The enhanced holographic superconductor: is it possible?

Makoto Natsuume (KEK)

arXiv: 1307.nnnn [hep-th]

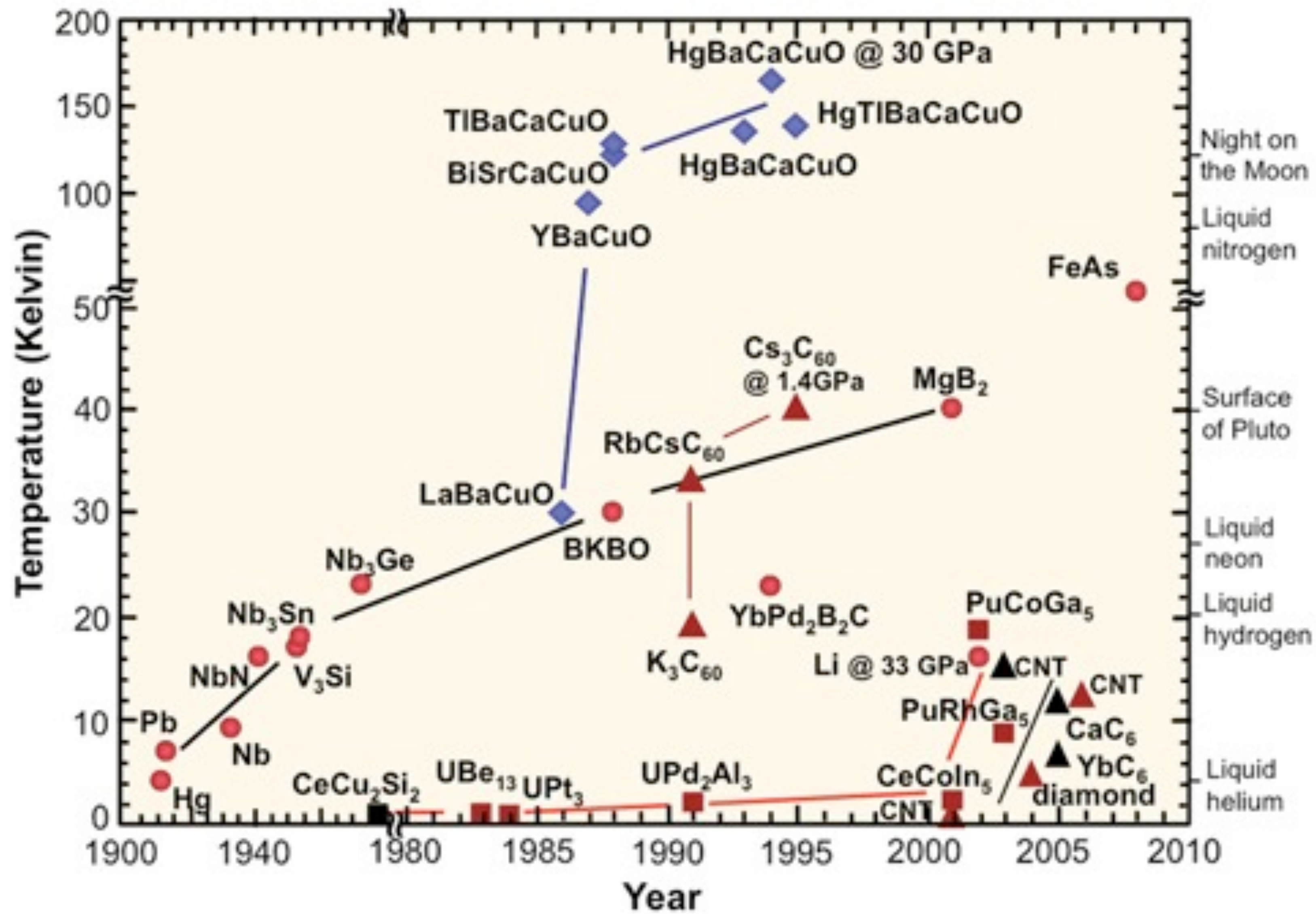
w/ Takashi Okamura (Kwansei Gakuin Univ.)

2013/ 7 KIAS-YITP

Is room-temperature superconductivity possible?

I won't answer it, but our work is motivated by this question.

History of highest Tc



Highest record: 133-164K (HgBaCaCuO)
although impressive progress e.g. on Fe-based materials in recent years
Maybe we need to try a completely different way...

http://www.ccas-web.org/superconductivity/

One interesting possibility: “enhanced superconductivity”

Time-dependent source → raise Tc

"Enhanced superconductivity"

Enhanced superconductivity: somewhat counterintuitive:

