{\mathrm{H}}^{(k)} \\ \Big\uparrow i \\ \mathcal{M}_{\mathrm{H}}^{(k)} \end{array}$$

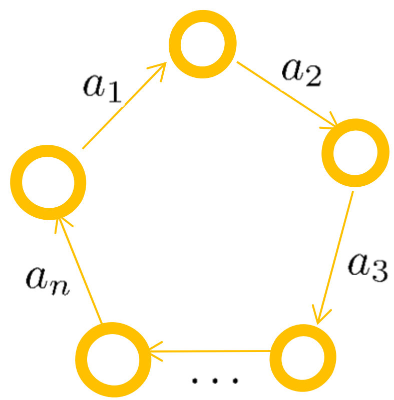


and decomposed into two parts

$$\Omega^{(k)}_{\text{Higgs}}(y) = \boxed{(-1)^{d_k} y^{a_k} \prod_{i\neq k} \frac{y^{a_i} - y^{-a_i}}{y - y^{-1}} + \Delta\Omega_{\{a_i\}}(y)}$$

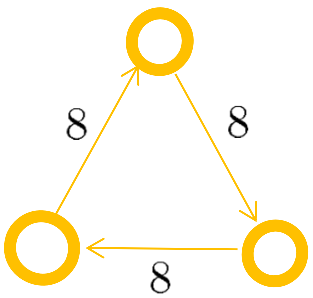
$$+ \boxed{\frac{(-y)^{n+2-\sum_i a_i}}{(y^2-1)^n} \prod_i \oint_{\omega_i=1} \frac{d\omega_i}{2\pi i} \left[\prod_i \left(\frac{1-y^2\omega_i}{1-\omega_i} \right)^{a_i} \right] \cdot \frac{1}{1-y^2\prod_i \omega_i} - \Delta\Omega_{\{a_i\}}(y)}$$

$$\begin{array}{c} X_{\mathrm{H}}^{(k)} = \prod_{i\neq k} CP^{a_i-1} \\ \Big\uparrow i \\ \mathcal{M}_{\mathrm{H}}^{(k)} \end{array}$$



wall-crossing states vs. wall-crossing-safe states

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$



$\dim H^{(p,q)}(\mathcal{M}_H)$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 2 & & 0 & & & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 3 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
1 & & 322 & & 6803 & & 4+18212 & & 6803 & & 322 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & 3 & & 0 & & 0 & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & & & 0 & & 2 & & 0 & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & & & 1 & & & & & &
\end{array}$$

wall-crossing states vs. wall-crossing-safe states

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$

$\dim H^{(p,q)}(\mathcal{M}_H)$

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & 0 & & 2 & & 0 & & & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 3 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
1 & & 322 & & 6803 & & 4+18212 & & 6803 & & 322 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & 3 & & 0 & & 0 & & \\
 & & & 0 & & 0 & & 0 & & 0 & & & \\
 & & & & 0 & & 2 & & 0 & & & & \\
 & & & & & 0 & & 0 & & & & & \\
 & & & & & & 1 & & & & & &
\end{array}$$

