

基研研究会「量子情報物理学」2013年12月3-5日

# トポロジカル秩序と量子エンタングルメント

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東京大学 理学系研究科 物理学専攻

古川 俊輔



トポロジカル秩序：局所的秩序変数では捉えられない秩序？？  
実は共通の性質（「何かの構造」）がある

波動関数の構造を定量化して捉えることができないか？

## 第一部

- 「トポロジカル相」とは？「トポロジカル秩序相」とは？
- トポロジカル秩序と普通の秩序の違いとは？  
(縮約密度行列の視点)

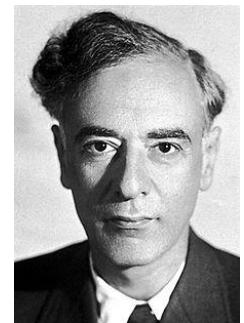
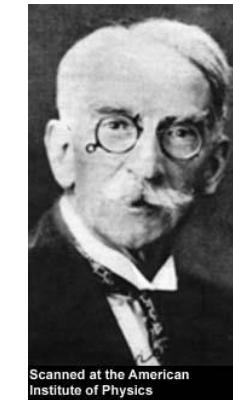
## 第二部

- エンタングルメント・エントロピーのトポロジカル秩序相への応用
- エンタングルメント・スペクトルのトポロジカル相への応用

# Conventional paradigm of phase transitions

## ➤ Symmetry breaking and order parameter (Weiss, Landau, ...)

- Magnets: breaking of spin rotational symmetry, magnetization
- Liquid-solid transition: translational symmetry breaking, density

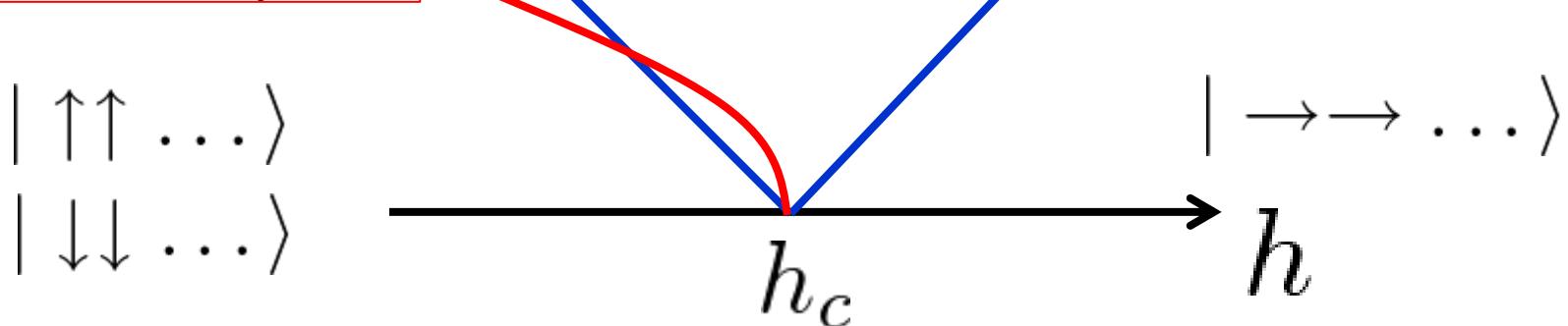


## ➤ Quantum phase transitions

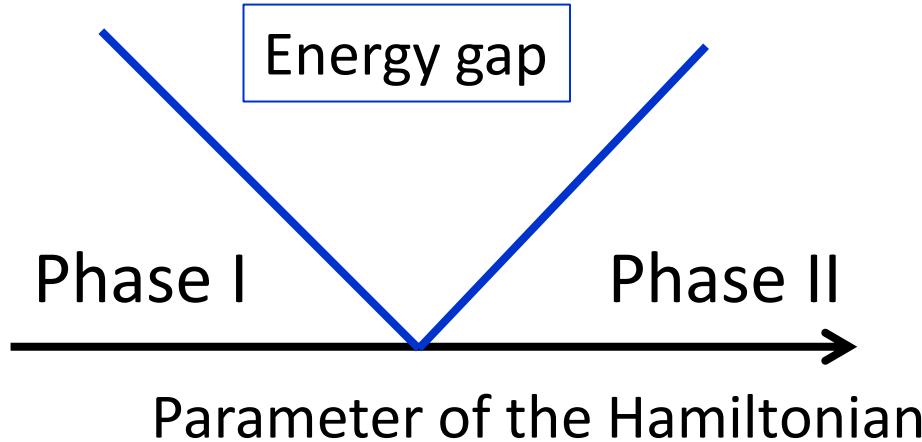
$$H = - \sum_{\langle ij \rangle} \sigma_i^z \sigma_j^z - h \sum_i \sigma_i^x$$

$$m = \frac{1}{N} \sum_i \langle \sigma_i^z \rangle$$

Energy gap



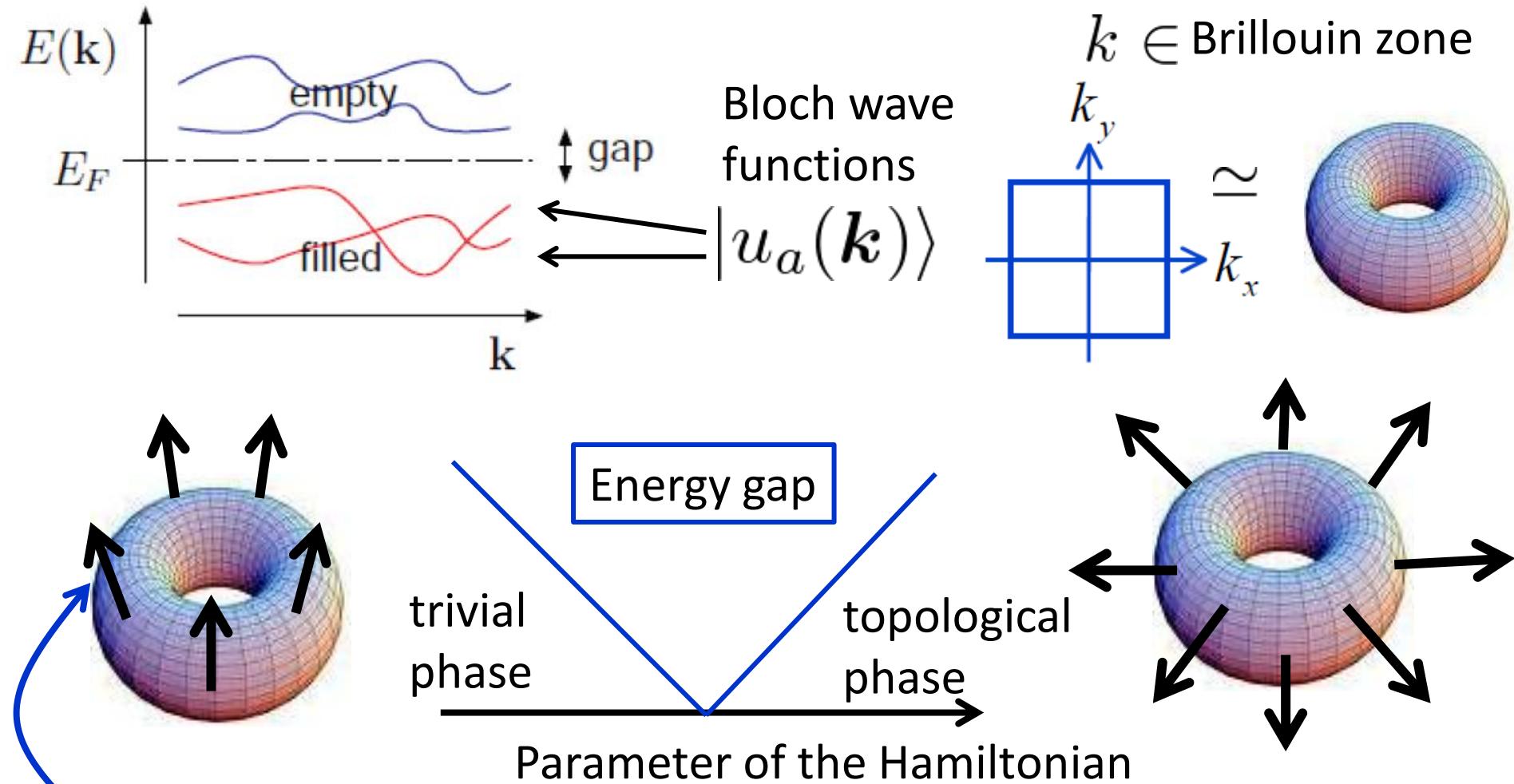
# Topological distinction of phases



- No local order parameter can distinguish the two phases.
- Different phases are distinguished by a certain topological property of the ground-state wave function
- Topological phases = Phases which are not smoothly connected to trivial phases

How to define?  
Product state, no winding, etc.

# Topological phases of band insulators



$$\mathbf{B}_a(\mathbf{k}) = \nabla_{\mathbf{k}} \times \mathbf{A}_a(\mathbf{k})$$

Berry curvature

$$\mathbf{A}_a(\mathbf{k}) = i\langle u_a(\mathbf{k}) | \nabla_{\mathbf{k}} | u_a(\mathbf{k}) \rangle$$

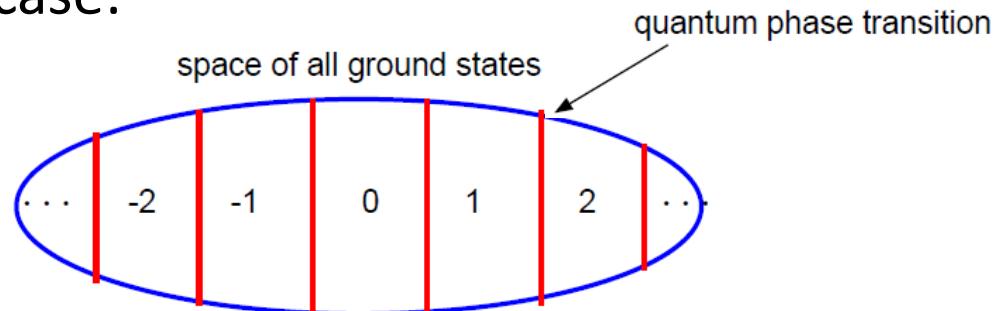
Berry connection

# Topological band insulators in two dimensions

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- Time-reversal-broken case:

