M5-brane indices from 5d gauge theories

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Hee-Cheol Kim, S.K. "M5-branes from gauge theories on S^5" arXiv:1206.6339

Hee-Cheol Kim, Joonho Kim, S.K. "Instantons on S^5 and M5-branes" arXiv:1211.0144

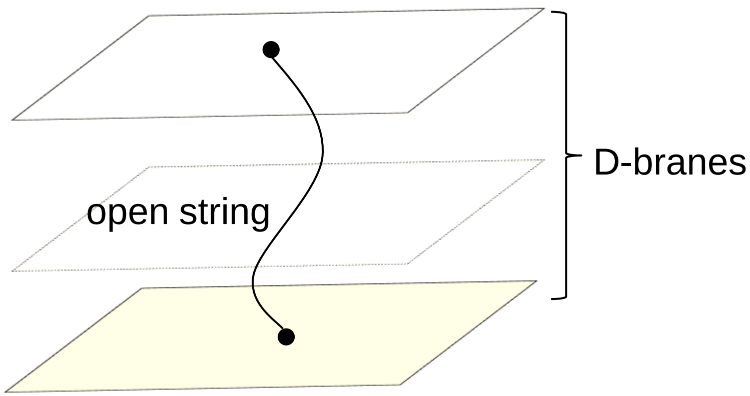
Hee-Cheol Kim, S.K., Sung-Soo Kim, Kimyeong Lee, to appear

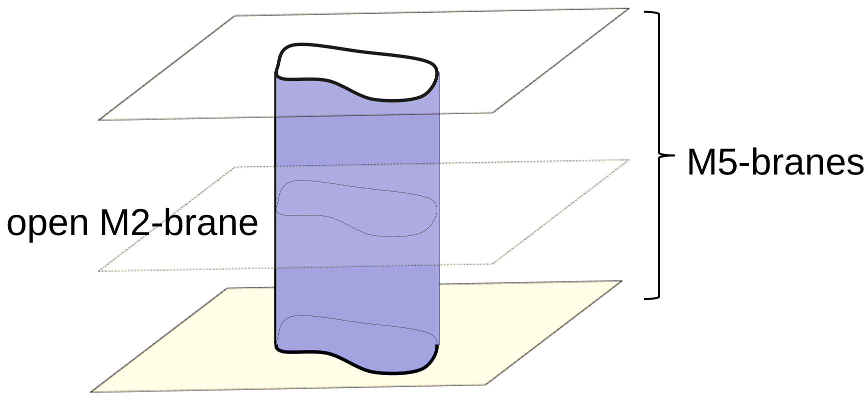
related works: [Kallen, Zabzine] 1202.1956; [Hosomichi, Seong, Terashima] 1203.0371;
[Kallen,Qiu,Zabzine] 1206.6008; [Kallen,Minahan,Nedeline,Zabzine] 1207.3763;
[Imamura] 1209.0561; [Lockhart, Vafa] 1210.5909; [Imamura] 1210.6308;
[H.-C.Kim, K.Lee] 1210.0853; [Haghighat,Iqbal,Kozcaz,Lockhart,Vafa] 1305.6322

M5-branes and 6d (2,0) SCFT

- M5-branes host 6d (2,0) SCFT: very unique quantum field theory.
- Not like the familiar Yang-Mills. It should be a “tensor gauge theory”

$$\int_{\Sigma_1} A_\mu dx^\mu \qquad\qquad \int_{\Sigma_2} B_{\mu\nu} dx^\mu dx^\nu$$





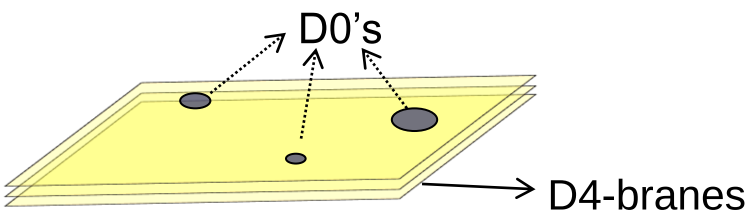
- N^3 light degrees of freedom should live on N M5’s.
- Its compactification leads to many interesting quantum systems.
- “M-theory of QFT.” A unifying framework to understand QFT dualities

- However, we know almost nothing on its microscopic definition.