J_3

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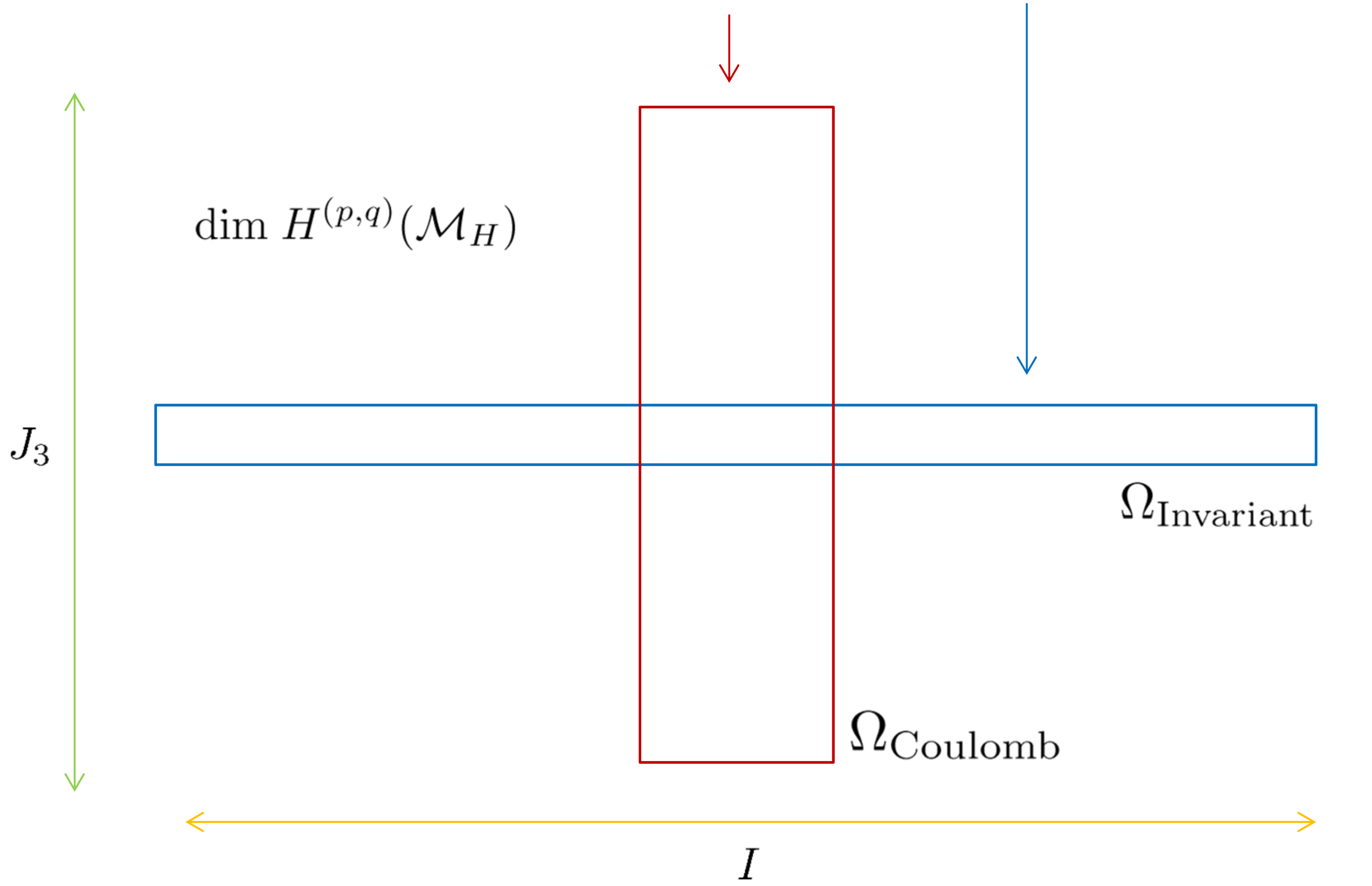