source → heat up system → ~~destroy~~ superconductivity

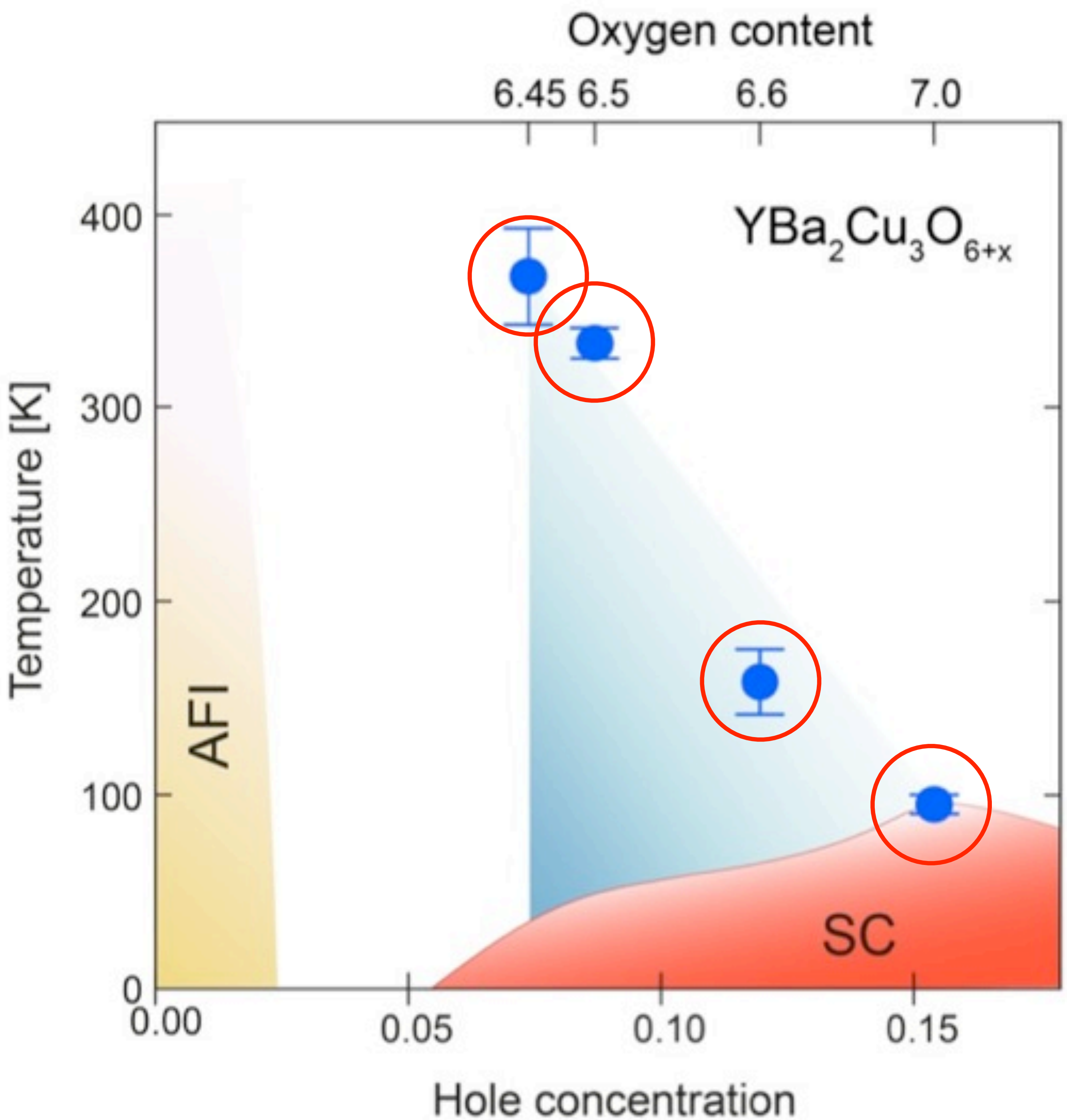
Theory: Eliashberg (1970)
Exp: '70s

Not a huge effect, but if happens to high-Tc material, room temp. Tc?

A recent attempt (2012/5):
YBCO: laser stimulation → superfluid component at room temp

S. Kaiser, D. Nicoletti, C. R. Hunt, W. Hu, I. Gierz, H. Y. Liu, M. Le Tacon, T. Loew, D. Haug, B. Keimer, A. Cavalleri, 1205.4661 [cond-mat.supr-con]

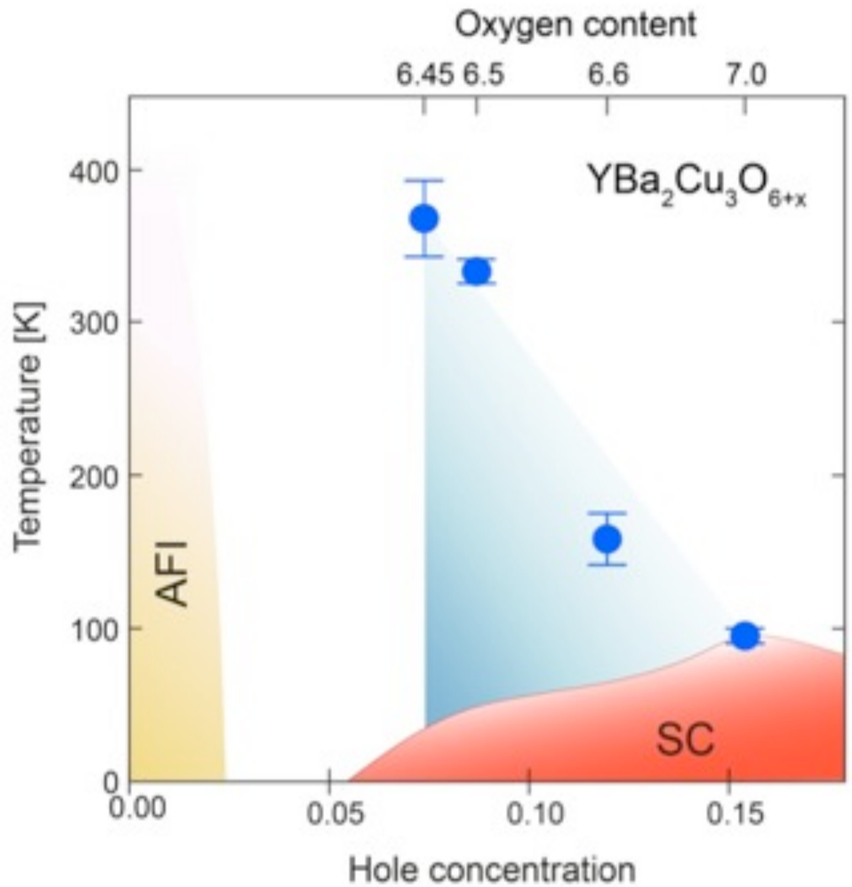
1205.4661v5 [cond-mat.supr-con]

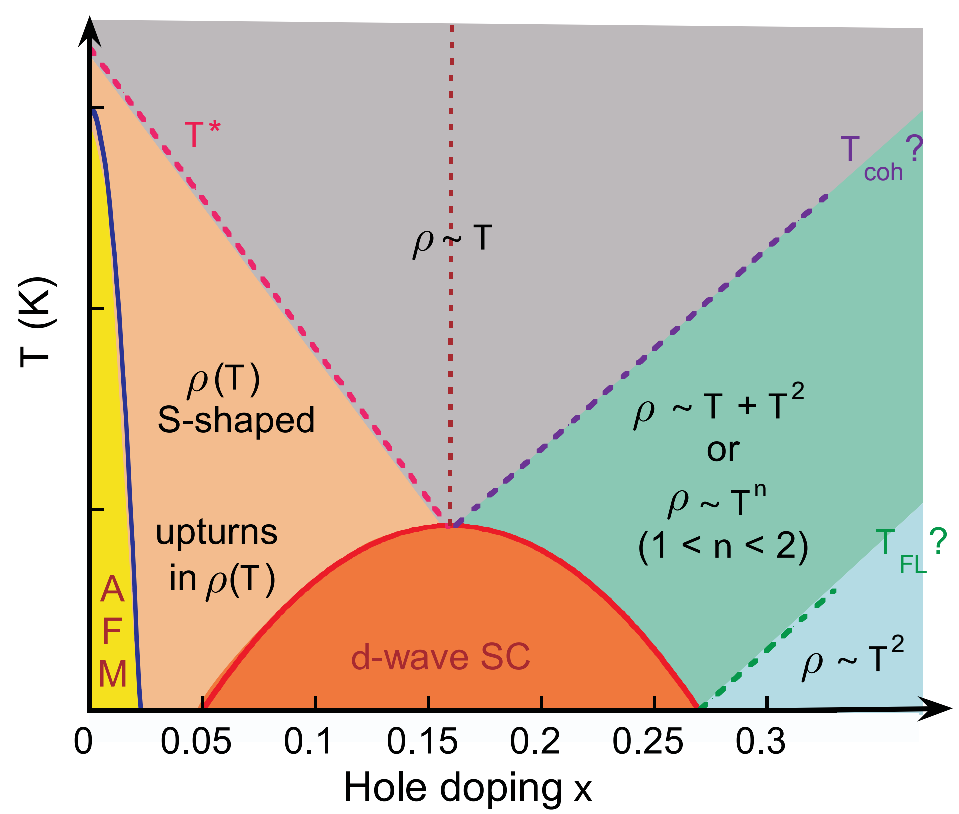


Traditional theory clearly inadequate to explain it because

- non-Fermi liquid phase
- Likely to be related to yet mysterious "pseudo-gap" physics

So, here is an interesting problem to explore





Hussey, J. Phys: Condens. Matter **20**, 123201 (2008).