Circle compactification and 5d SYM

- Compactify on S^1, 5d (S)YM at low E: non-perturbative study, some 6d physics.
- Instanton solitons: D0's on D4's = KK modes

$$F_{\mu\nu} = \star_4 F_{\mu\nu} \quad \text{on } R^4 \qquad \frac{4\pi^2}{g_{YM}^2} = \frac{1}{r_1}$$



- Nonrenormalizable: consistent quantum calculation possible?

- BPS quantities are often calculable with low E effective theories.
- SUSY path integrals: (secretly) Gaussian, almost trivial UV divergence
- Is it complete? At least, can it be consistently used, in BPS sector?
- Almost. Still with a few "small" ambiguities that we shall fix...

- Concrete study: 6d (2,0) theory on $S^5 \times S^1$ from SYM on S^5 or $\boxed{CP^2 \times S^1}$.

Kimyeong Lee's talk (Wed) for more detailed investigation on it.

Contents

Indices for 6d (2,0) theories

- Put the theory on S^5 x R: energy E ; SO(6) j_1, j_2, j_3 ; $SO(5)_R$ R_1, R_2

- Choose a pair of Q, S (= Q^+)

$$Q^{(R_1,R_2)}_{(j_1,j_2,j_3)} \rightarrow \quad Q^{(\frac{1}{2},\frac{1}{2})}_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} \quad : \quad \text{BPS bound } E = 2R_1 + 2R_2 + j_1 + j_2 + j_3$$

[Kinney,Maldacena,Minwalla,Raju] [Bhattacharya,Bhattacharyya,Minwalla,Raju]

- Index partition function on S^5 x S^1: counts local BPS operators on R^6

$$I(\beta, m, \epsilon_1, \epsilon_2) = \text{Tr}\left[(-1)^F e^{-\beta'\{Q,S\}} e^{-\beta(E-\frac{R_1+R_2}{2})} e^{\beta m(R_1-R_2)} e^{-\gamma_1(j_1-j_3)} e^{-\gamma_2(j_2-j_3)}\right]$$

- S^5 QFT interpretation: naïve reduction on S^1 with twistings (clear in Abelian theory)

1. β ~ S^1 radius ~ 5d or "type IIA" coupling: $\frac{4\pi^2}{g_{YM}^2} = \frac{1}{r_1} = \frac{2\pi}{r\beta}$
2. m: hypermultiplet mass (Scherk-Schwarz reduction) $\frac{\partial}{\partial\tau} \rightarrow \frac{\partial}{\partial\tau} - ia_i\frac{\partial}{\partial\phi_i} + \frac{R_1+R_2}{2} - m(R_1-R_2)$
3. a_i = (a, b, c), satisfying a+b+c=0, squash S^5:

$$e^{-\gamma_1(j_1-j_3)}e^{-\gamma_2(j_2-j_3)} = e^{-\beta(aj_1+bj_2+cj_3)}$$

Limits & unrefined indices

- Unrefined index with a=b=c=0: compute it from 5d N=1 SYM on round S^5

$$I(\beta, m, \epsilon_1, \epsilon_2) = \mathrm{Tr}\left[(-1)^F e^{-\beta'\{Q,S\}} e^{-\beta(E-\frac{R_1+R_2}{2})} e^{\beta m(R_1-R_2)} \cancel{\boxed{e^{-\gamma_1(j_1-j_3)} e^{-\gamma_2(j_2-j_3)}}}\right]$$

- “Simplest” unrefined index: take m = ½ & a=b=c=0 (m = -1/2 is similar)

$$\mathrm{tr}[(-1)^F e^{-\beta(E-R_1)}] \qquad Q^{(+\frac{1}{2}, R_2)}_{(j_1, j_2, j_3)}$$

- E - R_1 commutes with half of the 6d SUSY: maximal SYM on S^5
- Vector/hyper cancelation: technically easier to study

SYM on S^5

- Fields of maximal SYM: w/ SU(1|1), SU(4|1) (at $a_i = 0$), SU(4|2) (at $a_i = 0$, $m = ½$ or $- ½$)

- Data needed to construct SYM on S^5: $n_1^2+n_2^2+n_3^2 = 1$

background: $$ds_5^2 = dn_i^2 + n_i^2 d\phi_i^2 + \alpha^2 (a_i n_i^2 d\phi_i)^2 \qquad \alpha^{-2} = 1 - a_i^2 n_i^2 \qquad C = i\sum_{i=1}^{3} a_i n_i^2 d\phi_i$$

