Bose-Fermi Duality in Chern Simons theories with a single fundamental fermion and scalar

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- It has recently been realized that 3d Chern Simons gauge theories coupled to fundamental matter are 'solvable' in the large N limit [1,3,4,6] .
- This exact 'solution' reveals that the two theories

$$S = \int d^3x \left[i\varepsilon^{\mu\nu\rho} \frac{k_F}{4\pi} \mathrm{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) + \bar{\psi}\gamma^\mu \widetilde{D}_\mu \psi \right].$$

$$S = \int d^3x \left[i\varepsilon^{\mu\nu\rho} \frac{k_B}{4\pi} \mathrm{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) + \widetilde{D}_\mu \bar{\phi} \widetilde{D}^\mu \phi + \sigma \bar{\phi} \phi \right] \tag{1}$$

have identical spectrum of single trace operators, 3 point functions of these operators, and thermal partition functions in the large N limit, if we make the idenitifications

$$k_B = -k_F, \quad N_B = |k_F| - N_F$$

- This observation has led to the conjecture that the two theories on the previous slide are dual to each other, atleast in the large N limit [6] using [4]; see [1] for an initial suggestion .
- This conjecture is the first precise conjecture for a (nonsupersymmetric) bosonization duality in 3 dimensions. A three dimensional analogue the 2d Bosonization $\psi = e^{\phi}$? Sounds interesting and important
- Note that calculational evidence for this duality has so far been obtained only in the large N limit and for theories with fundamental matter fields. In this talk, based on [13] we will provide a viewpoint on this duality that connects it to a well known susy duality suggesting that it remains valid at finite N and perhaps also for other representations of the matter fields.

A Larger Class of Theories

The strategy we follow is to get new perspective on the bosonization duality is to recover it as a special case of a broader set of dualities. To that end we study the theory

$$S = \int d^3x \left[i\varepsilon^{\mu\nu\rho} \frac{k}{4\pi} \mathrm{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) + D_\mu \bar{\phi} D^\mu \phi + \bar{\psi}\gamma^\mu D_\mu \psi + m_B^2 \bar{\phi}\phi + m_F \bar{\psi}\psi + \frac{4\pi b_4}{k}(\bar{\phi}\phi)^2 + \frac{4\pi^2 x_6}{k^2}(\bar{\phi}\phi)^3 + \frac{4\pi x_4}{k}(\bar{\psi}\psi)(\bar{\phi}\phi) + \frac{2\pi y_4'}{k}(\bar{\psi}\phi)(\bar{\phi}\psi) + \frac{2\pi y_4''}{k}\left((\bar{\psi}\phi)(\bar{\psi}\phi) + (\bar{\phi}\psi)(\bar{\phi}\psi)\right) \right].$$

Parameters: m_B, m_F, b_4 (massive) x_6,x_4 y_4', y_4'' (dimensionless), + N, k (discrete). In the t' Hooft limit new effective continuous parameter $\lambda = \frac{N}{k}$.

N=2 point

This general class of theories has a very special point

$$m_F = m_B = b_4 = y_4'' = 0 \quad x_4 = x_6 = y_4' = 1 \tag{2}$$

$$S = \int d^3x \Big[i\varepsilon^{\mu\nu\rho} \frac{k}{4\pi} \mathrm{Tr}(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho) + \widetilde{D}_\mu \bar{\phi} \widetilde{D}^\mu \phi + \bar{\psi} \gamma^\mu \widetilde{D}_\mu \psi$$
$$+ \frac{4\pi}{k} (\bar{\psi}\psi)(\bar{\phi}\phi) + \frac{2\pi}{k} (\bar{\psi}\phi)(\bar{\phi}\psi) + \frac{4\pi^2}{k^2} (\bar{\phi}\phi)^3 \Big]$$

$\mathcal{N} = 2$ susy CS interacting with a single fundamental chiral multiplet; no superpotential. Superconformal, well studied. Gaiotto, Yin. Will be important later.

Thermal Partition Function

- In [13] we have computed the finite temperature thermal partition function of this general, non susy, class of theories on S^2 using the method developed in [1],[10],[11] We now describe the method and then state our results.
- Wish to compute path integral on $S^2 \times S^1$. We organize the computation as follows. First compute the effective action as a function of the holonomy and commuting 2d gauge fields. Next integrate over these fields.
- Effective action sum over bubble graphs; may be graded in powers of N (roughly by number of fundamental index loops). Graphs with no index loops - pure Chern Simons theory - contribute at $\mathcal{O}(N^2)$. Graphs at $\mathcal{O}(N)$ (roughly those with one fundamental index loop). Graphs at $\mathcal{O}(N^0)$ or lower; negligible in the large N limit.