$\mathbb{Z}$  classification

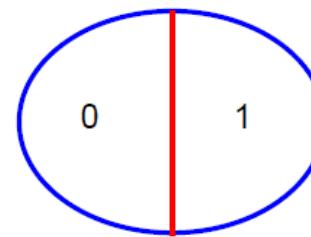


Chern number  $\leftrightarrow$  integer quantum Hall effects

$$\sigma_{xy} = \frac{e^2}{h} \times \text{Ch}$$

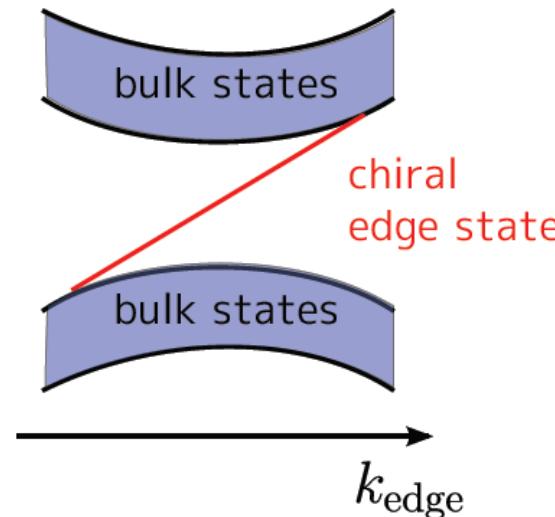
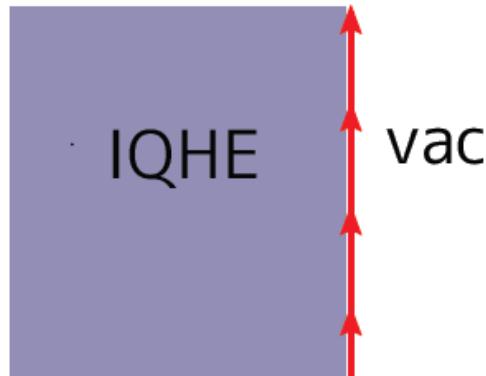
- Time-reversal-invariant case:

$\mathbb{Z}_2$  topological insulators

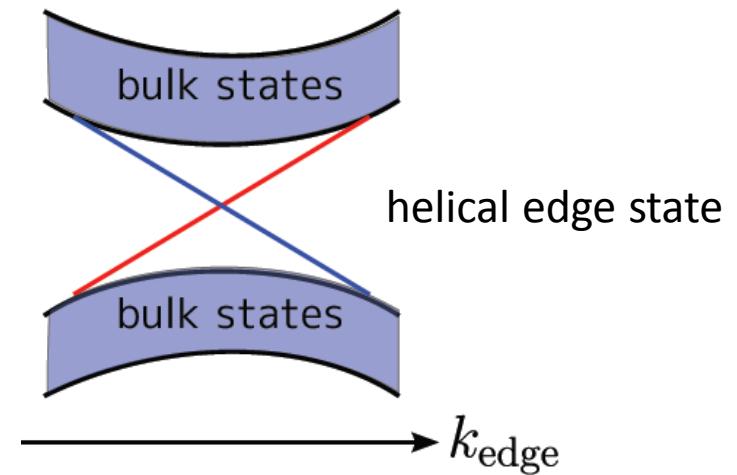
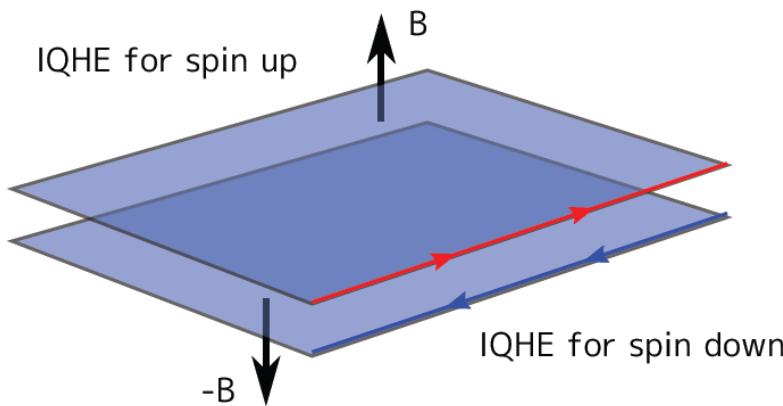


# Gapless edge modes

Chern number =  $n \rightarrow n$  gapless chiral (unidirectional) edge modes



$Z_2$  number = 0 or 1  $\rightarrow$  even or odd gapless helical edge modes



# Topologically ordered phases

➤ In strongly interacting systems, there are topological phases which cannot be understood even qualitatively by the band topology.

➤ Examples

Fractional quantum Hall (FQH) states: Laughlin, Moore-Read, etc.

Quantum spin liquids: toric code model, kagome antiferromagnet, etc.  
(\*still controversial\*)

➤ Fractional excitations

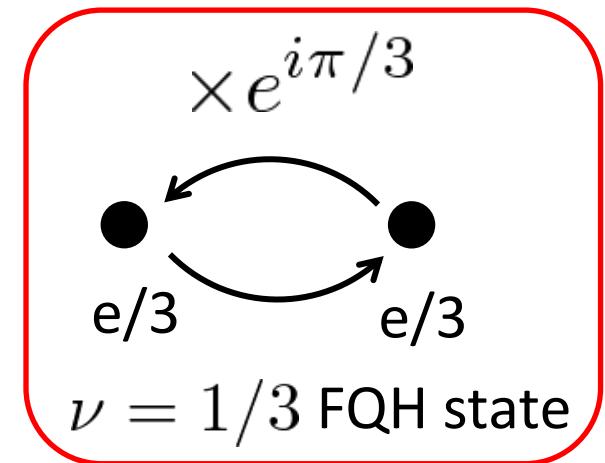
fractional charge and statistics

(some also show non-Abelian statistics)

➤ Ground-state degeneracy  $N$  that depends on the system's spatial topology

$$\nu = 1/3 \text{ FQH state}$$

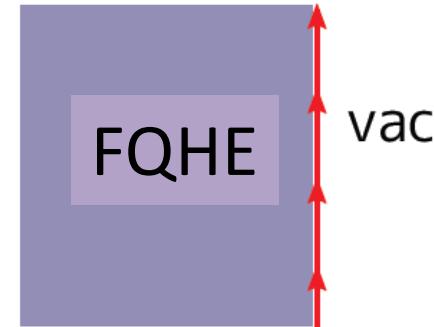
$$N = 3^g \quad g: \text{genus}$$



# Topologically ordered phases (cont'd)

- Strongly interacting gapless edge modes  
(in the chiral class)

e.g., chiral Tomonaga-Luttinger liquid



- Effective description: topological quantum field theory (TQFT)

$$S = - \int d^3x \frac{q}{4\pi} a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda} \quad \rightarrow \quad \left\{ \begin{array}{l} \text{Quantized Hall conductivity} \\ \text{Gapless edge modes} \end{array} \right.$$

Surface of 2+1D topological field theory  
→ 1+1D conformal field theory (Witten, 1989)

- How can we identify and classify topologically ordered phases starting from a microscopic Hamiltonian?

→ Certain entanglement structures of ground-state wave functions have turned out to be useful.

## Topological phases

### Integer topological phases

Topological band insulators

Haldane phase of spin chains  
(including AKLT model)

Symmetry-protected  
topological phases

### Fractional topological phases (topologically ordered phases)

Fractional quantum Hall states

Quantum spin liquids

Quasiparticles with unit charge

Quasiparticles with fractional charge

Unique ground state

Degenerate ground states  
on closed manifold

Topological entropy = 0

Topological entropy > 0

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- トポロジカル秩序と普通の秩序の違いとは？  
(縮約密度行列の視点)

SF, G. Misguich, M. Oshikawa, PRL,2006; J.Phys.C, 2007

## 第二部

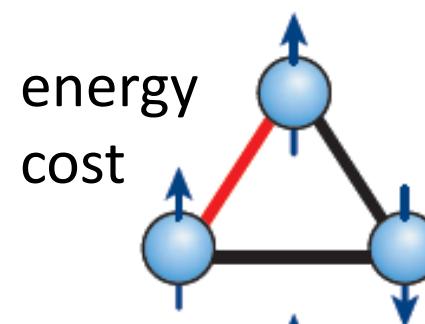
- エンタングルメント・エントロピーのトポロジカル秩序相への応用
- エンタングルメント・スペクトルのトポロジカル相への応用

# 量子スピン液体

- 絶対零度においても通常の秩序を一切示さない液体的状態  
 (局所的秩序変数で対称性の破れが検出できる秩序)  
**量子揺らぎが究極まで現れた状態**
- その探索はフラストレート磁性体研究における大きな目標の一つ

Review: L. Balents, *Nature* 464, 199-208 (2010)

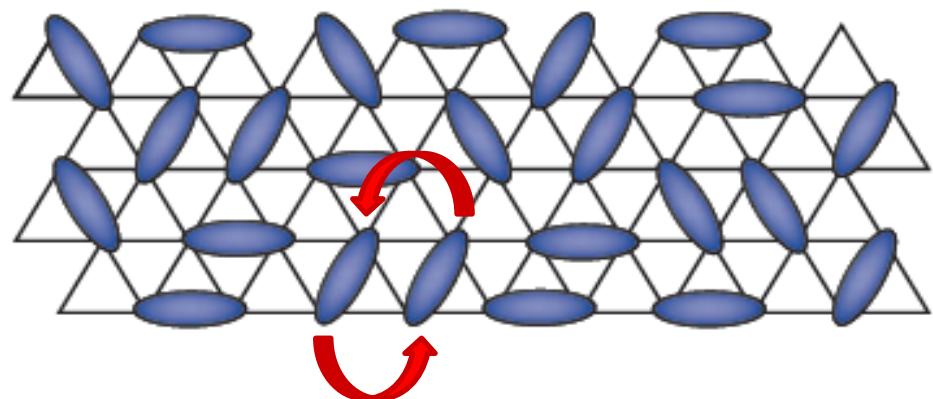
三角格子、カゴメ格子磁性体の合成が近年可能に。



フラストレーション:  
異なったボンドの相互作用を同時に満足できない

- Resonating valence bond (RVB)描像

P.W. Anderson,  
*Mater. Res. Bull.* 8, 153–160 (1973)