Killing spinor eqn: $$\left[\nabla_\mu - C_\mu \sigma_3 - \frac{i}{8\alpha}(dC)^{\nu\rho}\gamma_{\mu\nu\rho}\right]\epsilon = i\gamma_\mu \left[\alpha\sigma_3 - \frac{1}{4\alpha}(dC)_{\nu\rho}\gamma^{\nu\rho} + \frac{i}{2\alpha}\nabla_\nu \alpha \gamma^\nu\right]\epsilon$$

- Hybrid of SUGRA [Festuccia,Seiberg] + brutal [Hosomichi,Seong,Terashima] methods

- Action with off-shell SU(1|1): (bosonic) $\qquad D = 2(a_1^2 + a_2^2 + a_3^2)\alpha^2 \qquad V_{ab} = (dC)_{ab}$

$$\left.\begin{aligned} g_{YM}^2 e^{-1}\mathcal{L} \;=\; & \frac{1}{2}\left(\frac{3}{16\alpha^2}V^2 + \frac{1}{4}R + D\right)\alpha\phi^2 - \frac{1}{4\alpha}\phi^2 V^2 - \frac{1}{2}\phi V^{ab}F_{ab} \\ & -2\phi\left(-\frac{1}{4}V^{ab}F_{ab} - \frac{1}{2}\partial^a \alpha D_a \phi + \frac{i}{4}\alpha^2 (\sigma^3)_{ij} D^{ij}\right) \\ & -\alpha\left(-\frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}D^a\phi D_a\phi - \frac{1}{4}D_{ij}D^{ij}\right) + e^{-1}\frac{i}{8}\epsilon^{\mu\nu\lambda\rho\sigma}C_\mu F_{\nu\lambda}F_{\rho\sigma} \end{aligned}\right\} \text{ vector multiplet}$$

$$\left. + |D_\mu q^i|^2 + \left(4 - \frac{\alpha^2}{4}\right)|q^i|^2 - \bar{F}_{i'}F^{i'} + ([\bar{q}_i, \phi] - im\alpha\bar{q}_i)([\phi, q^i] - im\alpha q^i) - \bar{q}_i(\sigma^I)^i{}_j([D^I, q^j] + m\alpha^2\delta_3^I q^j)\right\} \text{ adjoint hypermultiplet}$$

- A 5d ambiguity: constant shift at inverse-powers of g_{YM}^2 to be added & tuned.

$$\Delta S_0 = \underbrace{\frac{1}{g_{YM}^2}\left[a_1 \int R^2 + \text{other background fields}\right]}_{\sim T} + \underbrace{\frac{1}{g_{YM}^6}\left[a_3 \int R + \text{other background fields}\right]}_{\sim T^3 \text{ (maximally possible high T growth)}}$$

to account for high T asymptotics of “free energy”

Localization of path integral

- Localization: $Z(\beta) = \int e^{-S-tQV}$: t independent (V chosen s.t. $\{Q^2,V\}=0$)
- Saddle points: $f = \epsilon^\dagger\epsilon = \alpha(1+a_i n_i^2)$ $\xi^\mu = \epsilon^\dagger\gamma^\mu\epsilon = (1+a_i)\partial^\mu_{\phi_i} = (1,1,1)+(a_i) \equiv \zeta + \tilde{\xi}$

$$\boxed{f\hat{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta\gamma}\hat{F}^{\alpha\beta}\xi^\gamma}, \quad \hat{F}_{\mu\nu}\xi^\nu = 0 \ , \quad D_\mu(\alpha^{-1}\phi) = 0 \ , \quad D = i\alpha\phi\sigma_3$$ all other fields = 0

$$\hat{F}_{\mu\nu} = F_{\mu\nu} - i\alpha^{-1}\phi(d\tilde{\xi})_{\mu\nu} = F_{\mu\nu}(A - i\alpha^{-1}\phi\tilde{\xi})$$

round S^5: self-dual instantons on CP^2 (base of Hopf fibration generated by ζ)

squashed S^5: ξ doesn't yield closed orbits on T^3. Global problem restricts the saddle points

- Squashed S^5: instantons collapsed to "3 fixed points" (on squashed CP^2).

$$(n_1,n_2,n_3) = (1,0,0),\ (0,1,0) \text{ or } (0,0,1)$$ $U(1)^2$ fixed points on (squashed) CP^2

- solution away from fixed points:

$$\phi = \alpha\phi_0 \ , \quad D = i\alpha^2\phi_0\sigma_3 \ , \quad F_{\mu\nu} = \phi_0 V_{\mu\nu} = i\phi_0(d\tilde{\xi})_{\mu\nu}$$