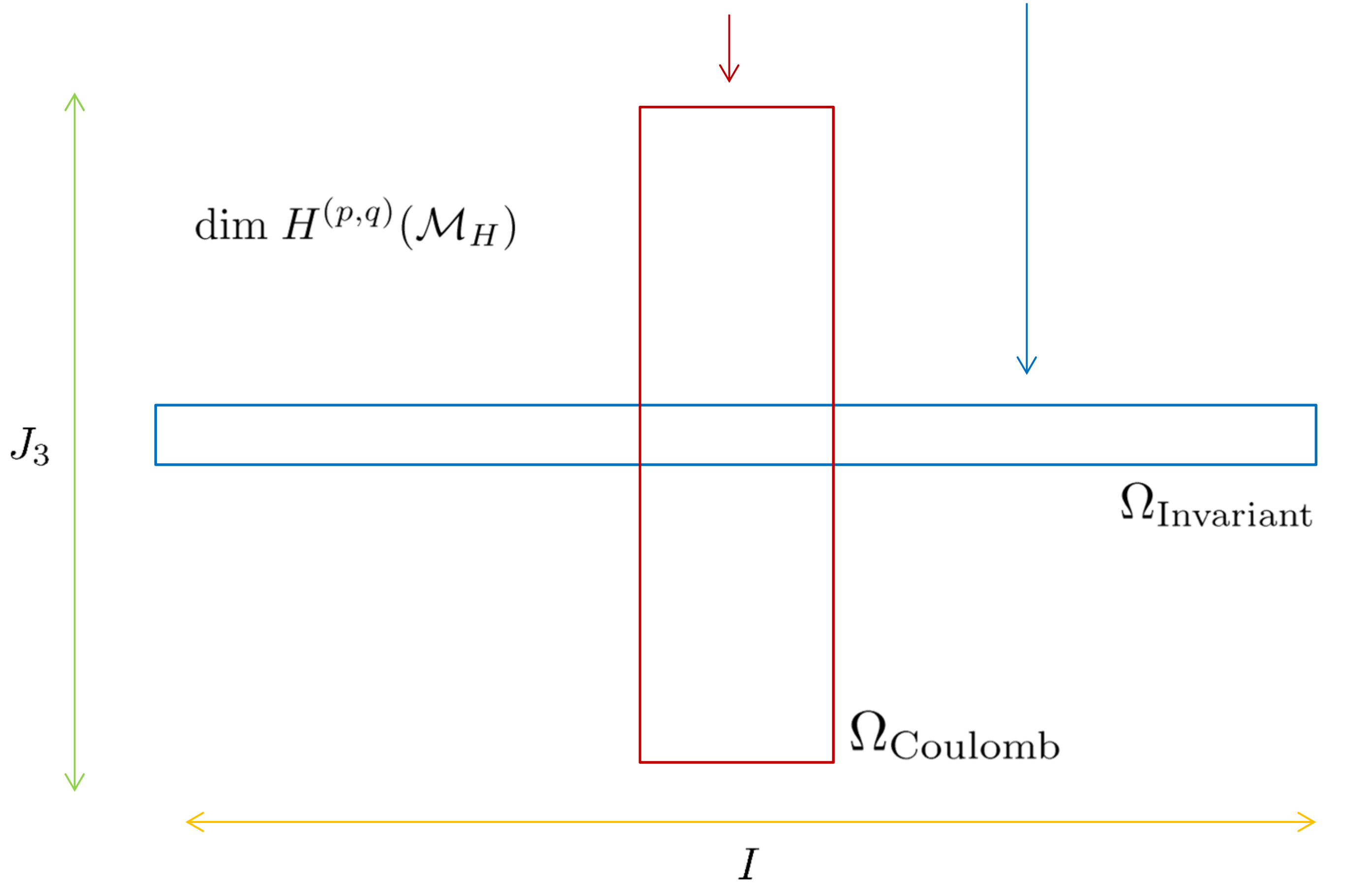
wall-crossing states vs. wall-crossing-safe states