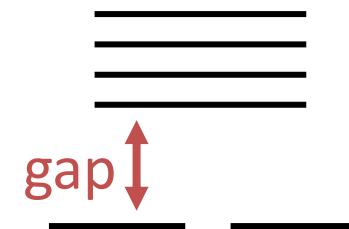


エネルギー・ギャップのある量子スピン液体を考える  
(トポロジカル秩序を持った量子スピン液体)。

Q. 与えられた模型に対し、量子スピン液体が  
形成されたことを示すにはどうすればよいか？

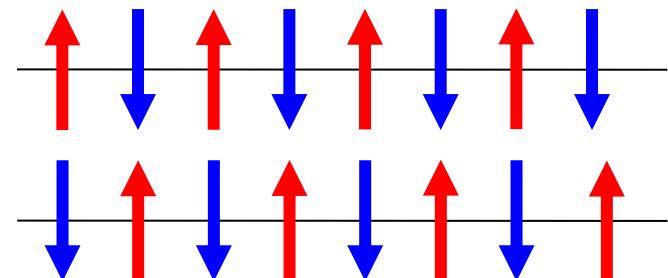
数値的に固有値、固有ベクトルを得る。

トーラス上で基底状態の縮退が見えるべき。



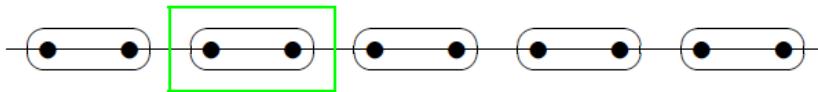
縮退の起源がトポロジカル秩序によることを示すには  
どうすれば良いか？

通常の秩序でも縮退を示す。  
例：ネール秩序



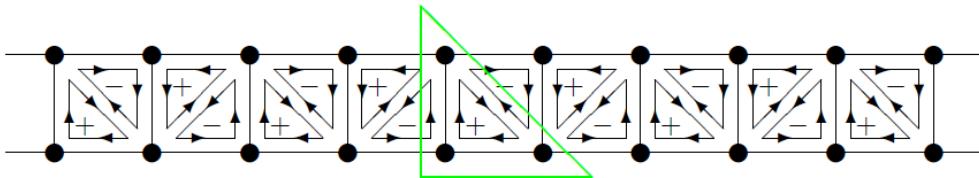
# フラストレート磁性体における様々な秩序

Dimer order  $S_j \cdot S_{j+1}$

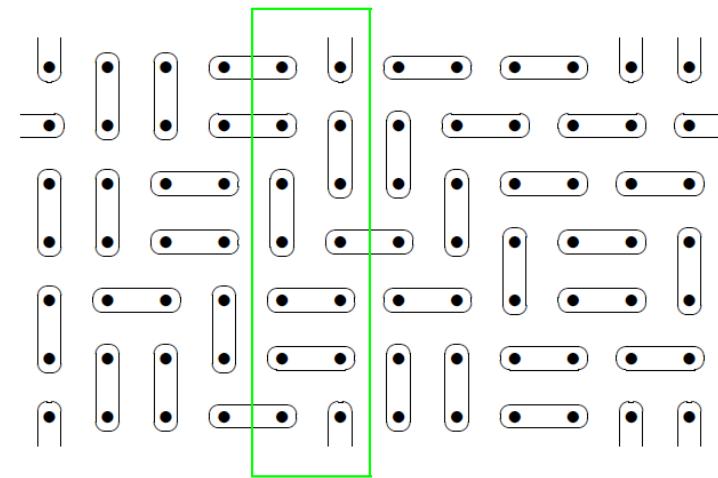


$$\text{---} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

Scalar-chiral order  $S_{1,j} \cdot (S_{2,j} \times S_{2,j+1})$



Resonating valence bond liquid



“topological order”

Q. 異なった秩序変数で特徴づけられる異なった秩序を  
系統的に取り扱う方法がないか？

# 秩序変数を決める方法

- ◆ Usual method: empirical method

$$\langle O_i O_j \rangle - \langle O_i \rangle \langle O_j \rangle = o_{\text{LRO}}^2 \neq 0 \quad (|i - j| \rightarrow \infty)$$

→ symmetry breaking in  $O_i$

General theorem  
 $o_{\text{LRO}} \leq o_{\text{SSB}}$   
Kaplan et al.  
Koma-Tasaki



What if the order parameter is beyond our imagination?

- ◆ Our proposal: direct determination from the ground states



# Setting of the problem

## Situation

We have obtained the low-energy spectrum and eigenstates of finite systems by numerical diagonalization.

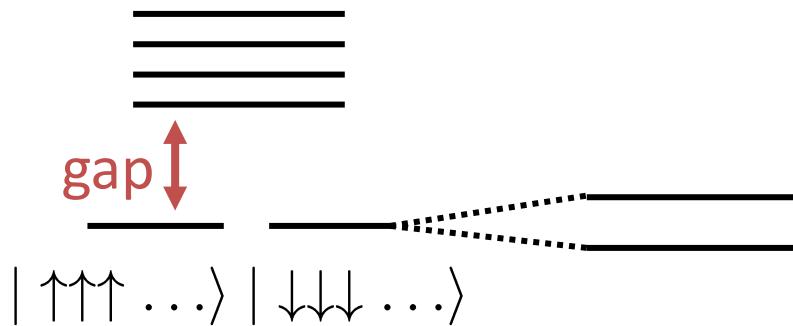
We have found finite-fold quasi-degenerate ground-states.

→ Breaking of some discrete symmetry is anticipated.

◆ Ising model with  
small perturbation

$$H = \sum_{i=1}^N [S_i^z S_{i+1}^z + \delta(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y)]$$

thermodynamic limit



finite-size system

$$\frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\dots\rangle - |\downarrow\downarrow\downarrow\dots\rangle)$$
$$\frac{1}{\sqrt{2}}(|\uparrow\uparrow\uparrow\dots\rangle + |\downarrow\downarrow\downarrow\dots\rangle)$$

Our method derives an order parameter directly from degenerate ground states (GS).

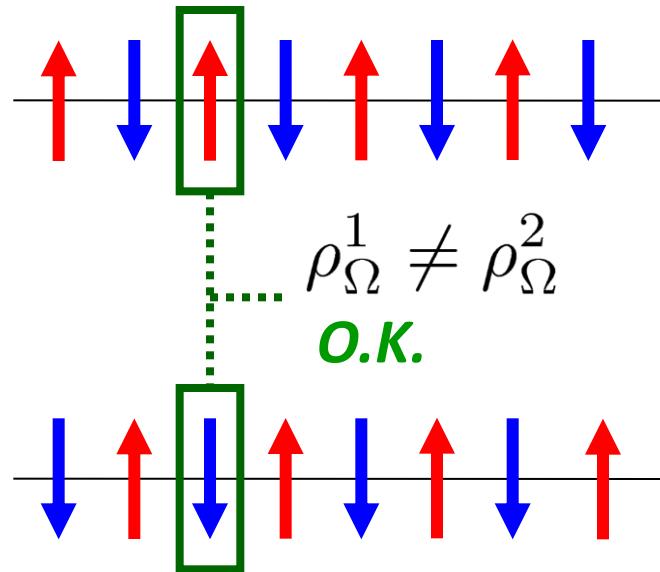
# Basic idea

We construct **symmetry-broken ground states**  $|\Psi_1\rangle, |\Psi_2\rangle$  as linear combinations of quasi-degenerate ground states.

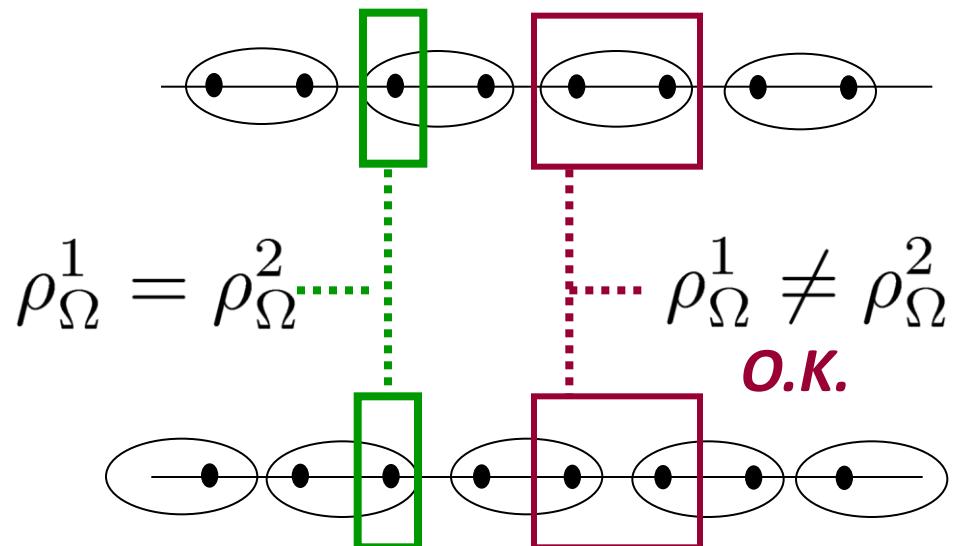
We examine in what area these states are distinguishable by looking at their **reduced density matrices (RDM)**.

$$\rho_{\Omega}^i = \text{Tr}_{\bar{\Omega}} |\Psi_i\rangle\langle\Psi_i| \quad (i = 1, 2). \quad \Omega: \text{subarea of the system}$$

Neel order



Dimer order



$$\rho_{\Omega}^1 \neq \rho_{\Omega}^2$$

O.K.

$$\rho_{\Omega}^1 = \rho_{\Omega}^2$$

$\rho_{\Omega}^1 \neq \rho_{\Omega}^2$

O.K.

# Measure of “Difference”

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$$\text{diff}(\rho_\Omega^1, \rho_\Omega^2) \equiv \max_{|\mathcal{O}_\Omega| \leq 1} \left| \underset{\Omega}{\text{Tr}}(\mathcal{O}_\Omega \rho_\Omega^1) - \underset{\Omega}{\text{Tr}}(\mathcal{O}_\Omega \rho_\Omega^2) \right|$$

$\mathcal{O}_\Omega$  : (variational) order parameter on  $\Omega$

$$= \sum_j |\lambda_j| \quad \{\lambda_j\} : \text{ eigenvalues of } \Delta\rho_\Omega \equiv \rho_\Omega^1 - \rho_\Omega^2$$

Maximum is achieved by the ***optimal order parameter***

$$\mathcal{O}_\Omega^{(\text{opt})} = \sum_j |j\rangle \operatorname{sgn} \lambda_j \langle j|$$

$\{|j\rangle\}$  : eigenvectors of  $\Delta\rho_\Omega \equiv \rho_\Omega^1 - \rho_\Omega^2$

## Properties

(a) Upper and lower bounds:  $0 \leq \text{diff}(\rho_\Omega^1, \rho_\Omega^2) \leq 2$

(b) Monotonicity:  $\Omega \subset \Lambda \Rightarrow \text{diff}(\rho_\Omega^1, \rho_\Omega^2) \leq \text{diff}(\rho_\Lambda^1, \rho_\Lambda^2)$

# Construction of symmetry-broken GSs

$\mathcal{H}$  : real in  $S^z$ -basis (“time-reversal” invariant)

2-fold finite-size (quasi-) degenerate GSs

$|\Phi_1\rangle$  and  $|\Phi_2\rangle$  : real (“time-reversal” invariant)