- determinant around saddle point: use suitable index theorems

Results

1. Classical measure: Gaussian measure & k-instanton factor $\lambda = r\phi$

$$e^{-S_0} \quad \text{with} \quad S_0 = \frac{2\pi^2 \mathrm{tr}\lambda^2}{\beta(1+a)(1+b)(1+c)} \qquad e^{-\frac{4\pi^2 k}{\beta(1+a_i)}} \quad (\ll 1) \quad \text{for} \quad i = 1, 2, 3$$

2. Result: determinant factorized to 3 fixed points

$$Z(\beta, m, \epsilon_1, \epsilon_2) = \frac{1}{|W|}\int_{-\infty}^{\infty}\left[\prod_{i=1}^{r} d\lambda_i\right] e^{-\frac{2\pi^2 \mathrm{tr}\lambda^2}{\beta(1+a)(1+b)(1+c)}}\, Z^{(1)}_{\rm pert} Z^{(1)}_{\rm inst} \cdot Z^{(2)}_{\rm pert} Z^{(2)}_{\rm inst} \cdot Z^{(3)}_{\rm pert} Z^{(3)}_{\rm inst}$$

(W: Weyl group, r: rank)

3. $Z_{\rm pert}$'s combine to:

$$\det_{\rm V} = \prod_{\alpha\in\text{root}} \prod_{p,q,r=0}^{\infty} \Big(p(1+a)+q(1+b)+r(1+c)+\alpha(\lambda)\Big)\Big((p+1)(1+a)+(q+1)(1+b)+(r+1)(1+c)+\alpha(\lambda)\Big)$$

$$\det_{\rm H} = \prod_{\mu\in\text{weight}} \prod_{p,q,r=0}^{\infty} \Big(p(1+a)+q(1+b)+r(1+c)+m+\frac{3}{2}+\mu(\lambda)\Big)^{-1} \times \Big((p+1)(1+a)+(q+1)(1+b)+(r+1)(1+c)-m-\frac{3}{2}+\mu(\lambda)\Big)^{-1}$$

a hyper in real rep. (e.g. adjoint)

4. $Z_{\rm inst} \sim Z_{\rm Nekrasov}$ on Omega-deformed $R^4 \times S^1$ (with careful parameter match). For U(N),

$$\epsilon_1 = a_1 - a_3\ , \quad \epsilon_2 = a_2 - a_3$$

$$Z^{(3)}_{\rm inst} = \sum_{k=0}^{\infty} e^{-\frac{4\pi^2 k}{\beta(1+c)}} \frac{(1+c)^{-k}}{k!} \oint \left[\prod_{I=1}^{k}\frac{d\phi_I}{2\pi}\right] \prod_{I=1}^{k}\prod_{i=1}^{N} \frac{\sin\pi\frac{\phi_I-\lambda_i-m-\frac{3(1+c)}{2}}{1+c}\sin\pi\frac{\phi_I-\lambda_i+m+\frac{3(1+c)}{2}}{1+c}}{\sin\pi\frac{\phi_I-\lambda_i-\epsilon_+}{1+c}\sin\pi\frac{\phi_I-\lambda_i+\epsilon_+}{1+c}}$$
$$\times \prod_{I\neq J}\sin\pi\frac{\phi_{IJ}}{1+c}\prod_{I,J}\frac{\sin\pi\frac{\phi_{IJ}+3c}{1+c}}{\sin\pi\frac{\phi_{IJ}-a+c}{1+c}\sin\pi\frac{\phi_{IJ}-b+c}{1+c}} \cdot \frac{\sin\pi\frac{\phi_{IJ}+m-\frac{1}{2}+a}{1+c}\sin\pi\frac{\phi_{IJ}+m-\frac{1}{2}+b}{1+c}}{\sin\pi\frac{\phi_{IJ}+m+\frac{3}{2}}{1+c}\sin\pi\frac{\phi_{IJ}+m+\frac{3}{2}+3c}{1+c}}$$

Weak-coupling expansion with small β. Index? Expand with $e^{-\beta}$ at strong coupling.