Partition Function : S_{eff}

- The contribution to the effective action of graphs at order N (roughly one fundamental loop) takes the schematic form

$$S_{eff} = NT^2 \int d^2x \left(f_1(U) + R\frac{f_2(U)}{T^2} + \frac{f_3(U)}{T^2}(\partial U)^2 \right. \\ \left. \frac{f_4(U)}{T^4} f_{ij} f^{ij} + \dots \right) \tag{3}$$

[10]

- S_{eff} is of the same order as the contribution from pure Chern Simons theory only when $VT^2 = \mathcal{O}(N)$. For this reason we keep $\frac{T^2V}{N}$ fixed in the limit $N \to \infty$.

Partition Function: Relation to pure Chern Simons

- With this scaling

$$S_{eff}(U(x)) = NT^2 \int d^2x\, f_1(U) + \text{subleading}.$$

  To compute S_{eff} it is thus sufficient to set $f_{ij} = 0$, $U(x) = U$ and work in flat space.

- To complete the determination of the partiton function we must now add in the contribution of all graphs with no fermion loops and then integrate over all U and f_{ij}. These two steps together, however, simply amount to computing

$$Z = \langle e^{-S_{eff}[U]} \rangle_{N,k}$$

  where the expectation value $\langle\rangle$ is taken in *pure* Chern Simons theory of rank N and level k. [11]

Partition Function

- The formula of the previous slide may be used as a starting point for the actual evaluation of the thermal partition function on S^2. Procedure plus results (e.g a cap on the density of eigenvalues of the holonomy matrix, phase transitions, new phases, presumably new classes of black holes) outlined in review talk at Strings 2013 in Seoul last week (see video recording at website).
- Focus of this talk is duality. Not interested computing thermal partition function but in demonstrating that it is identical for pairs of distinct theories.
- Turns out that this can be achieved by a set of formal manipulation on the result of the previous slide

Level Rank Duality

- Recall that $S_{eff}(U)$ is a gauge invariant function of the holonomy. Every such function can be expanded as a linear sum of Wilson loops (in arbitrary representations of $U(N)$) around the time circle.
- Let $\chi_Y(U)$ denote the character for the representation of $U(N)$ with Young Tableaux Y. It is a classic result (level rank duality) of pure Chern Simons theory that

$$< \chi_Y(U) >_{N,k} = < \chi_{Y^T}(\tilde{U}) >_{\tilde{N},\tilde{k}}$$

$$\tilde{N} = |k| - N, \quad \tilde{k} = -k$$

- By the Schur formula

$$\chi_Y(U) = \sum_p \chi_Y(p) \prod_j (\mathrm{tr}\, U^j)^{n_j}$$

where Y is a Young Tableaux with n boxes, p is an element of the permutation group S_n, $\chi_Y(p)$ is the character of the permutation element p in the representation labeled by Y, and n_j is the number of cycles of length j in the conjugacy class of the permutation p. Now

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$$\chi_Y(p) = \mathrm{sgn}(p)\chi_{Y^T}(p), \quad \mathrm{sgn}(p) = \prod_j (-1)^{n_j+1}$$

so that

$$\chi_{Y^T}(\mathrm{tr}\, U^n) = \chi_Y\left((-1)^{n+1}\mathrm{tr}\, U^n\right)$$

and

$$Z = \langle e^{-S_{eff}[\mathrm{Tr} U^n]} \rangle_{N,k} = \langle e^{-S_{eff}[(-1)^{n+1}\mathrm{Tr}\tilde{U}^n]} \rangle_{|k|-N,-k}$$

- In the large N limit $S_{eff} = S_{eff}[\rho]$ where $\rho(\alpha)$ is the eigenvalue density function. Now

$$\rho(\alpha) = \frac{1}{2\pi}\left(1 + \sum_{n\neq 0} \rho_n e^{in\alpha}\right)$$