Our goal: more modest

Just try to understand what holography would tell us about the enhanced superconductivity in general

Enhanced holographic superconductors: pioneered by Silverstein & collaborators

Bao - Dong - Silverstein - Torroba, 1104.4098

- They added "time-dependent chemical potential" $\mu(t)$ and saw an enhancement, but their setup itself is questionable.

- A correct way is to add time-dependent electric field or $A_i(t)$ but we see no enhancement unlike traditional superconductors.

Not just a matter of different setups, their enhancement is likely to come from improper setup & analysis.

cf. Li - Tian - Zhang, 1305.1600

Plan

- Introduction
- Holographic superconductors & AdS/CFT recipe

1 / 3

- Silverstein's setup: where is she wrong?
  - The Case of Missing Energy Flow
  - More problems

1 / 3

- Correct setup &
  no enhanced holographic superconductors

1 / 3

Holographic superconductors & AdS/CFT recipe

Holographic superconductors

Typically, Einstein-Maxwell-complex scalar system:

$$\mathcal{L} = \sqrt{-g}\left[R - 2\Lambda - \frac{1}{q^2}\left(F_{MN}^2 + \left|\nabla_M \psi - iA_M \psi\right|^2 + m^2 |\psi|^2\right)\right]$$

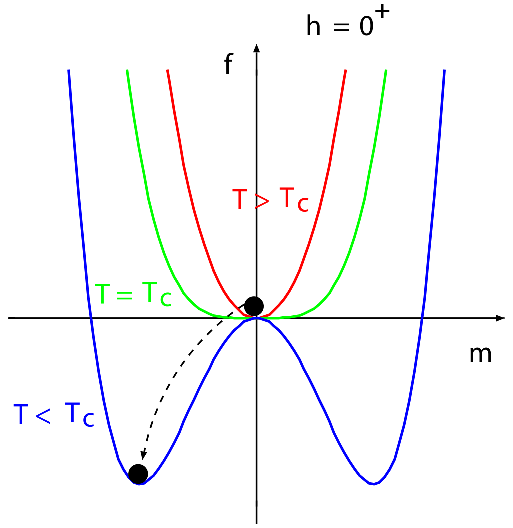
Hartnoll - Herzog - Horowitz, 0803.3295; 0810.1563
Gubser, 0801.2977

M,N...: bulk indices
μ,ν...: bdy indices

- ψ: order parameter
  Similar to Ginzburg-Landau,
  but V(ψ) is not necessary.
  ⇒ different way to achieve instability

$$F = \frac{1}{2m}\left|(-i\partial_i - qA_i)\psi\right|^2 + a|\psi|^2 + b|\psi|^4$$

- Hard to solve → “probe approx” $q \gg 1$
  Maxwell & scalar decouple from gravity



- Gravity

Pure gravity solution (Schwarzschild-AdS BH)
For 4d Bulk (3d Bdy)

$$ds_4^2 = \left(\frac{r}{L}\right)^2 \left(-f(r)dt^2 + dx^2 + dy^2\right) + L^2 \frac{dr^2}{r^2 f(r)}$$

$$= \left(\frac{T_* L}{u}\right)^2 \left(-f(u)dt^2 + dx^2 + dy^2\right) + L^2 \frac{du^2}{u^2 f(u)}$$

$$f(r) = 1 - \left(\frac{r_0}{r}\right)^3$$

$$f(u) = 1 - u^3, \quad u := r_0 / r$$

$$T_* = \frac{4\pi}{3} T$$

- Maxwell

SAdS: T only
Phase transition → introduce gauge potential $A_t = \mu_0 (1 - u)$

system: parametrized by T/μ_0

- Scalar & phase structure
  - T/μ_0 large: $\psi = 0$
  - T/μ_0 small: $\psi \neq 0 \rightarrow \psi$: order parameter
    ~ dual to "macroscopic wave fn"
- Instability because ψ mass effectively becomes tachyonic

$$m_{eff}^2 = m^2 - A_t^2(-g^{tt}) < 0 \qquad \text{for low enough } T/\mu_0$$

⇨ Useful criteria to understand more general situation

Source & response in AdS/CFT

In real experiments

$$\phi^{(0)}(t,x) \longrightarrow \delta\mathcal{O}(t,x)$$

“external source” “response”

We want to determine the response under the source

e.g.

magnetic field H → magnetization m

gauge (chemical) potential $\mu := A_t$ → charge density ρ

vector potential A_i → current J^i (Ohm’s law)

spacetime fluctuation $h_{\mu\nu}$ → EM tensor $T^{\mu\nu}$

Source & response in AdS/CFT

AdS/CFT does the job for us

4d bulk Maxwell field asymptotically (u→0) behaves as

$$A_i \sim A_i^{(0)} + \left\langle J^i \right\rangle u$$

4d Bulk fields
3d bdy quantities

"slow falloff" "fast falloff"
↓ ↓
source response
(vector potential) (current)

Justified by GKP-Witten relation, the most important eq in AdS/CFT

Then, the recipe is

- Specify your source $A_i^{(0)}$
- Solve bulk EOM
- Extract $\left\langle J^i \right\rangle$

Source & response in AdS/CFT

The other fields are similar:

$$A_i \sim A_i^{(0)} + \langle J^i \rangle u$$

4d Bulk fields
3d bdy quantities

vector ↑ potential　　　↑current

$$A_t \sim \mu + \langle \rho \rangle u$$

(in the gauge Au = 0)

chemical↑ potential　　↑charge density

$$\psi \sim \cancel{\psi^{(0)}} u + \langle \mathcal{O} \rangle u^2$$

for $L^2 m^2 = -2$

“macroscopic wave fn ↑ source”　(unrealistic)　　↑ “macroscopic wave fn”

“The Case of Missing Energy Flow”

Sherlock Homes would say so...