Abelian 6d index

- Free in 6d, exact index known [Bhattacharya,Bhattacharyya,Minwalla,Raju]

"letter index"

$$f(\beta, m, a, b, c) = \frac{e^{-\frac{3\beta}{2}}(e^{\beta m} + e^{-\beta m}) - e^{-2\beta}(e^{\beta a} + e^{\beta b} + e^{\beta c}) + e^{-3\beta}}{(1 - e^{-\beta(1+a)})(1 - e^{-\beta(1+b)})(1 - e^{-\beta(1+c)})}$$

full Abelian index

$$I \;=\; PE[f] \equiv \exp\left[\sum_{p=1}^{\infty} \frac{1}{p} f(p\beta, m, a, b, c)\right] \quad \text{x} \quad \boxed{e^{-\beta\epsilon_0}} \rightarrow \text{0-point “energy”}$$

1. Weak-coupling dual of above: [Gopakumar, Vafa]; see also [Lockhart, Vafa (2012)].

$$I = \underbrace{\left[\frac{\beta}{2\pi}(1+a)(1+b)(1+c)\right]^{\frac{1}{2}}}_{2.} e^{\underbrace{\frac{\pi^2}{6\beta}\frac{3/4 + m^2 - \frac{a^2+b^2+c^2}{2}}{(1+a)(1+b)(1+c)}}_{4.}} \times Z_{\text{pert}} \exp\left[\underbrace{\sum_{p=1}^{\infty} \frac{1}{p}\left(\frac{e^{-\frac{4\pi^2 p}{\beta(1+a)}}}{1 - e^{-\frac{4\pi^2 p}{\beta(1+a)}}} f(p, m, a, b, c) + (a, b, c \to b, c, a) + (a, b, c \to c, a, b)\right)}_{3.}\right]$$

2. Gaussian integration factor for U(1) eigenvalue λ

$$f(p, a, b, c) \equiv \frac{\sin\frac{p\pi(m - \frac{1}{2} + b)}{1+a}\,\sin\frac{p\pi(m - \frac{1}{2} + c)}{1+a}}{\sin\frac{p\pi(a-b)}{1+a}\,\sin\frac{p\pi(a-c)}{1+a}}$$

3. Abelian instantons, exactly summed over: [Iqbal-Kozcaz-Shabbir]

4. High T asymptotics: can't come from 5d dynamics, "fit" by constant shift of action

[Similar shift by Vafa-Witten (1994) to twisted 4d maximal SYM for S-duality]

Non-abelian 6d index (w/ 16 SUSY)

- Perturbative part: (instanton parts will be λ independent)

$$Z_{\rm pert} = \frac{1}{|W|}\int d\lambda \prod_{\alpha\in\Delta_+}\left(2\sinh\frac{\alpha\cdot\lambda}{2}\right)^2 e^{-\frac{2\pi^2}{\beta}{\rm tr}\lambda^2} = \left(\frac{\beta}{2\pi}\right)^{\frac{r}{2}} e^{\frac{\beta}{6}c_2|G|}\prod_{\alpha\in\Delta_+}\left(1-e^{-\beta(\alpha\cdot\rho)}\right)$$

(c_2: dual Coxeter number, |G|: dimension of semi-simple part, ρ: Weyl vector, r: rank)

- $Z_{\rm inst}$ also simplifies: U(N) & SO(2N) again, tune constant shift $e^{-\Delta S_0} = e^{\frac{\pi^2 N}{6\beta}}$

$$Z_{\rm inst} = \eta(e^{-\frac{4\pi^2}{\beta}})^{-N} = \left(\frac{2\pi}{\beta}\right)^{N/2}\eta(e^{-\beta})^{-N} = \left(\frac{2\pi}{\beta}\right)^{N/2} e^{\frac{N\beta}{24}}\prod_{n=1}^{\infty}\frac{1}{(1-e^{-n\beta})^N}$$

- Full finite N indices (they are indices, indeed…):

$$Z^{U(N)} = e^{\beta\left(\frac{N(N^2-1)}{6}+\frac{N}{24}\right)}\prod_{n=0}^{\infty}\prod_{s=1}^{N}\frac{1}{1-e^{-\beta(n+s)}}$$

$$Z^{SO(2N)} = e^{\beta\left(\frac{c_2|G|}{6}+\frac{N}{24}\right)}\prod_{n=0}^{\infty}\left[\frac{1}{1-e^{-\beta(n+N)}}\prod_{s=1}^{N-1}\frac{1}{1-e^{-\beta(n+2s)}}\right]$$

Which states are counted?