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$



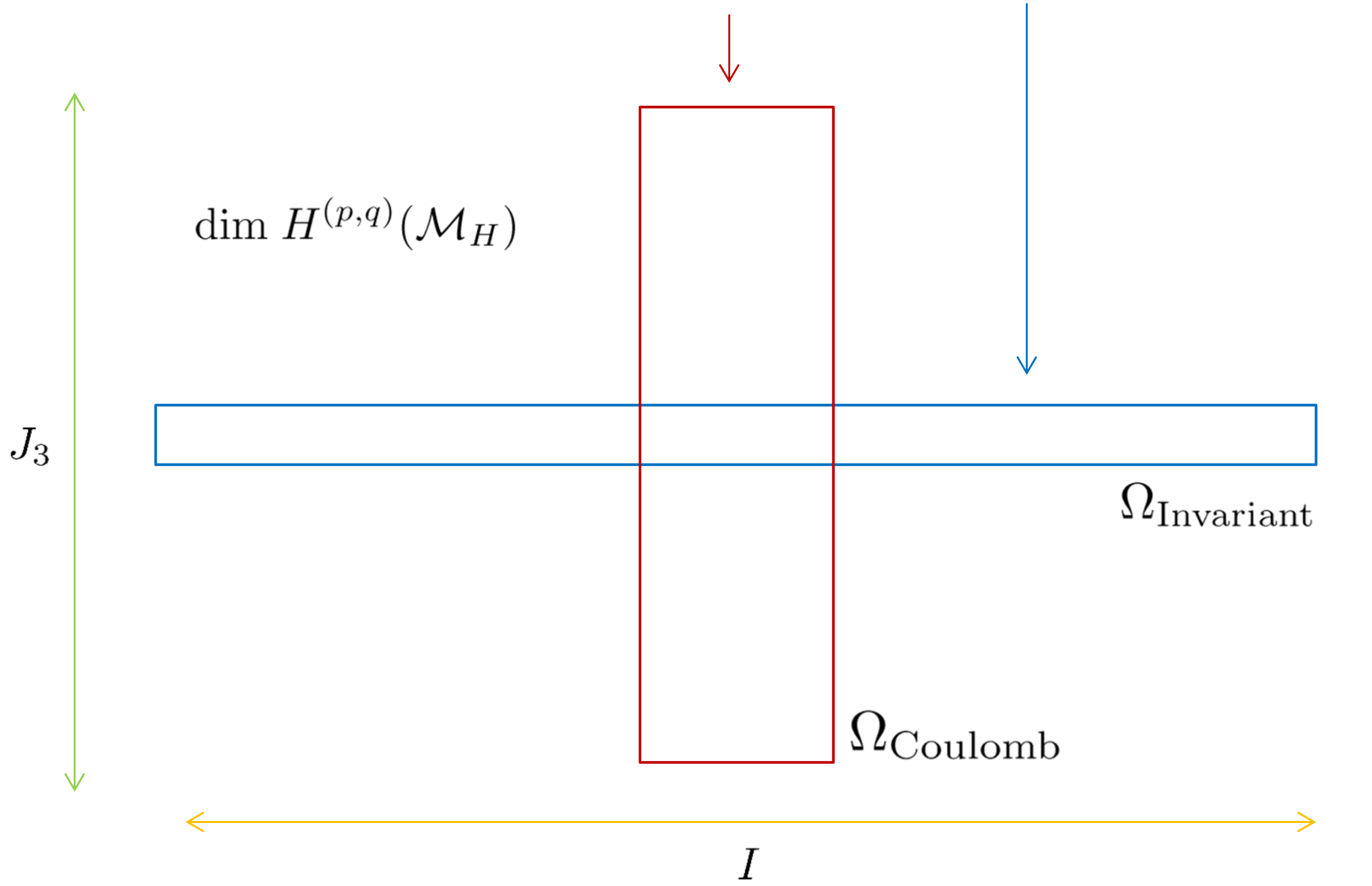
this simple dichotomy, due to the Lefschetz hyperplane theorem,
is literally true only for cyclic Abelian quivers:
for general quivers, the cohomology is far more intricate

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$



intrinsic Higgs states are likely to remain angular momentum singlets

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$



wall-crossing states vs. wall-crossing-safe states

$$H^*(\mathcal{M}_H) = i^*\left[H^*(X_H)\right] \oplus H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Updownarrow \qquad \mathrm{tr}(-1)^{p+q-d} y^{2p-d} \qquad \Updownarrow$$

$$\Omega_{\text{Coulomb}} \qquad \qquad \Omega_{\text{Invariant}} = \Omega_{\text{Higgs}} - \Omega_{\text{Coulomb}}$$

many-body bound states wall-crossing	single-center states wall-crossing-safe
angular momentum multiplets	angular momentum singlets

wall-crossing states vs. wall-crossing-safe states

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$

$$\Updownarrow \qquad \mathrm{tr}(-1)^{p+q-d}y^{2p-d} \qquad \Updownarrow$$

$$\Omega_{\mathrm{Coulomb}} \qquad\qquad \Omega_{\mathrm{Invariant}} = \Omega_{\mathrm{Higgs}} - \Omega_{\mathrm{Coulomb}}$$

many-body bound states
wall-crossing

polynomial degeneracy:
most of familiar BPS states in
field theories belong here

single-center states
wall-crossing-safe

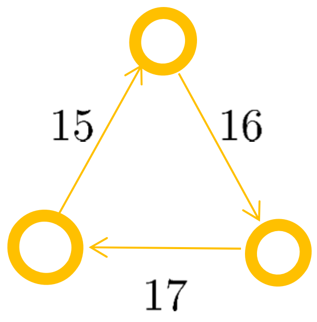
exponential degeneracy:
single-center BH's
belong here