Using $N\rho_n = \mathrm{tr}\, U^n$, and the level rank duality map, it follows that $|\lambda|\rho_n = (-1)^{n+1}|\tilde{\lambda}|\tilde{\rho}_n$ so that

$$Z = \langle e^{-S_{eff}[\rho]}\rangle_{N,\lambda} = \langle e^{-S_{eff}[\tilde{\rho}]}\rangle_{\tilde{N},\tilde{\lambda}}$$

where [11]

$$\tilde{N} = |k| - N \quad \tilde{\lambda} = \lambda - \mathrm{sgn}(\lambda), \quad |\tilde{\lambda}|\tilde{\rho}(\alpha) + |\lambda|\rho(\alpha+\pi) = \frac{1}{2\pi}$$

Consequence of level rank duality

- Now consider a class of Chern Simons theories labeled by a collection of parameters p, in addition to N and λ. In the context of this talk p represents the collection of the 7 field theory coupling constants described earlier.
- Let us suppose that there exists a map, $\tilde{p} = \tilde{p}(p)$ under which

$$S_{eff}(N, \lambda, p, [\tilde{\rho}]) = S_{eff}(\tilde{N}, \tilde{\lambda}, \tilde{p}, [\rho])$$

  It then follows that the partition functions of the theories N, λ, p and $\tilde{N}$, $\tilde{\lambda}$, $\tilde{p}$ are identical. This result holds at all temperatures (of order N). In other words there is a simple way for duality to work at the level of thermal partition functions. This is how it actually works, as we now explain.

Partition Function: Determination of $S_{eff}(U)$

- As explained, $S_{eff}(U)$ refers to the sum of all bubble graphs at $\mathcal{O}(N)$ (roughly one fundamental index loop). Quite remarkably, it turns out to be possible to analytically sum these graphs, in an unusual 'lightcone' gauge, in the most general fundamental Chern Simons matter theory. [1]
- The result of this summation is given in terms of a 'free energy functional'

$$F_{eff}(c_S, c_F, \rho)$$

[7], [1], [11], [13] where c_B and c_F respectively are the thermal masses of the bosonic and fermionic fundamental fields. S_{eff} is determined by extremizing this free energy functional w.r.t the thermal masses c_F, c_B. The equations for thermal masses that follow from the extremization of the free energy functional are called gap equations.

E.g.: Pure Fermi Theory

$$F[\rho, c_F] =$$
$$\frac{1}{6\pi}\Bigg[|c_F|^3\frac{(\lambda - \mathrm{sgn}(\lambda))}{\lambda} + \frac{3}{2\lambda}c_F^2\hat{m}_F^{\mathrm{reg}} - \frac{1}{2\lambda}\frac{(\hat{m}_F^{\mathrm{reg}})^3}{(\lambda - \mathrm{sgn}(m_F^{\mathrm{reg}}))^2}$$
$$-3\int_{-\pi}^{\pi} d\alpha\rho(\alpha)\int_{|c_F|}^{\infty} dy\, y(\ln(1+e^{-y-i\alpha-\nu_F}) + \ln(1+e^{-y+i\alpha+\nu_F}))\Bigg].$$

Gap Equation

$$\mathcal{C} = \frac{\mathrm{sgn}(\lambda)\,|c_F| - \hat{m}_F^{\mathrm{reg}}}{2\lambda}$$

$$\mathcal{C} = \frac{1}{2}\int d\alpha\rho(\alpha)$$
$$\left(\log(2\cosh\frac{|c_F| + i\alpha + \nu_F}{2}) + \log(2\cosh\frac{|c_F| - i\alpha - \nu_F}{2})\right)$$

[1] [10] [11] [13]

Duality

In [13] we have computed $F_{eff}[c, \rho]$ for the most general renormalizable theory with one fundamental scalar and fermion. It turns out that

$$F_{eff}(c_F, c_B, \rho) = F_{\tilde{p}}(\tilde{c}_F, \tilde{c}_B, \tilde{\rho})$$

$$\begin{aligned}
&\tilde{c}_F = c_B, \quad \tilde{c}_B = c_F \\
&\tilde{N} = |k| - N, \quad \tilde{k} = -k \quad \tilde{x}_4 = \frac{1}{x_4}, \quad \tilde{m}_F = -\frac{m_F}{x_4} \\
&\tilde{x}_6 = 1 + \frac{1 - x_6}{x_4^3}, \quad \tilde{b}_4 = -\frac{1}{x_4^2}(b_4 + \frac{3}{4}\frac{1 - x_6}{x_4} m_F) \\
&\tilde{m}_B^2 = -\frac{1}{x_4} m_B^2 + \frac{3}{4}\frac{1 - x_6}{x_4^3} m_F^2 + \frac{2}{x_4^2} b_4 m_F
\end{aligned}$$

It follow that theories related by this parameter map have identical partition functions. Suggests duality.