Symmetry-broken GSs: two possibilities

$$|\Psi_{1,2}\rangle = \frac{|\Phi_1\rangle \pm |\Phi_2\rangle}{\sqrt{2}} \quad \text{“time-reversal” invariant}$$

$$|\Psi_{1,2}\rangle = \frac{|\Phi_1\rangle \pm i|\Phi_2\rangle}{\sqrt{2}} \quad \text{“time-reversal” broken}$$

We can't know *a priori* which is the case;

so calculate both “diff1” and “diff2” separately

# Procedure

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## - case of two-fold GS degeneracy -

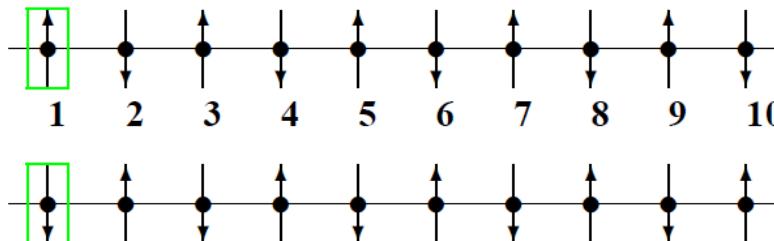
1. Exact diagonalization → degenerate GSs  $|\Phi_1\rangle, |\Phi_2\rangle$
  2. Construct symmetry-broken GSs  $|\Psi_1\rangle, |\Psi_2\rangle$   
as linear combinations of  $|\Phi_1\rangle, |\Phi_2\rangle$
- Two possibilities: “time-reversal” **invariant** / **broken**
3. Search for an appropriate area with “**diff1**” and “**diff2**”
  4. Construct the optimal order parameter

# Simple examples revisited

## ◆ Neel order

Area $A$	diff1	diff2
$\{1\}$	2	0
$\{1, 2\}$	2	0
$\{1, 2, 3\}$	2	0

$$|\Phi_{1(2)}\rangle = \frac{1}{\sqrt{2}} (| \uparrow\downarrow\uparrow\dots\rangle \pm | \downarrow\uparrow\downarrow\dots\rangle)$$



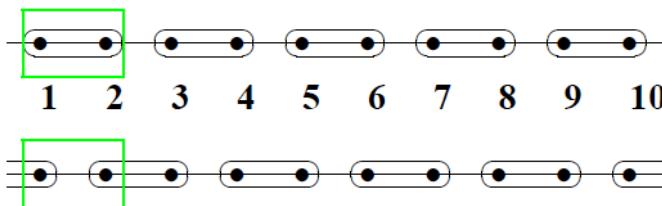
$$\Delta\rho_\Omega = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|$$

$$\mathcal{O}_\Omega^{(\text{opt})} = |\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow| = 2S_1^z$$

## ◆ Dimer order

$$|\Phi_{1(2)}\rangle = \frac{1}{\sqrt{2}} (|s_{12}\rangle|s_{34}\rangle\dots|s_{N-1,N}\rangle \pm |s_{23}\rangle|s_{45}\rangle\dots|s_{N,1}\rangle)$$

Area $A$	diff1	diff2
$\{1\}$	0	0
$\{1, 2\}$	$3/2 = 1.5$	0
$\{1, 2, 3\}$	$\sqrt{3} \cong 1.73$	0

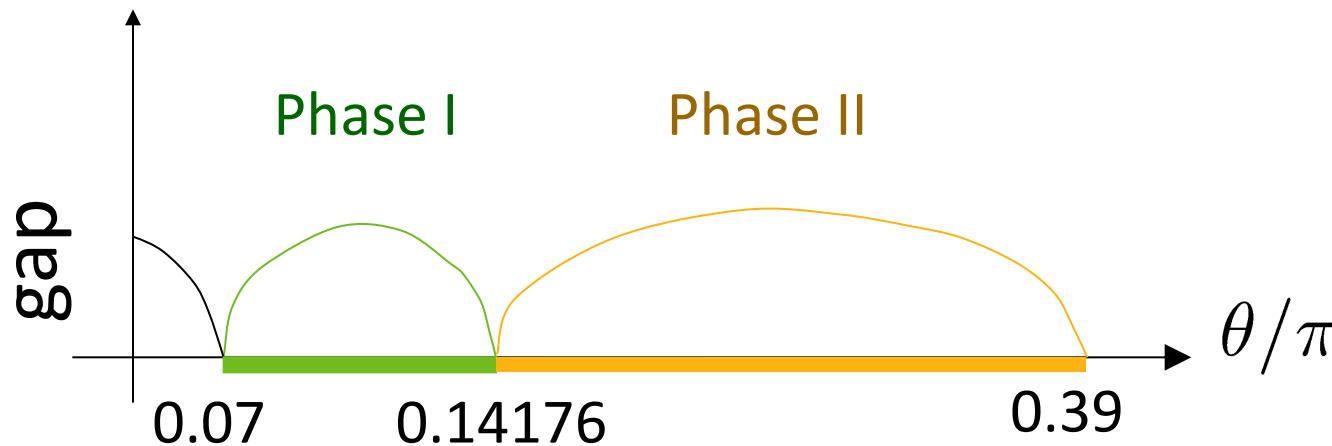
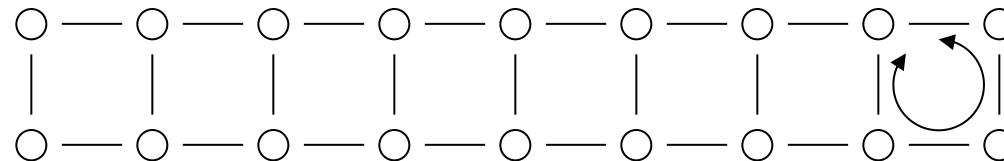


$$\Delta\rho_\Omega = \frac{3}{4} |s_{12}\rangle\langle s_{12}| - \frac{1}{4} \sum_\mu |t_{12}^\mu\rangle\langle t_{12}^\mu|$$

$$\mathcal{O}_{\{1,2\}}^{(\text{opt})} = |s_{12}\rangle\langle s_{12}| - \sum_\mu |t_{12}^\mu\rangle\langle t_{12}^\mu| = -2S_1 \cdot S_2 - 1/2$$

# Ladder spin model with ring-exchanges

$$\mathcal{H} = \cos \theta \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j + \sin \theta \sum_{\square} (P_4 + P_4^{-1})$$



2-fold degeneracy in both phases (wave-vectors  $k=0$  and  $k=\pi$ )

What are the corresponding order parameters?

Previous studies

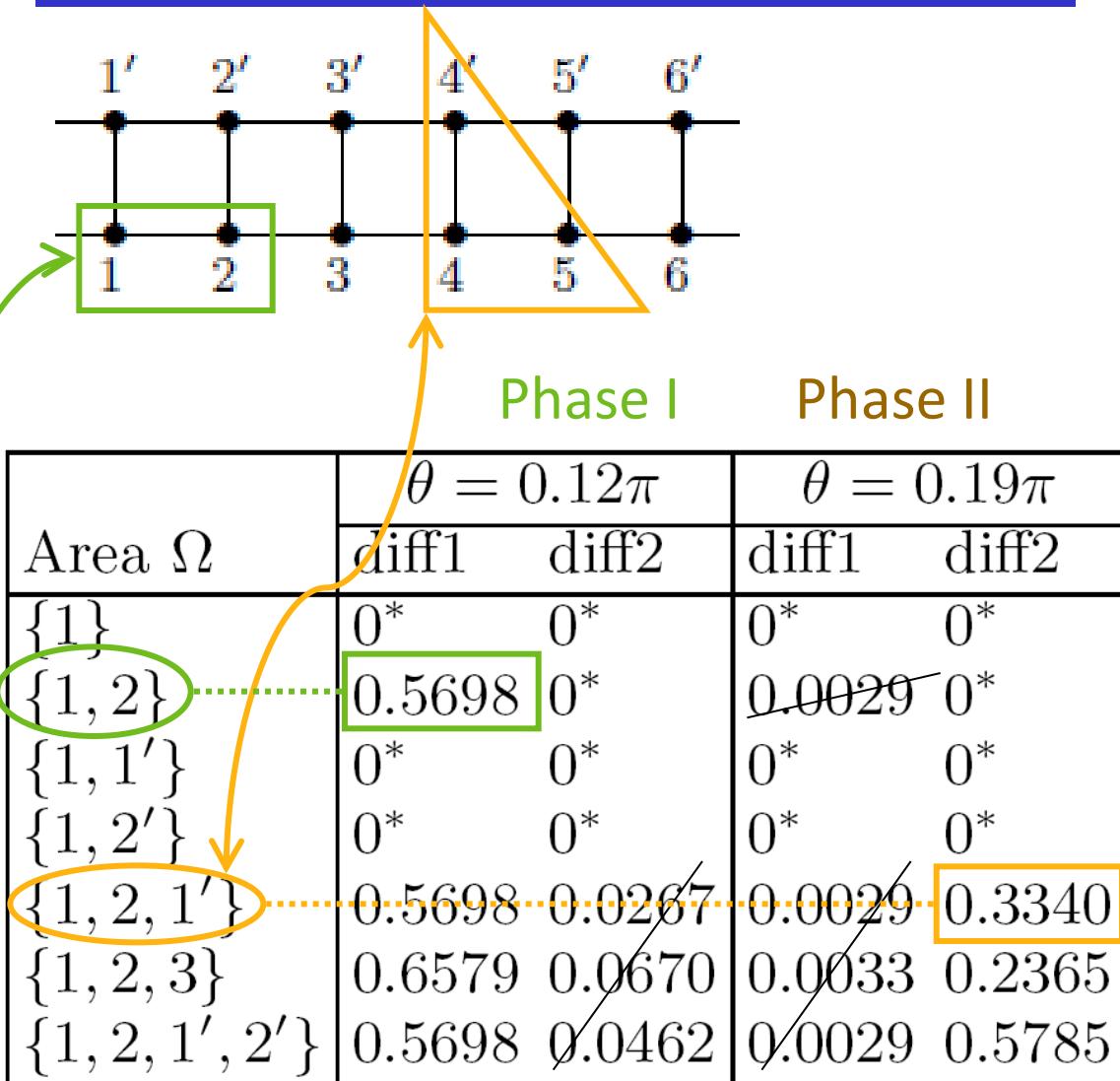
Laeuchli, Schmid, Troyer, PRB 67, 100409(R) (2003)

Hikihara, Momoi, Hu, PRL 90, 087204 (2003)