more examples of quiver invariants

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\text{Intrinsic}}$$

$$\Omega(y)\Big|^{\{a_{i,i+1}\}=(15,16,17)}_{\text{Intrinsic}} = \mathrm{tr}_{\text{Intrinsic}}(-1)^{2J_3}y^{2J_3+2I} = \sum(-1)^{p+q-d}y^{2p-d}h^{(p.q)}_{\text{Intrinsic}}$$

$$\begin{aligned}
&= 1665y^{-12}\\
&+ 724674y^{-10}\\
&+ 60686563y^{-8}\\
&+ 1523273844y^{-6}\\
&+ 13886938949y^{-4}\\
&+ 50685934038y^{-2}\\
&+ 77668453887\\
&+ 50685934038y^{2}\\
&+ 13886938949y^{4}\\
&+ 1523273844y^{6}\\
&+ 60686563y^{8}\\
&+ 724674y^{10}\\
&+ 1665y^{12}
\end{aligned}$$

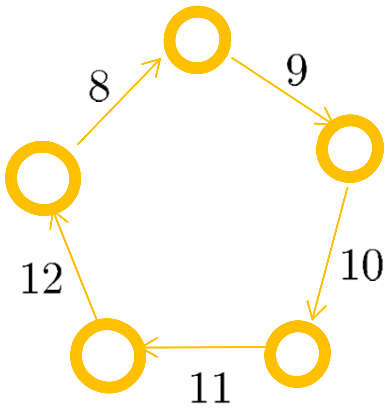


more examples of quiver invariants

$$H^*(\mathcal{M}_H) \;=\; i^*\left[H^*(X_H)\right] \;\oplus\; H^*(\mathcal{M}_H)_{\mathrm{Intrinsic}}$$

$$\Omega(y)\Big|_{\mathrm{Intrinsic}}^{\{a_{i,i+1}\}=(8,9,10,11,12)} = \mathrm{tr}_{\mathrm{Intrinsic}}(-1)^{2J_3}y^{2J_3+2I} = \sum(-1)^{p+q-d}y^{2p-d}h^{(p,q)}_{\mathrm{Intrinsic}}$$

$$\begin{aligned} = \; & 32294250/y^{22} + 58872952926/y^{20} + 23086762587054/y^{18} \\ & +3146301650299568/y^{16} + 186529800766285403/y^{14} \\ & +5480846262397291070/y^{12} + 86780383421802203555/y^{10} \\ & +783408269154731872224/y^{8} + 4192271239441338802849/y^{6} \\ & +13657486692285216220742/y^{4} + 27560691162972524163666/y^{2} \\ & +34791235315880411958041 + 27560691162972524163666y^{2} \\ & +13657486692285216220742y^{4} + 4192271239441338802849y^{6} \\ & +783408269154731872224y^{8} + 86780383421802203555y^{10} \\ & +5480846262397291070y^{12} + 186529800766285403y^{14} \\ & +3146301650299568y^{16} + 23086762587054y^{18} \\ & +58872952926y^{20} + 32294250y^{22} \end{aligned}$$



$$\Omega_{\mathrm{Invariant}} \sim \exp(\sum a_{ij})$$

→ black hole entropy ?

outstanding issues

origin & validity of the MPS Coulomb prescription for scaling cases ?

is the Coulomb-like Abelianization routine true
even for Higgs "phase" with quiver invariants ?

(in-)dependence of index on superpotential choices ?

detailed string theory embeddings and
microscopic counting of BH entropy ?

summary

d=1 N=4 quiver quantum mechanics offers
a universal framework for wall-crossing / counting of 4d BPS states
with the intuitive Coulomb "phase" for wall-crossing &
the comprehensive Higgs "phase" for faithful state counting

quiver invariants must/can be computed separately as input data
for wall-crossing, and appear everywhere from the BPS quiver
of N=2* theories to single-center BPS black holes

complete derivation of the index for non-Abelian quivers,
in the presence of quiver invariants, is not yet available but existing
Abelianization proposals suggest the quiver invariant
as a measure of single-center black hole microstates