- Full finite N indices: $$Z^{U(N)} = e^{\beta\left(\frac{N(N^2-1)}{6}+\frac{N}{24}\right)} \boxed{\prod_{n=0}^{\infty}\prod_{s=1}^{N}\frac{1}{1-e^{-\beta(n+s)}}}$$
$$Z^{SO(2N)} = e^{\beta\left(\frac{c_2|G|}{6}+\frac{N}{24}\right)} \boxed{\prod_{n=0}^{\infty}\left[\frac{1}{1-e^{-\beta(n+N)}}\prod_{s=1}^{N-1}\frac{1}{1-e^{-\beta(n+2s)}}\right]}$$
- We only know Z_{pert} for E_6, E_7, E_8. But we conjecture $Z_{\text{inst}}^{E_n} = \eta(e^{-\frac{4\pi^2}{\beta}})^{-n}$
- Then… $$Z^{ADE} = e^{\beta\left(\frac{c_2|G|}{6}+\frac{r}{24}\right)} \boxed{\prod_{n=0}^{\infty}\prod_{\text{Casimir op.}}\frac{1}{1-e^{-\beta(n+d)}}}$$ d: degree of the Casimir operator
- Counts ½ -BPS operators w/ one kind of holomorphic derivatives
- Analogy: similar operators in 4d N=4 SYM

U(N): $(\partial_1)^n \mathrm{tr} Z^s$ and their multiplications

SO(2N): $(\partial_1)^n \mathrm{tr} Z^{2s}$, $(\partial_1)^n \sqrt{\det Z}$ and their multiplications

- Large N limits agree with SUGRA indices on $AdS_7 \times S^4$ & $AdS_7 \times S^4/Z_2$

Casimir "energies"

- The overall prefactor: $e^{-\beta\epsilon_0} \equiv e^{\beta\left(\frac{c_2|G|}{6}+\frac{r}{24}\right)}$
- Casimir "energy": index version (regulator/renormalization dependent)

$$\epsilon_0 = \lim_{\beta'\to 0} \mathrm{tr}\left[(-1)^F \frac{E}{2} e^{-\beta' E}\right] \qquad (\epsilon_0)_{\text{index}} = \lim_{\beta'\to 0} \mathrm{tr}\left[(-1)^F \frac{E-R_1}{2} e^{-\beta'(E-R_1)}\right]$$

- For instance, the 6d Abelian theory illustrates the difference.

$$(\epsilon_0)_{\text{index}} = -\frac{1}{24} \neq -\frac{25}{384} = \epsilon_0$$

- Also, G = U(N) at large N: both exhibits N^3 scalings

$$(\epsilon_0)_{\text{index}} = -\frac{N^3}{6} \neq (\epsilon_0)_{\text{gravity}} = -\frac{5N^3}{24}$$

- AdS_7 dual of the index version? ("SUSY manifest" in holographic renormalization)

QFT on CP^2 x R

- A supersymmetric reduction along S^1/Z_K Hopf fiber of S^5/Z_K.

2π/K rotation with $\boxed{k \equiv j_1 + j_2 + j_3 + \frac{3}{2}(R_1 + R_2) + n(R_1 - R_2)}$

- The 5d theory is a low energy effective QFT for large K.
- Half-an-odd integer n: different twisted reduction, infinitely many 5d QFT
- Our interest: strong-coupling QFT at K=1. Instantons provide KK towers
- On-shell (Euclidean) action: can again make it off-shell w/ 2 SUSY

$$\begin{aligned} S \;=\; & \frac{1}{g_{YM}^2}\int d^5x\sqrt{g}\,\mathrm{tr}\Big[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi^I D^\mu\phi^I - \frac{i}{2}\lambda^\dagger\gamma^\mu D_\mu\lambda - \frac{1}{4}[\phi^I,\phi^J]^2 - \frac{i}{2}\lambda^\dagger\hat{\gamma}^I[\lambda,\phi^I] \\ & + \frac{2}{r^2}(\phi_I)^2 - \frac{1}{2r^2}(M_n\phi^I)^2 + \frac{1}{8r}\lambda^\dagger J_{\mu\nu}\gamma^{\mu\nu}\lambda - \frac{i}{2r}\lambda^\dagger M_n\lambda - \frac{i}{r}(3-2n)\,[\phi^1,\phi^2]\phi^3 - \frac{i}{r}(3+2n)[\phi^4,\phi^5]\phi^3 \\ & - \frac{i}{2r}\epsilon^{\mu\nu\lambda\rho\sigma}\left(A_\mu\partial_\nu A_\lambda - \frac{2i}{3}A_\mu A_\nu A_\lambda\right)J_{\rho\sigma}\Big] \end{aligned}$$

$$M_n \equiv \tfrac{3}{2}(R_1+R_2) + n(R_1-R_2)$$

Index on $CP^2 \times S^1$

- The index structure is manifest. Saddle point structure is more involved.