Adding time-dependence

According to AdS/CFT,

$$A_t \sim \mu + \langle \rho \rangle u$$

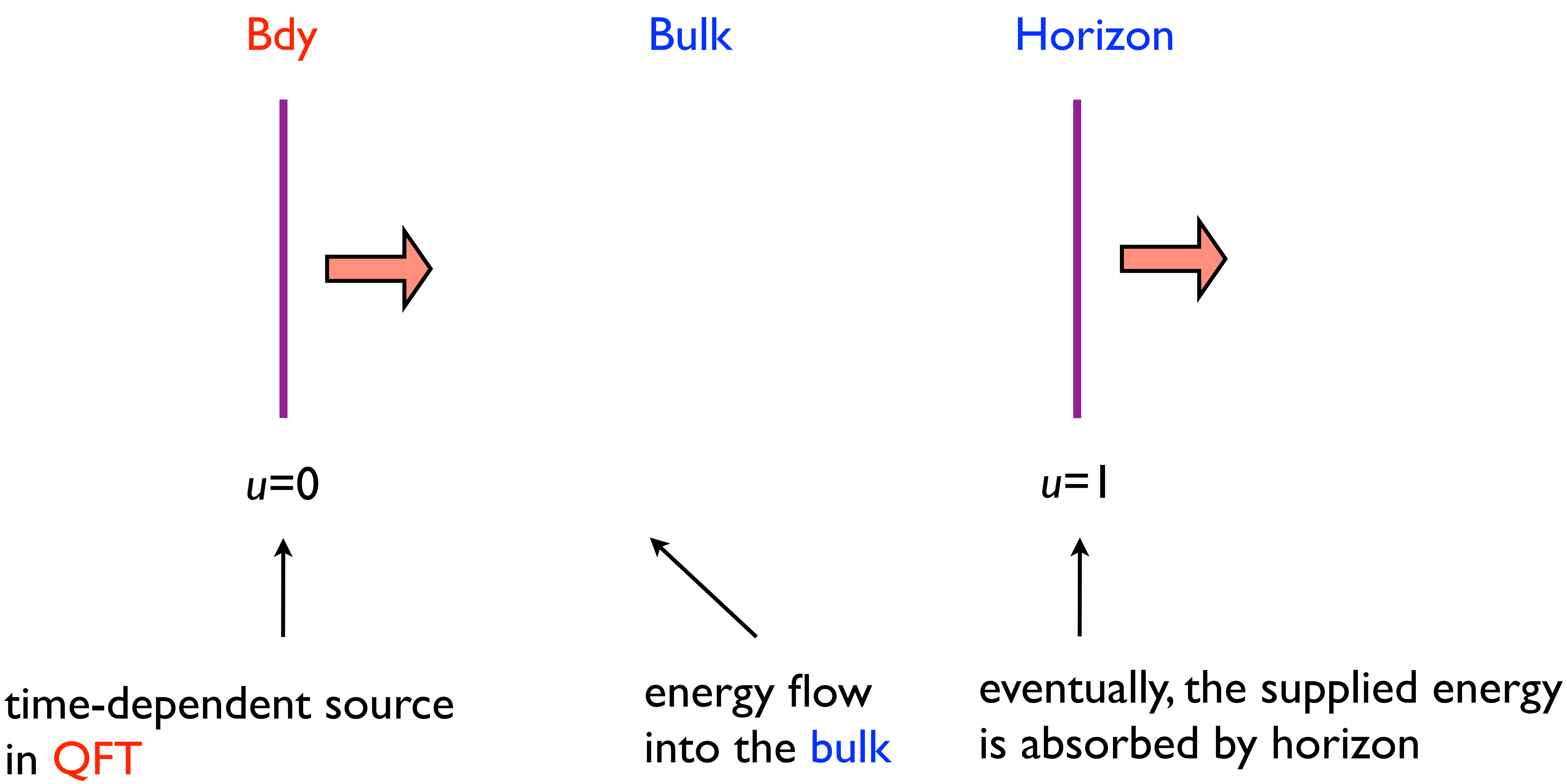
$$A_t(u) \sim \mu_0 + \cdots$$
$$\Downarrow$$
$$A_t(t,u) \sim \mu(t) + \cdots ?$$

Indeed adapted by Silverstein and collaborators

Bao - Dong - Silverstein - Torroba, 1104.4098

However, various problems

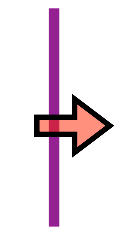
The problem



⇨ In this way, a dynamic equilibrium is achieved

Energy flow: bdy → bulk

The bdy should supply energy to bulk



u=0

$$\Delta E \sim -\int_{bdy} dt d^2x\, T_{ut}$$

$\sqrt{\dfrac{-h}{g_{uu}}}$ ignored

$$T_{MN} = F_M{}^L F_{NL} - \frac{1}{4} g_{MN} F^2 + \cdots$$

$$= -\int_{bdy} dt d^2x\, F_u{}^L F_{tL} + (scalar)$$

↳ vanish if no source $\psi^{(0)}$

$$F_u{}^i F_{ti}$$

$x^i = (x, y)$: bdy spatial coords

$$= \int_{bdy} dt d^2x \left\langle J^i \right\rangle E_i^{(0)}$$

$$\left\langle J^i \right\rangle \propto F^{ui}\Big|_{bdy} \Leftrightarrow A_i \sim A_i^{(0)} + \left\langle J^i \right\rangle u$$

$$F_{ti}^{(0)} = \partial_t A_i^{(0)} = -E_i^{(0)}$$

Energy flow from bdy to bulk:

$$\Delta E = \int_{bdy} dt d^2x \left\langle J^i \right\rangle E_i^{(0)} \sim -\int_{bdy} dt d^2x \, F_u{}^i F_{ti}$$

bulk magnetic field

- Bdy: Joule heat from bdy
- Bulk: Poynting vector $\vec{E} \times \vec{B}$

Silverstein: At(t,u) → Fti=Fui=0 No energy flow from bdy

Ai(t,u) is mandatory to supply energy to bulk

Energy flow: bulk → horizon

No energy supply from bdy, but energy absorbed at horizon
(incoming wave BC@horizon)

quasinormal modes

→ ψ simply decays to const

Amado - Kaminski - Landsteiner, 0903.2209
Maeda - Natsuume - Okamura, 0904.1914

→ No interesting dynamic equilibrium

$\Updownarrow$

Silverstein et al. actually impose the regularity BC@horizon on static ψ

Their reasoning: high frequency limit → time-dependent solution replaced by its average

→ True dynamic equilibrium?

So far

- Prob 1: No energy flow from AdS bdy to bulk?
- Prob 2: No energy flow from bulk to horizon?

The vector potential Ai(t,u) is mandatory to discuss enhancement

Source & response in AdS/CFT

The other fields are similar:

4d Bulk fields
3d bdy quantities

$$A_i \sim A_i^{(0)} + \langle J^i \rangle u$$

vector ↑ potential ↑current

$$A_t \sim \mu + \langle \rho \rangle u$$

(in the gauge Au = 0)

chemical↑ potential ↑charge density

Ours

Silverstein

$$\psi \sim \psi^{(0)} u + \langle \mathcal{O} \rangle u^2$$

for $L^2 m^2 = -2$

"macroscopic wave fn ↑ source" (unrealistic)

↑ "macroscopic wave fn"

More problems

Near critical pt

We will not add At(t,u), but we have more to say about their work.