Consistency checks

- $\tilde{p} = p$ for the special $\mathcal{N} = 2$ susy point. Reduces to the Giveon Kutasov level rank duality for the $\mathcal{N} = 2$ superconformal theory.
- Linearizing about the $\mathcal{N} = 2$ consistent with the operator transformation $\bar{\phi}\phi \to -\bar{\phi}\phi$ and $\bar{\psi}\psi \to -\bar{\psi}\psi$ and trace factorization.
- Proposed duality relations map the $\mathcal{N} = 1$ manifold of theories

$$\begin{aligned} & m_F = \mu, m_B^2 = \mu^2, b_4 = \mu w, x_4 = \frac{1+w}{2}, \\ & x_6 = w^2, y_4' = w, y_4'' = w - 1 \end{aligned} \tag{4}$$

onto themselves with Jain, S.M. , Yokoyama

$$w' = \frac{3-w}{1+w}, \quad \mu' = -\frac{2\mu}{1+w}.$$

- Results consistent with decoupling limits, see below.

Exact Pole Masses

- The procedure for the computation of the thermal free energy also yields the thermal masses, c_F and c_B. At $T = 0$ these are the exact pole masses the propagating fields. Explicitly
-

$$
\begin{aligned}
& c_{B,0}^2 = \lambda^2(1+3x_6)\frac{c_{B,0}^2}{4} - 2\lambda b_4|c_{B,0}| + \\
& x_4\frac{\lambda(-\lambda+2\,\text{sgn}(\mathrm{X}_0))}{(\lambda-\text{sgn}(\mathrm{X}_0))^2}(-m_F + x_4\lambda|c_{B,0}|)^2 + m_B^2 \\
& c_{F,0}^2 = \left(\frac{m_F - x_4|c_{B,0}|\lambda}{-\lambda+\text{sgn}(\mathrm{X}_0)}\right)^2 \\
& \mathrm{X}_0 = \lambda(|c_{F,0}| - x_4|c_{B,0}|) + m_F
\end{aligned}
\tag{5}
$$

Fermionic Decoupling Limit

- Consider the limit in which m_F, m_B and b_4, are scaled as

$$
\begin{aligned}
m_F &\to \infty \\
m_B^2 &= a_1 m_F^2 + a_2 m_F + a_3 \\
b_4 &= g_1 m_F + g_2
\end{aligned}
\qquad (6) \qquad \boxed{\texttt{fer}}
$$

  holding $x_4, x_6, a_1, a_2, a_3, g_1, g_2$ fixed.
- In the fermionic scaling limit we require $c_{B,0} \to \infty$ but $c_{F,0} = \tilde{c}_{F,0}$.
- Plugging into the gap equations we find $c_{F,0} = A m_F^2 + B m_F + C$ with A, B, C functions of a_i, g_i. Our condition $c_{F,0} = \tilde{c}_{F,0}$ requires $A = B = 0$ and $C = \tilde{c}_{F,0}$. Equations allow us to solve for a_1, a_2 and a_3 in terms of g_1, g_2 and $\tilde{c}_{F,0}$.

- Explicitly we find

$$a_1 = \frac{\lambda^2(8g_1x_4 - 3x_6 - 1) + 4}{4\lambda^2x_4^2}$$
$$a_2 = \frac{2g_2}{x_4} - \frac{m_F^{reg}\left(\lambda^2(4g_1x_4 - 3x_6 - 1) + 4\right)}{2\lambda^2x_4^2}$$
$$a_3 = (m_F^{reg})^2\left(x_4\frac{\lambda(\lambda - 2\mathrm{sgn}(X^{reg}))}{(\mathrm{sgn}(X^{reg}) - \lambda)^2} + \frac{\frac{4}{\lambda^2} - 3x_6 - 1}{4x_4^3}\right) - \frac{2g_2m_F^{reg}}{x_4} \tag{7}$$

with

$$\mathrm{sgn}(X^{reg}) = \mathrm{sgn}(\lambda). \tag{8}$$

Fermionic Scaling Limit

In this limit the thermal free energy turns out to be identical to that of the theory

$$S = \int d^3x \left[i\varepsilon^{\mu\nu\rho} \frac{k}{4\pi} \mathrm{Tr}\left(A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho\right) + \bar{\psi} \gamma^\mu \widetilde{D}_\mu \psi + m_F^{\mathrm{reg}} \bar{\psi}\psi \right]. \tag{9}$$

with

$$\tilde{c}_{F,0} = \frac{m_F^{\mathrm{reg}}}{\mathrm{sgn}(m_F^{\mathrm{reg}}) - \lambda}. \tag{10}$$

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In particular the free energy is indepedent of the parameters g_1, g_2, x_4, x_6 wich appear to be spurious in the fermionic scaling limit.