Phase I : staggered dimer order

Phase II: scalar-chiral order

# diff1 and diff2



Numerical results on 14x2 system - periodic BC

Phase I: leg dimer order

$$\Delta\rho_{\{1,2\}} \propto \vec{S}_1 \cdot \vec{S}_2$$

$$\mathcal{O}_{\{1,2\}}^{(\text{opt})} = 2\vec{S}_1 \cdot \vec{S}_2 + \frac{1}{2}$$

Phase II: scalar chiral order  
(broken time-reversal sym.)

$$\Delta\rho_{\{1,2,1'\}} \propto \vec{S}_1 \cdot (\vec{S}_2 \times \vec{S}_{1'})$$

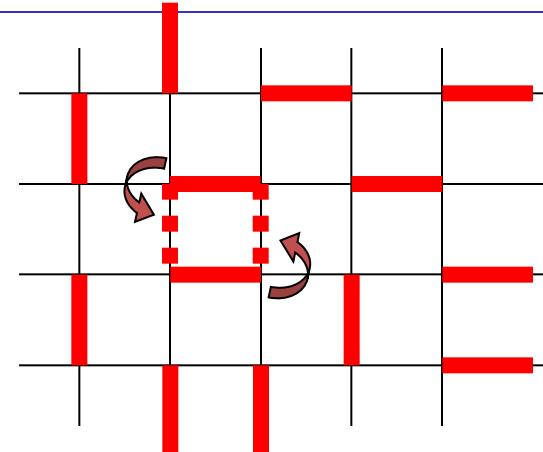
$$\mathcal{O}_{\{1,2,1'\}}^{(\text{opt})} = \frac{4}{\sqrt{3}} \vec{S}_1 \cdot (\vec{S}_2 \times \vec{S}_{1'})$$

*Reproduced known results !*

# Quantum dimer models

$$H = \sum_{\square} \left[ -t \left( \left| \begin{array}{c|c} \bullet & \bullet \\ \bullet & \bullet \end{array} \right\rangle \langle \begin{array}{c|c} \bullet & \bullet \\ \bullet & \bullet \end{array} \right| + \text{h.c.} \right) \right. \\ \left. + v \left( \left| \begin{array}{c|c} \bullet & \bullet \\ \bullet & \bullet \end{array} \right\rangle \langle \begin{array}{c|c} \bullet & \bullet \\ \bullet & \bullet \end{array} \right| + \left| \begin{array}{c|c} \bullet & \bullet \\ \bullet & \bullet \end{array} \right\rangle \langle \begin{array}{c|c} \bullet & \bullet \\ \bullet & \bullet \end{array} \right| \right) \right]$$

Rokhsar-Kivelson, 1988



Effective description of spin models restricted to the valence-bond subspace

Exact RVB-type GSs at  $t=v$  (Rokhsar-Kivelson point):

$$|\text{RK}\rangle = \frac{1}{\sqrt{|\mathcal{E}|}} \sum_{c \in \mathcal{E}} |c\rangle \quad \text{Rokhsar-Kivelson wave fn.}$$

RVB liquid phase on triangular lattice, kagome lattice,...

# Topological sectors

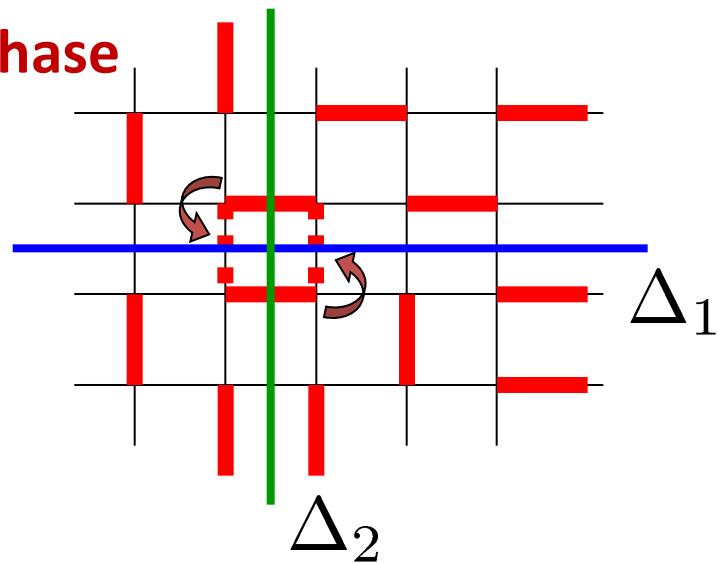
The parity of the number of dimers crossing a cut is a conserved quantity.

$\mathcal{E}$ : set of all the dimer coverings  $\rightarrow \mathcal{E}^p$ ,  $p = ++, +-, -+, --$

The spectrum can be determined separately in each sector.

## 4-fold degenerate ground states in RVB phase

$$\text{RK point: } |\Phi_p\rangle = \frac{1}{\sqrt{|\mathcal{E}^p|}} \sum_{c \in \mathcal{E}^p} |c\rangle$$



GS degeneracy depending on the topology:

2-fold on the cylinder, 4-fold on the torus, ...

What is the order parameter?

# Method for multiple GS degeneracy

Rather than calculating diff1, diff2, diff3, etc.,  
we maximize “diff” over the GS manifold.

$$D_\Omega = \max_{|\Psi\rangle} \text{diff} \left( \rho_\Omega, \rho_\Omega^{(\text{ref})} \right)$$

$$\rho_\Omega^{(\text{ref})} \equiv \frac{1}{q} \sum_i \text{Tr}_{\bar{\Omega}} |\Phi_i\rangle\langle\Phi_i| \quad : \text{reference} \quad \left[ \begin{array}{l} \text{RDM averaged} \\ \text{over the GS manifold} \end{array} \right]$$

$$\rho_\Omega \equiv \text{Tr}_{\bar{\Omega}} |\Psi\rangle\langle\Psi|$$

$$|\Psi\rangle = \sum_i \alpha_i |\Phi_i\rangle \quad : \text{a state in the GS manifold}$$

$$\left. \quad \left[ |\Phi_i\rangle \ (i = 1, \dots, q) \ : \text{degenerate GSs} \right] \right.$$

The existence of an order parameter is signaled by  $D_\Omega > 0$

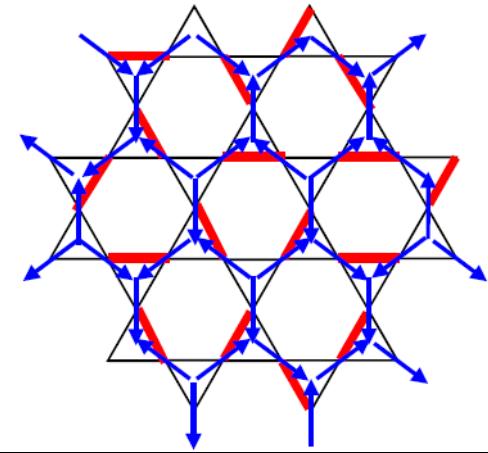
# Solvable QDM on the kagome lattice

$$|\Phi_p\rangle = \frac{1}{\sqrt{|\mathcal{E}^p|}} \sum_{c \in \mathcal{E}^p} |c\rangle \quad \text{introduced by Misguich et al., PRL,02}$$

zero dimer-dimer correlation length  
(due to the arrow representation)

→ no finite-size effect

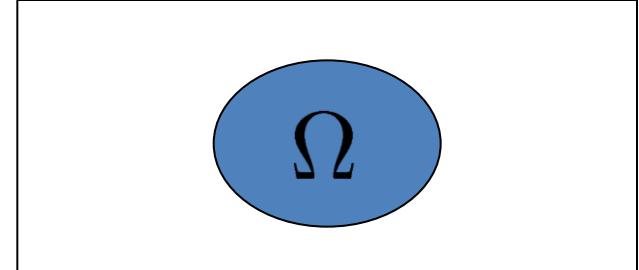
Rigorous argument is possible!



◆ Local (finite) area:

$$D_\Omega = 0$$

No local order parameter!

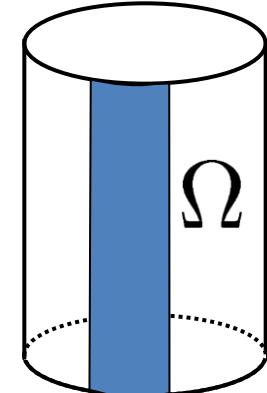


◆ Non-local area (extending over the system)

$$D_\Omega > 1$$

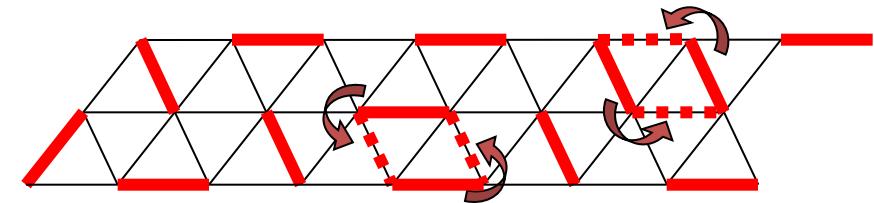
Existence of a non-local order parameter

(Similar result: Ioffe & Feigel'man, PRB, 2002)

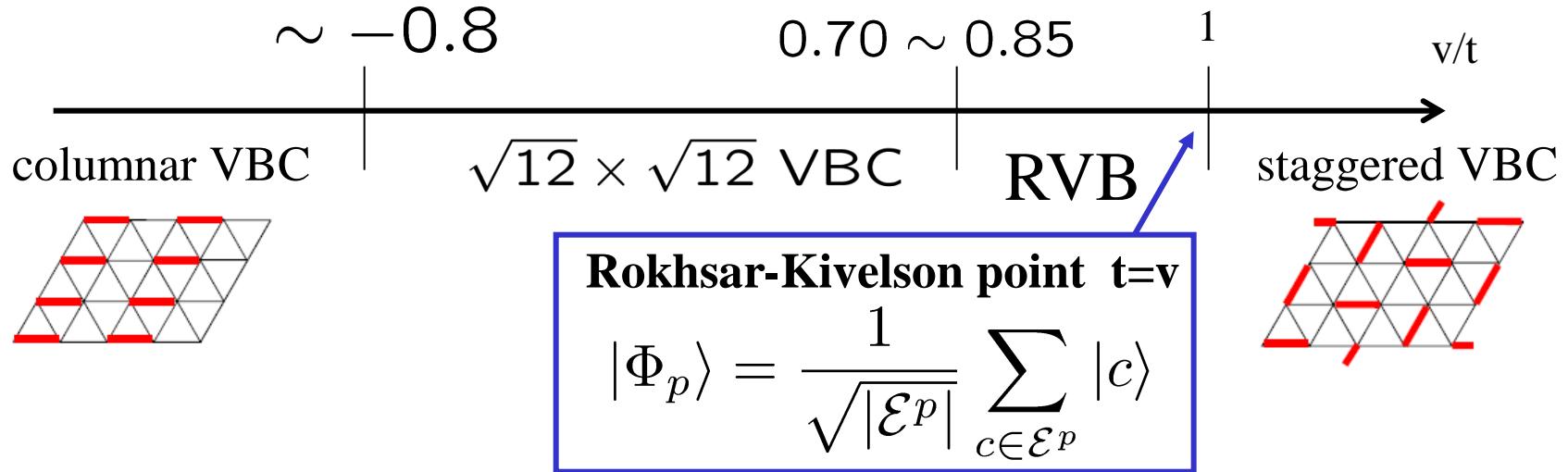


# QDM on the triangular lattice

$$H = \sum [-t(|\underline{\text{--}}\rangle\langle/\!/\!| + h.c.) + v(|\underline{\text{--}}\rangle\langle\underline{\text{--}}| + |\!/\!/\!\rangle\langle/\!/\!|)]$$



QMC: Moessner and Sondhi, PRL 86 (2001); Green fn. MC: Ralko et al., PRB 71 (2005)



**Rokhsar-Kivelson point  $t=v$**

$$|\Phi_p\rangle = \frac{1}{\sqrt{|\mathcal{E}^p|}} \sum_{c \in \mathcal{E}^p} |c\rangle$$

Finite dimer-dimer correlation length

→ Finite-size effects arise.