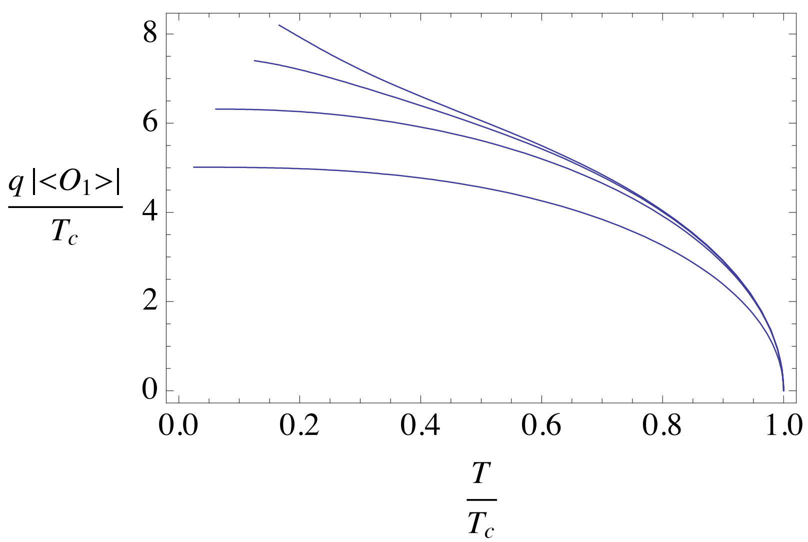
Near critical pt Tc, the order parameter ψ small → perturb in ψ

$$\psi = \varepsilon^{1/2}\psi_1 + \cdots$$
$$A_M = \boldsymbol{A}_M + \varepsilon A_{1,M} + \cdots$$

0th order

$$\partial_u \boldsymbol{F}_{ut} = \partial_t \boldsymbol{F}_{tu} = 0 \rightarrow \boldsymbol{F}_{tu} = \mu_0$$

→ Standard solution: $\boldsymbol{A}_t = \mu_0(1-u)$
$\boldsymbol{A}_u = 0$



Adapted from 0810.1563

Silverstein et al. chooses a solution

$$\boldsymbol{A}_t = \mu_0(1-u) + \partial_t \gamma(t,u) = \mu(t)(1-u)$$
$$\boldsymbol{A}_u = \qquad \partial_u \gamma(t,u)$$

$$\gamma(t,u) := (1-u)\int dt'(\mu(t') - \mu_0)$$

Note: $A_u \neq 0$ (normally $A_u = 0$ gauge)

But the bulk Maxwell field has the gauge sym.

$$A_M(t,u) \to A_M(t,u) - \partial_M \Lambda(t,u)$$

or

$$A_t(t,u) \to A_t(t,u) - \partial_t \Lambda(t,u)$$
$$A_u(t,u) \to A_u \qquad - \partial_u \Lambda$$

Their choice is gauge equivalent to the static case (by Λ=γ)

$$\boldsymbol{A}_t = \mu_0(1-u)$$
$$\boldsymbol{A}_u = 0$$

In other words, their choice is a gauge choice

Issue of chemical potential

But if $\mu_{\rm naive} := A_t(u=0),\ A_t = \mu(t)(1-u)$ looks OK?

This is not a gauge-inv def.

$$\begin{cases} A_t \to A_t - \partial_t \Lambda(t,u) \\ A_u \to A_u - \partial_u \Lambda \end{cases}$$

Is it really a time-dependent μ?
What is μ when $A_u(t,u) \neq 0$?

(1) Transform back to Au=0 gauge → const μ_0

(2) Find a gauge-inv. def.

Gauge inv. chemical potential

As the gauge-inv def., we propose

$$\mu_{inv} := \int_1^0 du\, F_{ut}$$

$$= A_t(t,u=0) - A_t(t,u=1) - \partial_t \int_1^0 du\, A_u(t,u)$$

$\mu_{inv} = \mu_0$ even for their choice

In Au=0 gauge,

$$\mu_{inv} \to A_t(t,u=0) - A_t(t,u=1)$$

Residual gauge sym

In Au=0 gauge, bulk gauge sym is not completely fixed

$$A_t(t,u) \to A_t(t,u) - \partial_t \Lambda(t)$$

→ Bdy gauge sym

One may fix it by $\quad A_t(t,u=1)=0$

Then, $\quad \mu_{inv} \to A_t(t,u=0) - \cancel{A_t(t,u=1)}$ reduces μ_{naive}

1st order analysis

$$(\boldsymbol{D}^2 - m^2)\psi_1 = 0 \qquad \boldsymbol{D}_M := \nabla_M - i\boldsymbol{A}_M$$

$$\nabla_N F_1^{MN} = j_1^M \qquad j_1^M = 2\,\mathrm{Im}\left(\psi_1^\dagger \boldsymbol{D}^M \psi_1\right)$$

To solve 1st order eq., they impose the unitary gauge where $\theta=0$

$$\psi_1 = |\psi_1| e^{i\theta_1}$$

- But we expand around $\psi=0$, so this cannot fix a gauge at leading order.
- In fact, their Maxwell field itself is a gauge choice $\quad \boldsymbol{A}_t = \mu(t)(1-u)$
- Then, too restrictive to impose an additional gauge condition

The prob becomes apparent in the imaginary part of EOM

$$(\boldsymbol{D}^2 - m^2)\psi_1 = 0$$

Let $\psi_1 = |\psi_1| e^{i\theta_1}$

$$\hat{\boldsymbol{A}}_M = \boldsymbol{A}_M - \nabla_M \theta_1$$

EOM

$$\nabla^2 |\psi_1| - (m^2 + \hat{\boldsymbol{A}}_M^2)|\psi_1| - \frac{i}{|\psi_1|} \underbrace{\nabla_M j_1^M}_{\text{Bulk current conserv}} = 0$$

$$j_1^M = 2\,\mathrm{Im}\left(\psi_1^\dagger \boldsymbol{D}^M \psi_1\right) = -2|\psi_1|^2\, \hat{\boldsymbol{A}}^M$$

- Even in unitary gauge, Im(EOM) is not absent, but they do not take the eq. into account.
- They determine

$\boldsymbol{A}_t$ & $\boldsymbol{A}_u$: leading order

$|\psi_1|$: 1st order

Im(EOM) gives a nontrivial condition which may not be satisfied
Put differently, one cannot choose $\boldsymbol{A}_M$ freely in unitary gauge.

$$\nabla_M j_1^M \propto -\partial_t\left(|\psi_1|^2\,\hat{\boldsymbol{A}}_t\right) + fu^2\partial_u\left(\frac{T_*^2 f}{u^2}|\psi_1|^2\,\hat{\boldsymbol{A}}_u\right) = 0$$

Problem summary of Silverstein et al.

- Prob 1: No energy flow from AdS bdy to bulk?
- Prob 2: No energy flow from bulk to horizon?
- Prob 3: Problem on bulk gauge sym.?
  - Their time-dependence is just a gauge choice?
  - The def of μ is not gauge inv.?
  - Impose an additional gauge (unitary gauge)?
- Prob 4: Lack of bulk conserv. eq.?

No enhanced holographic superconductor

Then, what is the alternative?

Ai(t) is mandatory to supply energy from bdy to bulk, so add Ai(t).