$$D^1 = D^2 = 0\,,\ \boxed{F^- = 2sJ}\,,\quad \phi + D = 4s$$

anti-self-dual instantons allowed on CP^2, proportional to Kahler 2-form

- After a localization calculation, one obtains a contour integral: U(N)

$$\boxed{\sum_{s_1,s_2,\cdots s_N=-\infty}^{\infty}} \frac{1}{|W_s|} \boxed{\oint \left[\frac{d\lambda_i}{2\pi}\right]} e^{\frac{\beta}{2}\sum_{i=1}^N s_i^2 - i\sum_i s_i\lambda_i} Z^{(1)}_{\rm pert} Z^{(1)}_{\rm inst} \cdot Z^{(2)}_{\rm pert} Z^{(2)}_{\rm inst} \cdot Z^{(3)}_{\rm pert} Z^{(3)}_{\rm inst}$$

sum over anti-self-dual fluxes

Integral over S^1 holonomies

- $Z_{\rm pert}$:

$$Z^{(1)}_{\rm pert} Z^{(2)}_{\rm pert} Z^{(3)}_{\rm pert} = \prod_{\alpha\in\Delta_+} \frac{\prod_{\sum_{i=1}^3 p_i = \alpha(s)} 2\sin\frac{\alpha(\lambda+i\sigma)+\beta p_i a_i}{2} \cdot \prod_{\sum_{i=1}^3 p_i=\alpha(s)-3} 2\sin\frac{\alpha(\lambda+i\sigma)+\beta p_i a_i}{2}}{\prod_{\sum_{i=1}^3 p_i=\alpha(s)-1} 2\sin\frac{\alpha(\lambda+i\sigma)+\beta p_i a_i-\beta\hat{m}}{2} \cdot \prod_{\sum_{i=1}^3 p_i=\alpha(s)-2} 2\sin\frac{\alpha(\lambda+i\sigma)+\beta p_i a_i+\beta\hat{m}}{2}}$$

- $Z_{\rm inst}$: product of 3 Nekrasov's $Z_{\rm inst}$ on $R^4 \times S^1$, with suitable identifications of paramters

The contour

- Naïve localization obscures the correct contour: Why is this an issue…?
- The local determinant calculus on $R^4 \times S^1$ partly forgot our original problem.
- Restoring the forgotten information: signs of BPS charges from 6d unitarity bounds

$$R_1 \geq 0\ ,\quad R_2 \geq 0\ ,\quad j_1 + j_2 \geq 0\ ,\quad j_2 + j_3 \geq 0\ ,\quad j_3 + j_1 \geq 0$$

- 3rd local determinant: BPS states weighted by

$$\epsilon_\pm = \frac{\epsilon_1 \pm \epsilon_2}{2}\ ,\quad \epsilon_1 = b - a\ ,\quad \epsilon_2 = c - a$$

$$e^{-\beta\epsilon_+ \overbrace{\boxed{(R_1+R_2+j_2+j_3)}}^{\geq 0}} e^{-\beta\epsilon_-(j_2-j_3)} e^{\beta(m+n(1+a))(R_1-R_2)}$$

- Expand all $Z^{(i)}_{\text{pert}}\ Z^{(i)}_{\text{inst}}$ in positive powers of $e^{-\beta\epsilon_+} = e^{\frac{3\beta a_i}{2}}$: constrain the contour
- Example: U(2) gauge theory with 1 self-dual instantons and flux (0,0) $\quad \zeta \equiv e^{-i(\lambda_1 - \lambda_2)}$

$$\oint \frac{d\zeta}{2\pi i \boxed{\zeta^2}} (\zeta-1)^2 e^{-\beta(1+a)} \frac{\sinh\frac{\beta(m-\frac{1}{2}+b)}{2}\sinh\frac{\beta(m-\frac{1}{2}+c)}{2}}{\sinh\frac{\beta(b-a)}{2}\sinh\frac{\beta(c-a)}{2}} \left[2 + \frac{\zeta(e^{-\frac{3\beta a}{2}} + e^{\frac{3\beta a}{2}})(e^{-\frac{3\beta a}{2}} + e^{\frac{3\beta a}{2}} - e^{\beta(m-\frac{1+a}{2})} - e^{-\beta(m-\frac{1+a}{2})})}{\boxed{(\zeta - e^{3\beta a})}(\zeta - e^{-3\beta a})}\right]$$

$$+(a,b,c \to b,c,a) + (a,b,c \to c,a,b)$$

- Vacuum comes with nonzero flux and negative "energy".