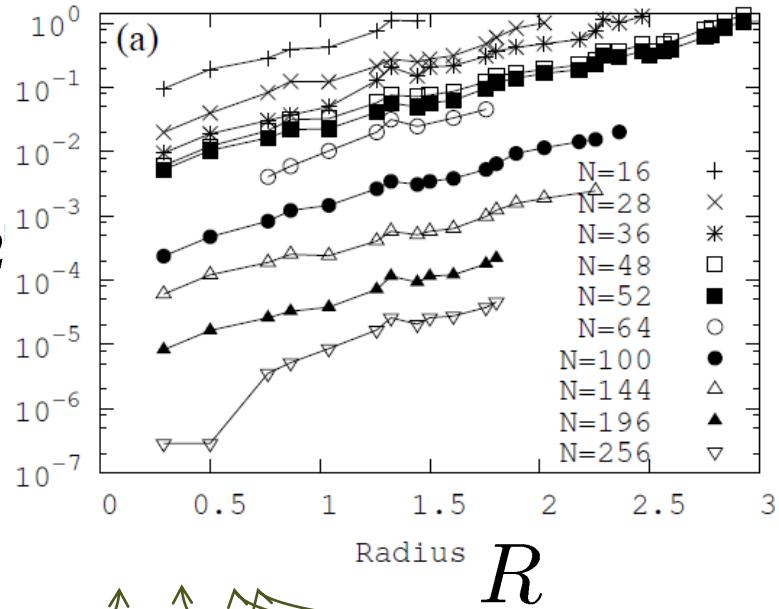
A systematic treatment of finite-size effects is necessary.

# QDM on the triangular lattice: result at RK point

Numerical methods:

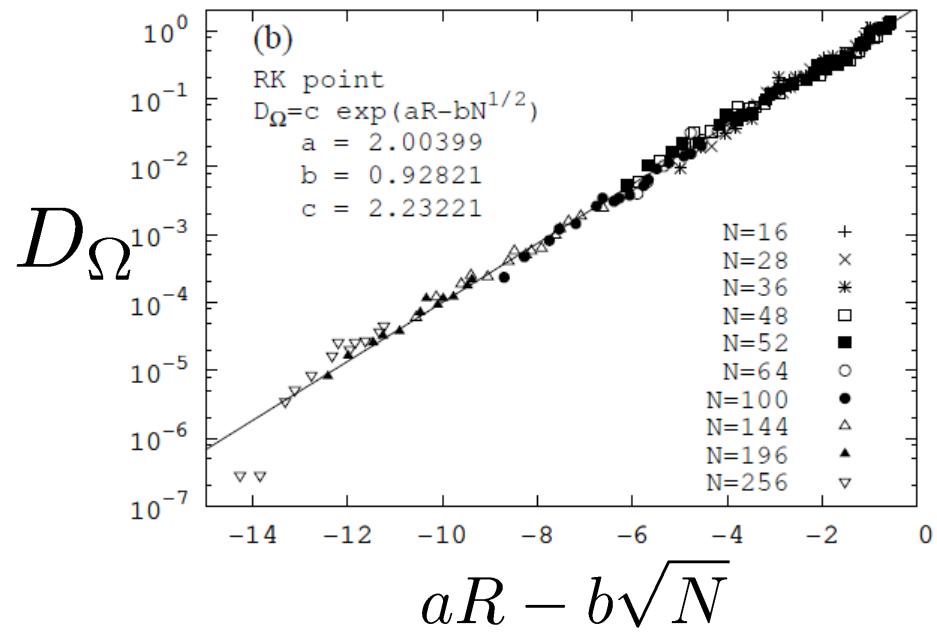
direct enumeration of dimer coverings (up to  $N=52$ )  
 evaluation of Pfaffians (up to  $N=256$ )

$D_\Omega$



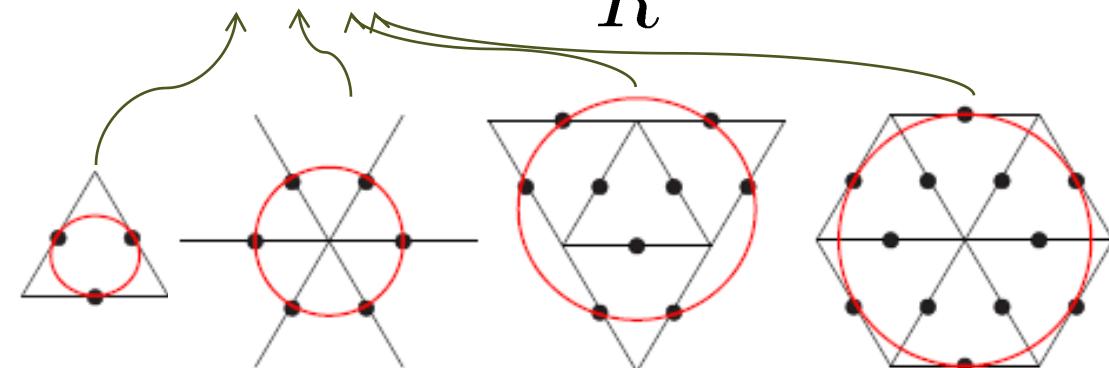
Simple exponential fitting

$$D_\Omega = c \exp(aR - b\sqrt{N})$$

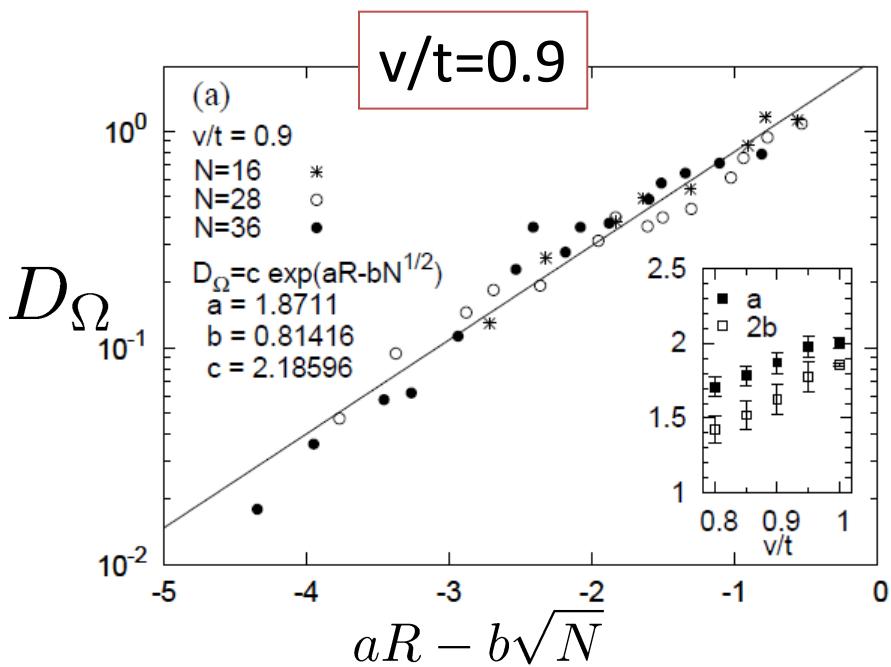


$D_\Omega$  goes to zero exponentially with  $N^{1/2}$  for any fixed  $R$ .

→ **Absence of any local order parameter**



# Away from the RK point



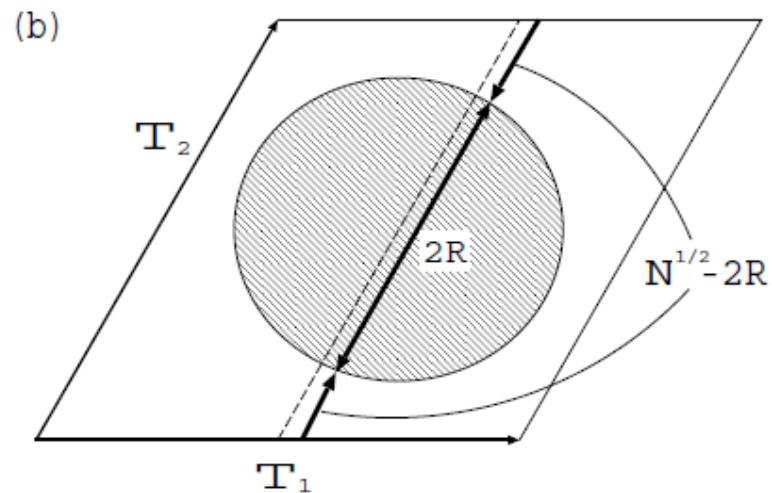
$$\begin{aligned} D_\Omega &= c \exp(aR - b\sqrt{N}) \\ &\approx c \exp[-b(\sqrt{N} - 2R)] \end{aligned}$$

$D_\Omega$  goes to zero exponentially with  $N^{1/2}$  for any fixed  $R$ .

→ Absence of any local order parameter

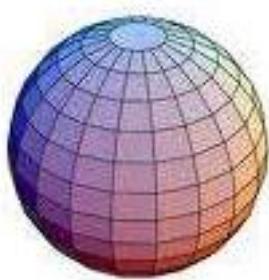
Other  $v/t$ : exact diagonalization  
(up to  $N=36$ )

$a \approx 2b$  holds in the RVB phase.