However, no enhancement unlike traditional superconductors

Go back to effective mass

Ai does not seem very useful to enhance superconductivity

Effective mass for ψ:

$$m_{eff}^2 = m^2 + \left\{ -A_t^2(-g^{tt}) + A_i^2 g^{ii} \right\}$$

↓ destabilize normal state

↓ stabilize normal state

In fact, a large enough magnetic field Ai(x) destroys superconductivity (critical magnetic field)

True for holographic superconductors too

e.g. H^3, 0810.1563, Nakano - Wen, 0804.3180, Albdash - Johnson, 0804.3466, 0906.0519, Montull - Pomarol - Silva, 0906.2396, **Maeda - Natsuume - Okamura, 0910.4475, ...**

A complication:

- In static case, various Fourier modes for ψ: decouple
- Because of Ai(t,u), they are no longer decoupled, hard to analyze

Decompose Ai:

$$A_i^2 = \left\langle A_i^2 \right\rangle + \mathcal{A}_2(t)$$

time-average oscillatory part

$$m_{eff}^2 = m^2 + \left[-A_t^2(-g^{tt}) + \left\{ \left\langle A_i^2 \right\rangle + \mathcal{A}_2(t) \right\} g^{ii} \right]$$

↓ ↓

naive argument applies

Couple to various Fourier modes for ψ
Our job is to estimate this term

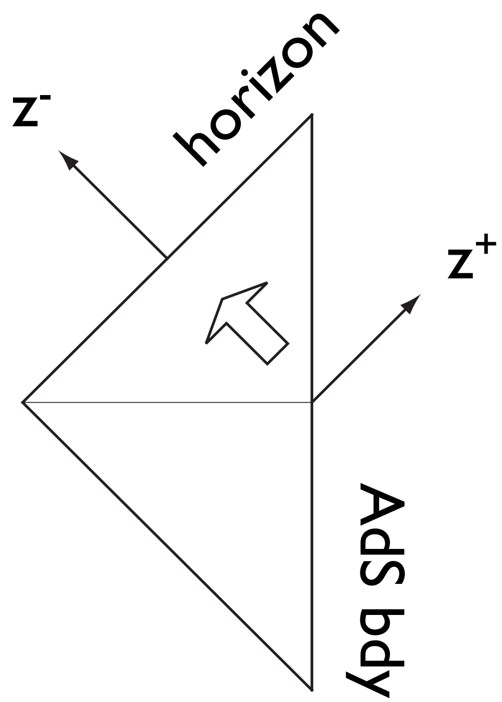
Setup

Again expand at critical pt:

$$\psi = \varepsilon^{1/2}\psi_1 + \cdots$$
$$A_M = \boldsymbol{A}_M + \cdots$$

Leading order:

$$\boldsymbol{A}_t = \mu_0(1-u) \rightarrow \text{standard}$$
$$\boldsymbol{A}_u = 0 \rightarrow \text{standard gauge choice}$$
$$\left|\boldsymbol{A}_i\right| = \left|\boldsymbol{A}_i(z^+)\right| \rightarrow \frac{E_0}{\Omega}\sin(\Omega z^+) \quad \text{for example}$$



$z^\pm = t \pm u_*$ u*: tortoise coord

$$ds^2 = \left(\frac{T_* L}{u}\right)^2 f(-dt^2 + du_*^2) + \cdots$$

Leading order EOM:

$$\partial_+\partial_-\boldsymbol{A}_i = 0$$

1st order: $$(\boldsymbol{D}^2 - m^2)\psi_1(u, z^+) = 0$$

Hard to analyze:

(1) high-frequency analytically

(2) low-frequency analytically

(3) intermediate numerically

No enhancement in all regions

(1) high-frequency limit

Scalar likely to evolve w/ its time scale
oscillate rapidly w/ $1/\Omega$ simultaneously

cf. Landau - Lifshitz, *Mechanics*

$$\psi_1(u,z^+) = \psi_{slow}(u,z^+) + \psi_{fast}(u,z^+)$$

Effective mass from $\mathcal{A}_2$ after taking time-average over $2\pi/\Omega$:

$$\begin{aligned}\left\langle \mathcal{A}_2(z^+)\psi_1 \right\rangle &= \left\langle \mathcal{A}_2 \psi_{slow} \right\rangle + \left\langle \mathcal{A}_2 \psi_{fast} \right\rangle \\ &= \cancel{\left\langle \mathcal{A}_2 \right\rangle} \psi_{slow} + \cancel{\left\langle \mathcal{A}_2 \psi_{fast} \right\rangle}\end{aligned}$$

$\mathcal{A}_2$: periodic

Reduces to the static prob w/ time-average only (naive case)
→ no enhancement

(2) low-frequency limit

Various Fourier modes of ψ_I coupled

→ Truncate to a finite # (N) of Fourier modes

→ diagonalize N×N differential eqs (possible for low-Ω)

$$m_{eff}^2 \propto -f^{-1}\boldsymbol{A}_t^2 + \left\langle \boldsymbol{A}_i^2 \right\rangle \left(1 - \frac{1}{2}\lambda_k \right)$$

↳ eigenvalue w/ $\lambda_k \leq 2$

Indeed off-diagonal terms compensate effect of $\left\langle \boldsymbol{A}_i^2 \right\rangle$

But not enough to destabilize normal state → no enhancement

(3) intermediate

Numerically solve truncated EOM w/ a small # of modes

(such as N=3, 5) via shooting method

Haven't explored the full parameter space, but no enhancement so far

No enhancement in holographic superconductors

Comments

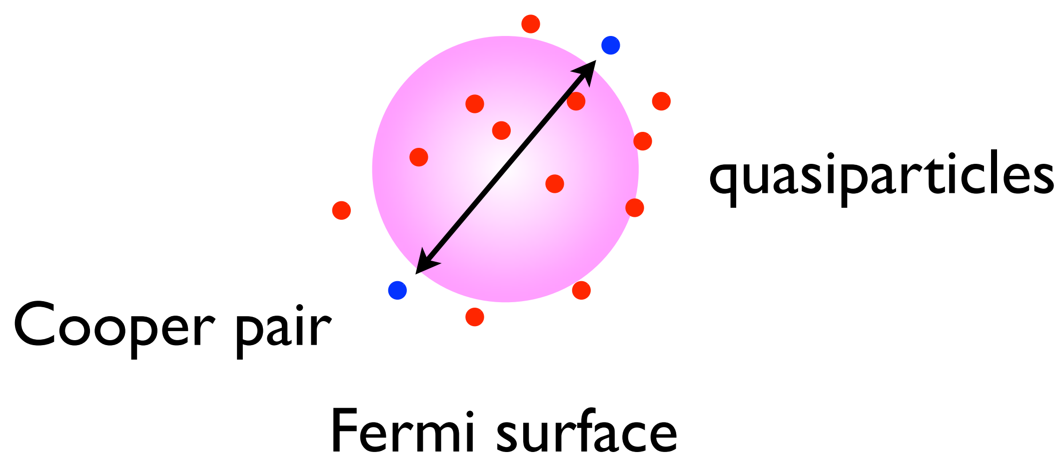
1. Why no enhancement?

2. Bulk fermion necessary?

3. Beyond the probe limit

Eliashberg theory

Increase T → more quasiparticles, block Cooper pair formation
Eventually, destroy superconductivity at Tc



Extract quasiparticles

Time-dependent source → excite quasiparticles to higher levels
leaving room for Cooper pair formation
→ quasiparticles decay to phonons

1. Why no enhancement?

In Eliashberg theory, hierarchy of scales necessary for enhancement:

$$\frac{1}{\tau} \ll T_c$$

↳ relaxation time of quasiparticle

But for BHs, natural to expect $1/\tau \sim O(T)$

Lack of hierarchy is the reason of no enhancement?