$$s = (s_1, s_2, \cdots, s_N) = (N-1, N-3, N-5, \cdots, -(N-1))$$

contributes to vacuum "energy" by $\quad \epsilon_0 \leftarrow -\dfrac{N(N^2-1)}{6}$

Some tests

(instanton number ~ energy)

- U(N) index agrees w/ large N gravity dual for k $\leq$ N: checked for N $\leq$ 3
- E.g. k = N = 3: (all multiplied by vacuum energy factor & q^3) $q = e^{-\beta},\ y_i = e^{-\beta a_i},\ y = e^{\beta(m-\frac{1}{2})}$

$$\begin{aligned}
Z_{(2,0,-2)} &= 3\left[y^2(y_1+y_2+y_3) + y(y_1^2+y_2^2+y_3^2) + y^{-1}(y_1+y_2+y_3) - (1 + \frac{y_1}{y_2} + \frac{y_2}{y_1} + \cdots) + y^3\right] \\
&\quad +6y\left[y(y_1+y_2+y_3) - (y_1^{-1}+y_2^{-1}+y_3^{-1}) + y^{-1} + y^2\right] + y^3 \\
Z_{(2,-1,-1)} + Z_{(1,1,-2)} &= -2y\left[y(y_1+y_2+y_3) - (y_1^{-1}+y_2^{-1}+y_3^{-1}) + y^{-1} + y^2\right] \\
&\quad -2y\left[y(y_1+y_2+y_3) - (y_1^{-1}+y_2^{-1}+y_3^{-1}) + y^{-1} + y^2\right] \\
&\quad -4y^3 - 4y^2(y_1+y_2+y_3) - 2y\left(y_1^2+y_2^2+y_3^2 - \frac{1}{y_1} - \frac{1}{y_2} - \frac{1}{y_3}\right) + 2\left(\frac{y_1}{y_2} + \frac{y_2}{y_3} + \cdots\right) - 2y^{-1}(y_1+y_2+y_3) \\
Z_{(1,0,-1)} &= y^3 + y^2(y_1+y_2+y_3) - y(y_1^{-1}+y_2^{-1}+y_3^{-1}) + 1
\end{aligned}$$

add all

$$Z_{SUGRA} = 3y^3 + 2y^2(y_1+y_2+y_3) + y\left(y_1^2+y_2^2+y_3^3 - \frac{1}{y_1} - \frac{1}{y_2} - \frac{1}{y_3}\right) - \left(\frac{y_1}{y_2} + \frac{y_2}{y_1} + \cdots\right) + y^{-1}(y_1+y_2+y_3)$$

- General U(N) index up to k$\leq$2. Large N agrees w/ SUGRA: e.g. k=2 example

Contributions from various anti-self-dual fluxes

$$\left\{\begin{aligned}
&q^2\left[\frac{N(N+1)}{2}y^2 + Ny(y_1+y_2+y_3) - N\left(y_1^{-1}+y_2^{-1}+y_3^{-1}\right) + Ny^{-1}\right] \\
&-(N-1)(N-2)q^2y^2 - (N-1)q^2\left[y^2 + y(y_1+y_2+y_3) - (y_1^{-1}+y_2^{-1}+y_3^{-1}) + y^{-1}\right] \\
&+\frac{(N-2)(N-3)}{2}q^2y^2 \qquad = q^2\left[2y^2 + y(y_1+y_2+y_3) - (y_1^{-1}+y_2^{-1}+y_3^{-1}) + y^{-1}\right]
\end{aligned}\right.$$

SUGRA index on AdS_7 x S^4

- New predictions of spectrum at k>N beyond SUGRA.

Concluding remarks

- So the QFT on S^5 and CP^2 x S^1 can be used to study the S^5 x S^1 index.

- Learn more about the (2,0) theory from this index.
- Study (2,0) theory on other manifolds with S^1 factor. (→ perhaps Sungjay Lee's talk)

- CP^2 x R approach will be useful to study 6d (1,0) SCFT's & their indices.

- Still, I feel we seemed to have obtained an inefficient expression for a simple result (division into 3 factors, etc.): improvement from different approaches…?

- $Z[S^5]$ could have different applications (with different field contents):

- Study of 5d SCFT on S^5: $\log Z[S^5] \sim N^{5/2}$ [Jafferis, Pufu] [Assel, Estes, Yamazaki] (2012)
- Relation to q-deformed 2d CFTs' correlators [Nieri, Pasquetti, Passerini] (2013)