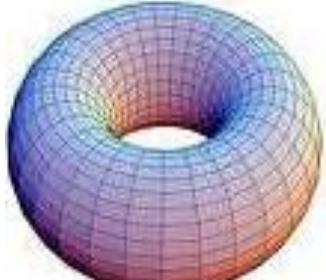


# まとめ : トポロジカル秩序とは ?

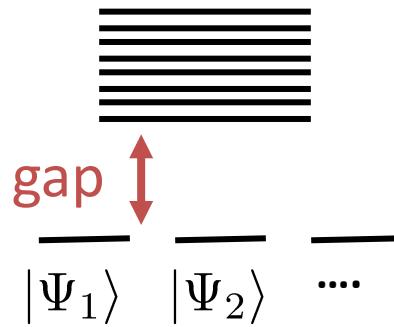
- 系の形状のトポロジーに依存した基底状態の縮退度



$g=0$



$g=1$



$\nu = 1/3$  分数量子ホール状態: degeneracy =  $3^g$

$\mathbb{Z}_2$  スピン液体(QDM, toric code): degeneracy =  $4^g$

- 縮退した基底状態は局所的に区別がつかない

- 縮約密度行列間の「距離」を用いて定量的に議論

$$D_\Omega = \max_{|\Psi\rangle} \text{diff} \left( \rho_\Omega, \rho_\Omega^{(\text{ref})} \right) \quad \text{diff}(\rho_\Omega^1, \rho_\Omega^2) \equiv \max_{|\mathcal{O}_\Omega| \leq 1} \left| \text{Tr}_\Omega(\mathcal{O}_\Omega \rho_\Omega^1) - \text{Tr}_\Omega(\mathcal{O}_\Omega \rho_\Omega^2) \right|$$

SF, Misguich, Oshikawa, PRL, 2006; J.Phys.C, 2007

## 第一部

- 「トポロジカル相」とは？「トポロジカル秩序相」とは？
- トポロジカル秩序と普通の秩序の違いとは？  
(縮約密度行列の視点)

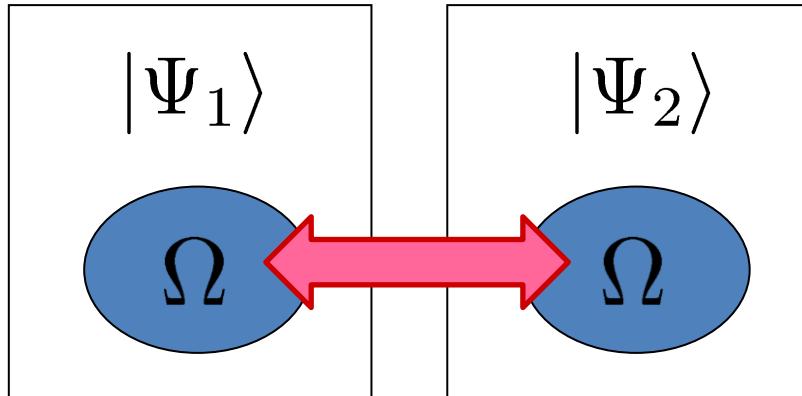
## 第二部

- エンタングルメント・エントロピーのトポロジカル秩序相への応用
- エンタングルメント・スペクトルのトポロジカル相への応用

# Preskill の提案

Quantum information and physics: some future directions, J.Mod.Opt. 47 (2000) 127-137

## トポロジカル秩序相の縮退した基底状態



局所的に区別がつかない  
非局所的領域で初めて区別がつく  
(Global encoding of information)

量子情報にも似た状況

数人集まって初めて  
秘密がわかる

Quantum secret sharing

Quantum error correction

量子情報の考え方 ⇒ 物性の多体基底状態を特徴づける新しい方法

参考: 臨界現象に関する提案

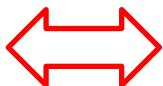
共形場理論のc-theorem, g-theoremを繰り込み群での情報のロスとして  
解釈できないか?

# What is entanglement?

Structure of a quantum state which cannot be represented as a product form

- $|\Psi\rangle = |\Psi_A\rangle|\Psi_B\rangle$

No entanglement between A and B

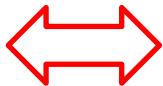


Reduced density matrix on A is a pure state.

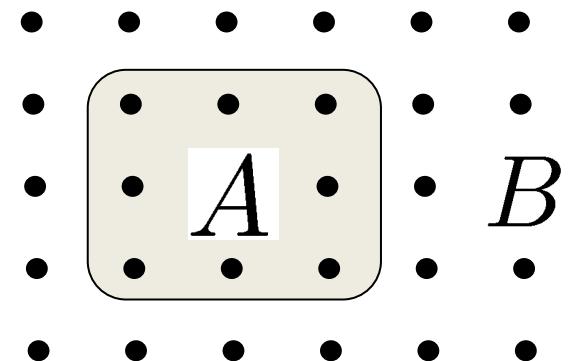
$$\rho_A = \text{Tr}_B |\Psi\rangle\langle\Psi| = |\Psi_A\rangle\langle\Psi_A|$$

- $|\Psi\rangle \neq |\Psi_A\rangle|\Psi_B\rangle$

There is entanglement between A and B



Reduced density matrix on A is a mixed state.



# How to quantify entanglement ?

→ Measure how mixed the reduced density matrix is.

**Entanglement entropy  
(von Neumann entropy)**

$$S_A = -\text{Tr } \rho_A \log \rho_A \quad (= S_B)$$
$$= - \sum_i p_i \log p_i \quad \{p_i\}: \text{eigenvalues of } \rho_A$$

- product state  $\longrightarrow$  pure state  $p_i = 1, 0$   
 $|\Psi\rangle = |00\rangle$   $\rho_A = |0\rangle\langle 0|$   $S_A = 0$
- entangled state  $\longrightarrow$  mixed state  $p_i = \frac{1}{2}, \frac{1}{2}$   
 $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$   $\rho_A = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$   $S_A = \log 2$

# Scaling of entanglement entropy

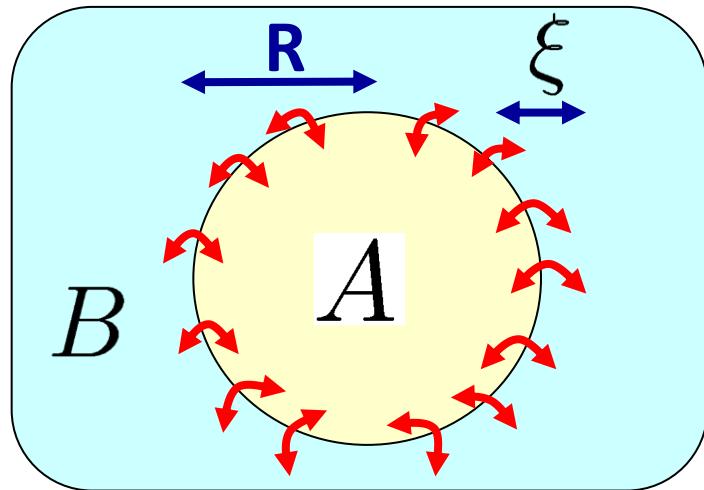
- If the system has only short-range correlations,

$$S_A \propto (\text{boundary size of } A)$$

*"boundary law" (or "area law")*

Srednicki, PRL, 1993

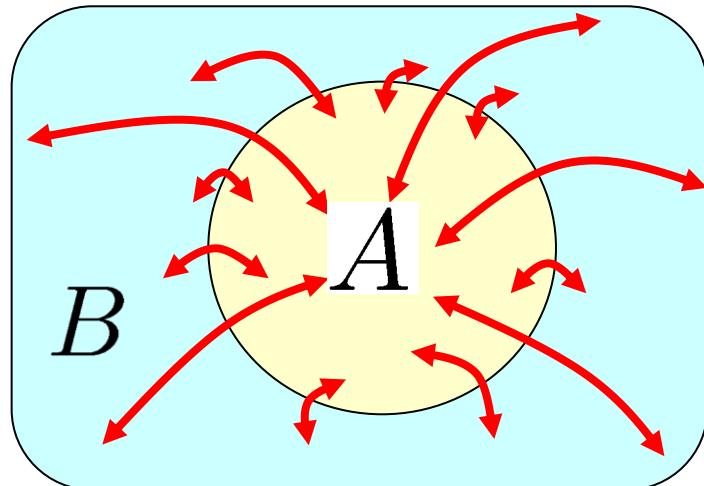
Wolf,Verstraete,Hastings,Cirac,PRL,2008



- Deviation from boundary law

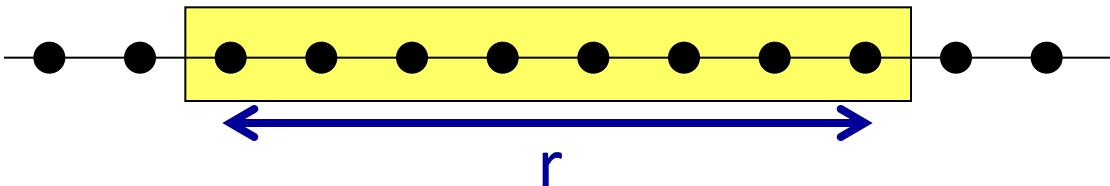
→ Signal of certain long-range or non-local correlations

Useful applications to critical systems, topologically ordered phases, etc.