Note: this is the condition for Fermi liquid

2. Bulk fermion necessary?

Our system: Einstein-Maxwell-scalar ~ Ginzburg-Landau
No bulk fermion

Condition for enhancement ← Condition for Fermi liquid

Bulk fermion: interesting possibility to explore
Not obvious if our setup is insufficient though:

- Eliashberg theory is summarized as a GL-like theory
- In low-Ω limit, the oscillatory part indeed tends to compensate the time-average part.
  In a sense, "enhancement." (Tc: higher than time-average only)
  Our trouble: the oscillatory part never larger than the time-average

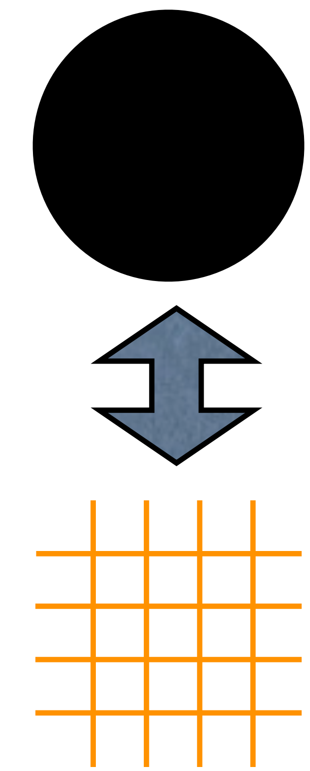
3. Beyond the probe limit

We take the probe limit.
What would happen if the backreaction is included?

Holography:

supplied energy → BH ⇒ heat bath: BH

↳ infinite heat bath in probe limit



Eliashberg:

supplied energy → quasiparticle → phonon ⇒ heat bath: lattice

In reality, no infinite heat bath
Heating effect of phonon: destroys the enhancement for $\Omega \gtrsim T_c$.

Backreaction does not help for enhancement.

Summary

- Enhanced holographic superconductors have been discussed previously, but we saw problems in previous work.
- Lessons:
  - Mind energy flow from bdy to bulk
  - Mind the def of chemical potential when $A_u(t,u) \neq 0$
- Correct analysis should involve $A_i(t)$, but enhancement does not happen unlike traditional superconductors.
- If holographic superconductors resemble cuprates in some way, our result may suggest that the observed enhancement in cuprates comes from a mechanism which is different from Eliashberg.

Advertisement

My AdS/real-world textbook (Sep. 2012)

Now working on the English edition (from Spr**g*r?), so if you find errors in Japanese edition, please let me know.

Corrections available at
http://www.h7.dion.ne.jp/~natsuume/ads-cft.html

From a review:
“Natsuume magic”
— Keiji Fukushima, *Keio Univ.*

Published Sep. 2012

Backup

Energy flow: full expression

$$\Delta E = -\int_{horizon} dz^+ d^2x \left(2\left|D_+\psi\right|^2 + F_{+i}F_+{}^i\right)$$
$$+\int_{bdy} dt d^2x \left\langle J^i \right\rangle E_i^{(0)} - 2\int_{bdy} \frac{d\Sigma}{\sqrt{g_{uu}}} \left(D_{(u}\psi\right)^\dagger \left(D_{t)}\psi\right)$$

↳ vanish if no source $\psi^{(0)}$

Comments on At(t,u)

We are not saying that *all* At(t,u) are meaningless

- Silverstein's At(t,u) seems meaningless since it is gauge-equiv to static case.
- At(t,u) cannot be used to exchange energy bet bdy and bulk

  But, in principle, this is not the only way to supply energy to bulk e.g. one can supply energy to bulk *directly*.

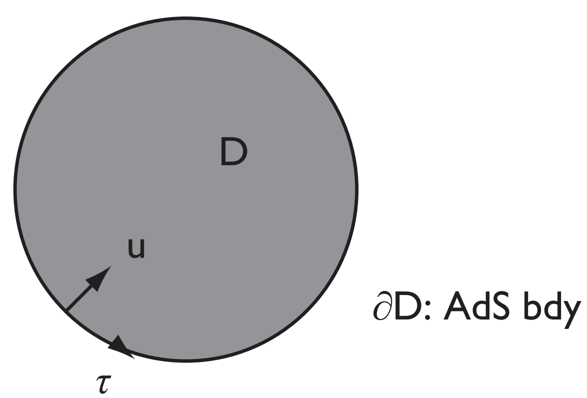
Gauge-inv chemical potential (Euclidean)

For the Euclidean BH, (τ, u)-plane forms a disk D

$$\frac{1}{\beta}\int_D F_2 = \frac{1}{\beta}\int_D dA = \frac{1}{\beta}\oint_{\partial D} A \qquad \text{:Wilson-loop}$$

For simplicity, take the gauge $A_\tau(\tau, u=1)=0$

Our proposed def → reduces to a common gauge-inv def of μ



Eliashberg theory (1)

Eliashberg theory is summarized by a GL-like eq:

$$\frac{T}{T_c} - 1 = -\frac{7\varsigma(3)}{8\pi^2}\left(\frac{\Delta}{T_c}\right)^2 + \frac{1}{4}\tau\alpha\frac{\Omega}{T_c}G\left(\frac{\Delta}{\Omega}\right) - \frac{\pi}{2}\frac{\alpha}{T_c}$$

↓ enhancement

↓ suppression effect of $\langle A_i^2 \rangle$

Δ : condensate
τ: relaxation time of quasiparticle
$\alpha \propto$ (electric field)^2
G: complicated fn but the maximum value is G(1/2)~3.6