Bosonic Scaling Limit

- In a similar manner, the bosonic pole mass is held fixed at m_B^{cri} while the fermionic mass is scaled to infinity if we choose
- $$
\begin{aligned}
a_1 &= x_4 - \frac{x_4}{(\mathrm{sgn}(m_F) - \lambda)^2} \\
a_2 &= 2 m_B^{\mathrm{cri}} \lambda \left(x_4^2 \left(\frac{1}{(\mathrm{sgn}(m_F) - \lambda)^2} - 1 \right) + g_1 \right), \\
a_3 &= \frac{1}{4} (m_B^{\mathrm{cri}})^2 \left(\lambda^2 x_4^3 \left(4 - \frac{4}{(\mathrm{sgn}(m_F) - \lambda)^2} \right) - (3x_6 + 1)\lambda^2 + 4 \right) \\
&+ 2 m_B^{\mathrm{cri}} g_2 \lambda
\end{aligned}
\tag{11}
$$

$\boxed{\mathrm{bmb}}$

Bosonic Scaling Limit

- In this limit the thermal free energy turns out to be identical to that of the theory

$$
\begin{aligned}
S = \int d^3x \Big[& i\varepsilon^{\mu\nu\rho}\frac{k}{4\pi}\mathrm{Tr}(A_\mu\partial_\nu A_\rho - \frac{2i}{3}A_\mu A_\nu A_\rho) \\
& + \widetilde{D}_\mu\bar{\phi}\widetilde{D}^\mu\phi + \sigma(\bar{\phi}\phi + \frac{m_B^{\text{cri}}}{4\pi})\Big]
\end{aligned}
\tag{12}
$$

- Why do we get the critical boson theory? Hold m_B fixed by hand. However coefficient of $(\bar{\phi}\phi)^2$ which is also massive, generated dynamically. Generically of order m_F and so goes to infinity. But critical theory = regular theory perturbed by an infinite ϕ^4 term.

Duality of fermionic and bosonic scaling limits

- Not difficult to check that the fermionic and bosonic decoupling limits are interchanged by our duality map. This implies a duality between the purely fermionic and purely bosonic theories with N and k interchanged according to the level rank relations, while [10]

$$m_B^{\mathrm{cri}} = -\frac{1}{\lambda - \mathrm{sgn}(\lambda)} m_F^{\mathrm{reg}} \tag{13}$$

cri

- Of course the equality of the thermal free energy of the bosonic and fermionic theories can be checked indepenently; this was done in [10,11]. The important element of the enlarged setting for duality is that it connects the bosonization duality to Giveon Kutasov duality. This has important consequences, as we now argue.

Bosonization from Giveon Kutasov

- Giveon Kutasov: Exact duality between superconformal field theories that applies even at finite N.
- An exact duality between two conformal theories immediately implies the existence of a set of dualities between the families of quantum field theories obtained by perturbing these CFTs by marginal and relevant operators.
- In this talk we have confirmed this general expectation, and *determined the explicit form of the map between parameters at infinite N.* In particular we have demonstrated that duality interchanges the bosonic and fermionic decoupling limits.
- If we assume that the duality map at large but finite N approximates the infinite N uniformly in couplings, it follows that bosonic and fermionic and purely fermionic decoupling limits continue to be interchanged by duality. Very reasonable. Limits exist. What else can duality do? Follows that bosonization duality holds at finite N.

Comments

- Particles that continuously interpolate between being bosonic and fermionic are presumably generically anyonic. Make precise?
- Can we compute an exact S matrix of the scattering particles?
- As Giveon Kutasov duality applies to bifundamental theories. Likely similar nonsusy duality. Possible to check at small $\frac{M}{N}$?
- Note that the susy duality also interchanges bosons and fermions. Consequences?
- Would be nice to perform a thorough investigation of the physics of bosonization (e.g. Fermi Sea= Bose Condensate? Anyonic statistics?).
- More examples of nonsusy dualities obtained from mass deformations of conformal susy dualities (e.g. $N = 4$ YM?)
- Dual of bifundamental fermions at $M = N$?

Sounds like an exciting programme.