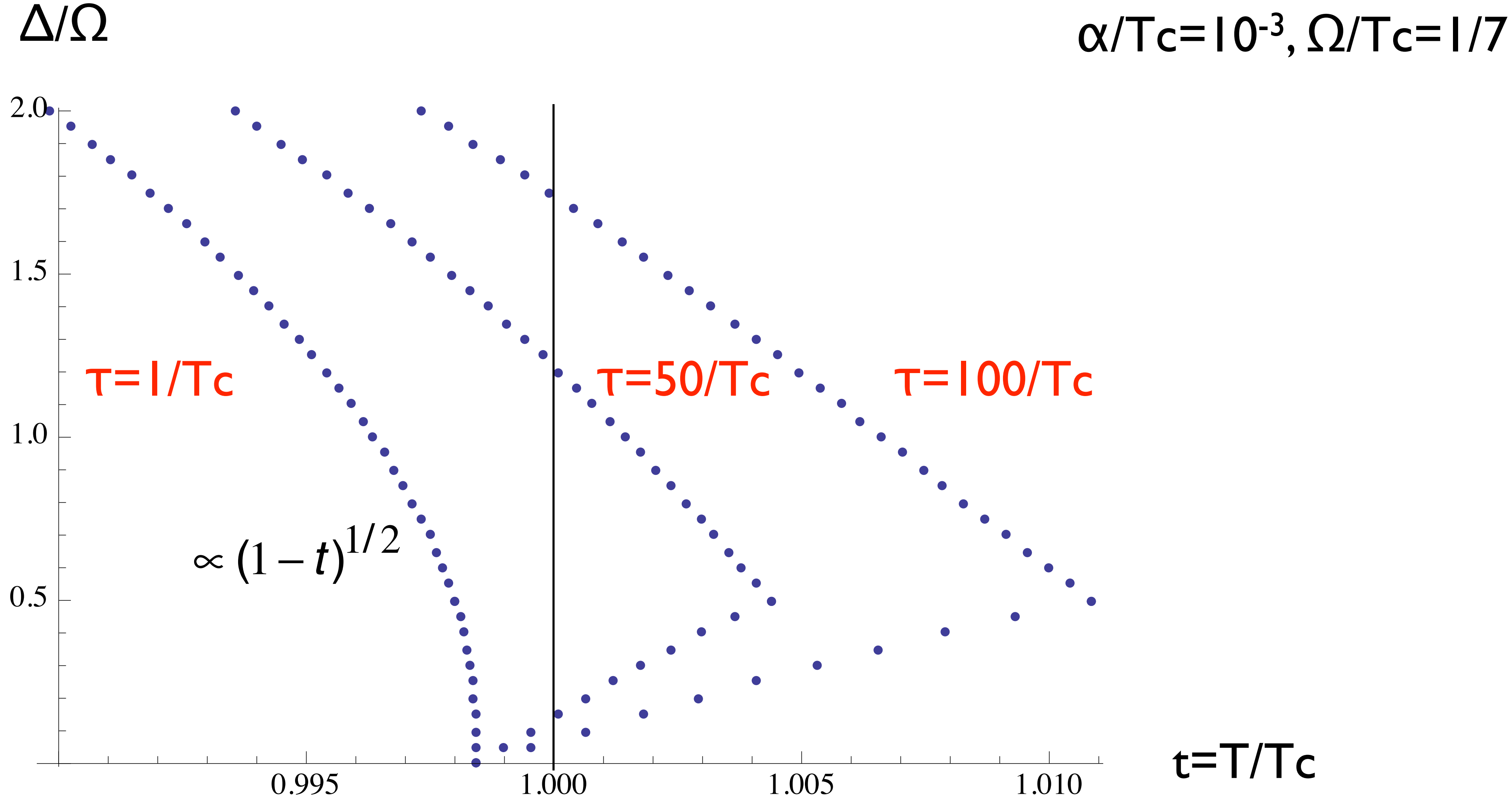
5 dimensionless parameters: T/Tc, Δ/Tc, Ω/Tc, α/Tc, τTc

- When α=0, standard mean-field behavior $\Delta \propto (1-t)^{1/2}$
- Large τ: favorable for enhancement

Back

Eliashberg theory (2)

Typical behavior of condensate for fixed electric field (α), Ω, and τ



- Large τ: favorable for enhancement
- lower branch actually unstable → 1st-order transition in reality

1st order (scalar)

$$\left\{\frac{2}{T_*}d_u\partial_+ + \mathcal{L}_u + \mathcal{A}_2(z^+)\right\}\varphi(u,z^+) = 0$$

$$\varphi := \frac{\psi_1}{u} \qquad d_u := \partial_u - i\frac{A_t}{T_* f}$$

$$L^2 m^2 = -2 \qquad \mathcal{L}_u := -\partial_u(f\partial_u) + V(u)$$

$$V(u) := u + \frac{1}{T_*^2}\left(-\frac{1}{f}A_t^2 + \left\langle A_i^2 \right\rangle\right)$$

$$\frac{A_i^2}{T_*^2} =: \mathcal{A}_2(z^+) + \frac{1}{T_*^2}\left\langle A_i^2 \right\rangle$$

Eq. we would like to solve, but difficult to handle analytically

Back

(I) high-Ω

$$\left\{\frac{2}{T_*}d_u\partial_+ + \mathcal{L}_u + \mathcal{A}_2(z^+)\right\}\varphi(u,z^+) = 0$$

$$\varphi(u,z^+) = \varphi_{slow}(u,z^+) + \varphi_{fast}(u,z^+)$$

$$\left\{\frac{2}{T_*}d_u\partial_+ + \mathcal{L}_u\right\}\varphi_{slow} \sim -\overline{\mathcal{A}_2\varphi_{fast}}$$ → depends on fast mode thru $\overline{\mathcal{A}_2\varphi_{fast}}$

$$\frac{2}{T_*}d_u\partial_+\varphi_{fast} \sim -\mathcal{A}_2\varphi_{slow}$$ → $\mathcal{A}_2$: source for fast mode

"-": time-average over 2π/Ω

One can solve fast mode and can show $\overline{\mathcal{A}_2\varphi_{fast}} \propto \overline{\mathcal{A}_2\int\mathcal{A}_2} = \overline{\partial_+\left(\int\mathcal{A}_2\right)^2} = 0$

→ slow mode: no contribution from the fast mode

After dynamic equilibrium is achieved, $\partial_+\varphi_{slow} \sim 0$

$$\mathcal{L}_u\varphi_{slow} \sim 0$$

Reduces to the static prob w/ time-average only → no enhancement

(2) low-Ω

$$\left\{\frac{2}{T_*}d_u\partial_+ + \mathcal{L}_u + \mathcal{A}_2(z^+)\right\}\varphi(u,z^+)=0$$

After Fourier transform in z^+:

$$\mathcal{L}_u\begin{pmatrix}\vdots\\ \varphi_{-2}\\ \varphi_{-1}\\ \varphi_0\\ \varphi_1\\ \varphi_2\\ \vdots\end{pmatrix}=\frac{1}{2T_*^2}\left\langle \boldsymbol{A}_i^2\right\rangle\begin{pmatrix}\ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots\\ \cdots & -4iwd_u & 1 & 0 & 0 & 0 & \cdots\\ \cdots & 1 & -2iwd_u & 1 & 0 & 0 & \cdots\\ \cdots & 0 & 1 & 0 & 1 & 0 & \cdots\\ \cdots & 0 & 0 & 1 & 2iwd_u & 1 & \cdots\\ \cdots & 0 & 0 & 0 & 1 & 4iwd_u & \cdots\\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}\vdots\\ \varphi_{-2}\\ \varphi_{-1}\\ \varphi_0\\ \varphi_1\\ \varphi_2\\ \vdots\end{pmatrix}$$

$$\varphi_n := \varphi(u,\omega=2n\Omega)$$

$$w := \frac{8T_*\Omega^3}{E_0^2}$$

Back

1. infinite matrix → (2N+1)×(2N+1)
2. low Ω → ignore du terms

$$\mathcal{L}_u\vec{\varphi} \sim \frac{1}{2T_*^2}\left\langle \boldsymbol{A}_i^2 \right\rangle \mathcal{M}\vec{\varphi}$$

$$\vec{\varphi} = {}^t\left(\varphi_{-N}\cdots\varphi_N\right)$$

$$\mathcal{M} = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$ tridiagonal

Diagonalize:

$$\left[-\partial_u(f\partial_u)+u+\frac{1}{T_*^2}\left\{-\frac{1}{f}\boldsymbol{A}_t^2+\left\langle \boldsymbol{A}_i^2\right\rangle\left(1-\frac{1}{2}\lambda_k\right)\right\}\right](\vec{v}_k\varphi_k)\sim 0$$

$$\mathcal{M}\vec{v}_k = \lambda_k\vec{v}_k$$

Off-diagonal terms indeed compensate the effect of Ai, but

$$\lambda_k = 2\cos\frac{\pi k}{2(N+1)} \leq 2$$

→ no enhancement