# One-dimension systems

A



Holzhey,Larsen,& Wilczek,  
Nucl.Phys.B,1994  
Vidal, Latorre, Rico, & Kitaev,  
PRL, 2003  
Calabrese & Cardy,  
J.Stat.Mech,2004

- Gapped (non-critical) system

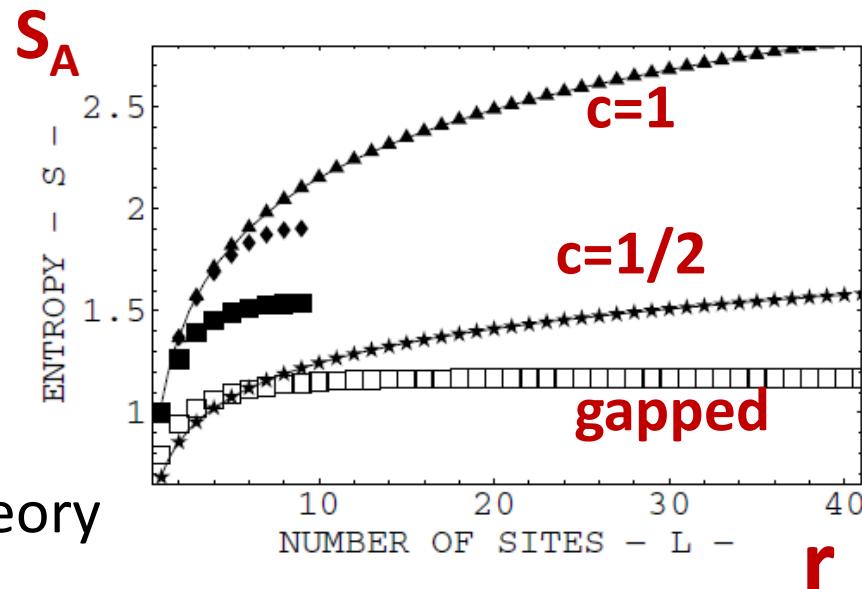
$$S_A \rightarrow \text{const. } (r \rightarrow \infty)$$

- Critical system

$$S_A \simeq \frac{c}{3} \log r + s_1$$

c: central charge of conformal field theory

$\simeq$  number of gapless modes

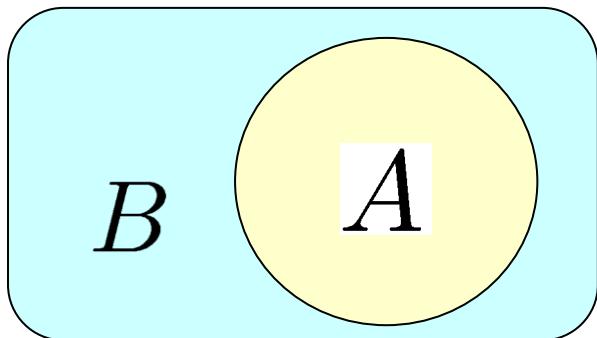


Underlying field theory



generic structure  
of ground state

# Entanglement entropy detects intrinsic topological order!



Kitaev & Preskill, PRL 96, 110404 (2006)

Levin & Wen, PRL 96, 110405 (2006)

(Also, Hamma et al., PRA, 2005)

$$S_A = \alpha L_A - \gamma \quad L_A: \text{perimeter}$$

boundary-law  
contribution

universal topological  
contribution

$\gamma = \ln D_{\text{topo}}$ : **topological entanglement entropy**

$D_{\text{topo}}$ : **total quantum dimension**

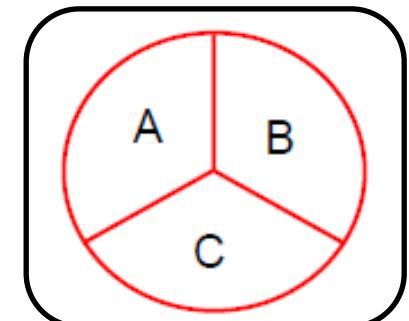
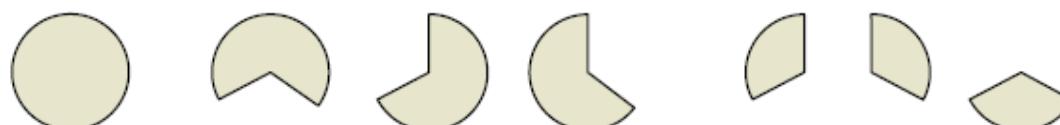
Universal constant related  
to the statistical properties of  
fractional quasiparticles

$Z_2$  spin liquid:

$D_{\text{topo}} = 2$

$\nu = 1/q$  FQH state:  $D_{\text{topo}} = \sqrt{q}$

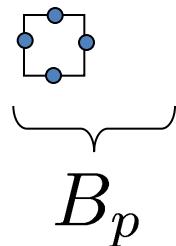
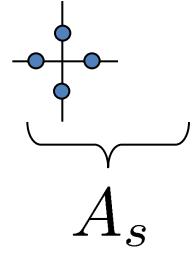
$$S_{ABC} - (S_{AB} + S_{BC} + S_{AC}) + (S_A + S_B + S_C) \rightarrow -\gamma$$



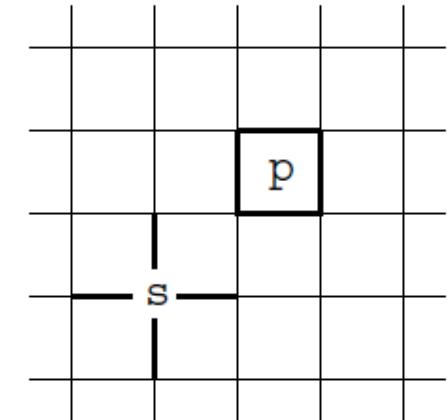
# Toric code model

Kitaev, quant-ph,1997; Annal.Phys,2003

$$H = -J_A \sum \prod \sigma_i^x - J_B \sum \prod \sigma_i^z$$



$$J_A, J_B > 0$$

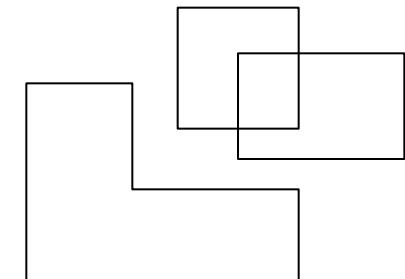


All terms commute.

→ Ground state:  $A_s = +1, B_p = +1$  for all  $s, p$

$J_B = 0$  → Degenerate manifold  $\mathcal{E}$   
Loop configs.

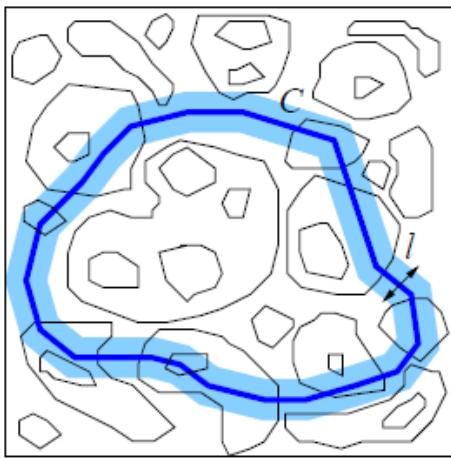
$J_B > 0$  → Resonace between loop configs.



$$\sigma_i^x = -1$$

→ Ground state:  $|\Psi\rangle = \sum_{c \in \mathcal{E}} |c\rangle$

# String correlations

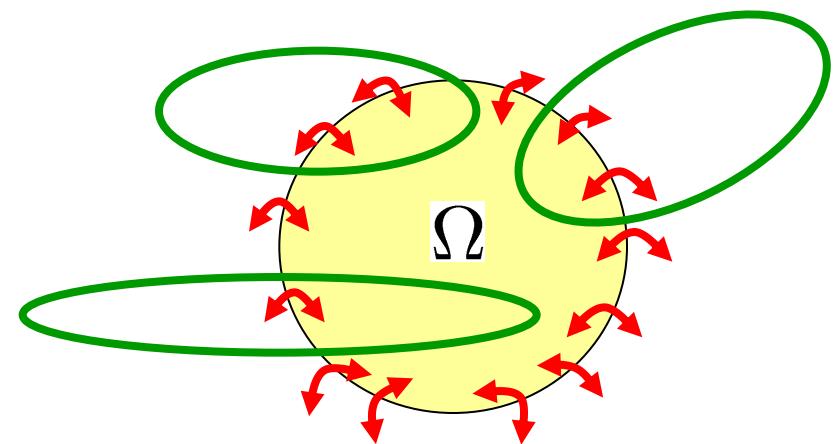


$$\langle W^x(C) \rangle = \left\langle \prod_{i \in C} \sigma_i^x \right\rangle = 1$$

$$\langle W^z(C') \rangle = \left\langle \prod_{i \in C'} \sigma_i^z \right\rangle = 1$$

C, C': closed loop

Hastings & Wen, PRB, 2005

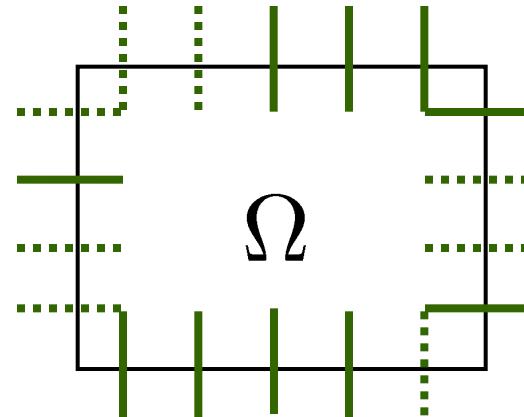


String correlations  
→ topological entropy

# Entanglement entropy in toric code model

Decomposition in terms of boundary configings. Hamma et al., PRA, 2005

$$\begin{aligned}
 |\Psi\rangle &= \frac{1}{\sqrt{|\mathcal{E}|}} \sum_{c \in \mathcal{E}} |c\rangle \\
 &= \frac{1}{\sqrt{|\mathcal{E}|}} \sum_{\alpha} \sum_{c_{\Omega} \in \mathcal{E}_{\Omega}^{\alpha}} |c_{\Omega}\rangle \sum_{c_{\bar{\Omega}} \in \mathcal{E}_{\bar{\Omega}}^{\alpha}} |c_{\bar{\Omega}}\rangle \\
 &= \sum_{\alpha} \underbrace{\left( \frac{|\mathcal{E}_{\Omega}^{\alpha}| \cdot |\mathcal{E}_{\bar{\Omega}}^{\alpha}|}{|\mathcal{E}|} \right)^{1/2}}_{p_{\alpha}} \underbrace{\left( \frac{1}{\sqrt{|\mathcal{E}_{\Omega}^{\alpha}|}} \sum_{c_{\Omega} \in \mathcal{E}_{\Omega}^{\alpha}} |c_{\Omega}\rangle \right)}_{|\psi_{\Omega}^{\alpha}\rangle} \underbrace{\left( \frac{1}{\sqrt{|\mathcal{E}_{\bar{\Omega}}^{\alpha}|}} \sum_{c_{\bar{\Omega}} \in \mathcal{E}_{\bar{\Omega}}^{\alpha}} |c_{\bar{\Omega}}\rangle \right)}_{|\psi_{\bar{\Omega}}^{\alpha}\rangle}
 \end{aligned}$$



“Schmidt decomposition”

$$S_{\Omega} = - \sum_{\alpha} p_{\alpha} \ln p_{\alpha} = L \ln 2 - \ln 2$$

← string correlation

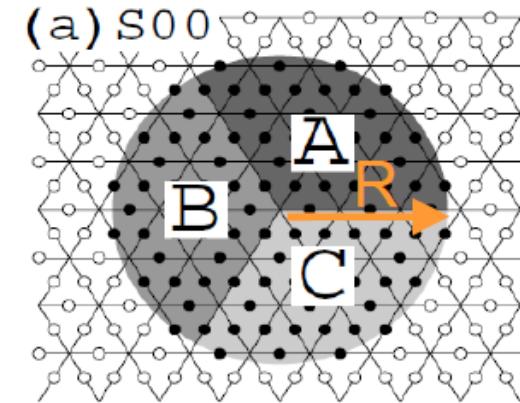
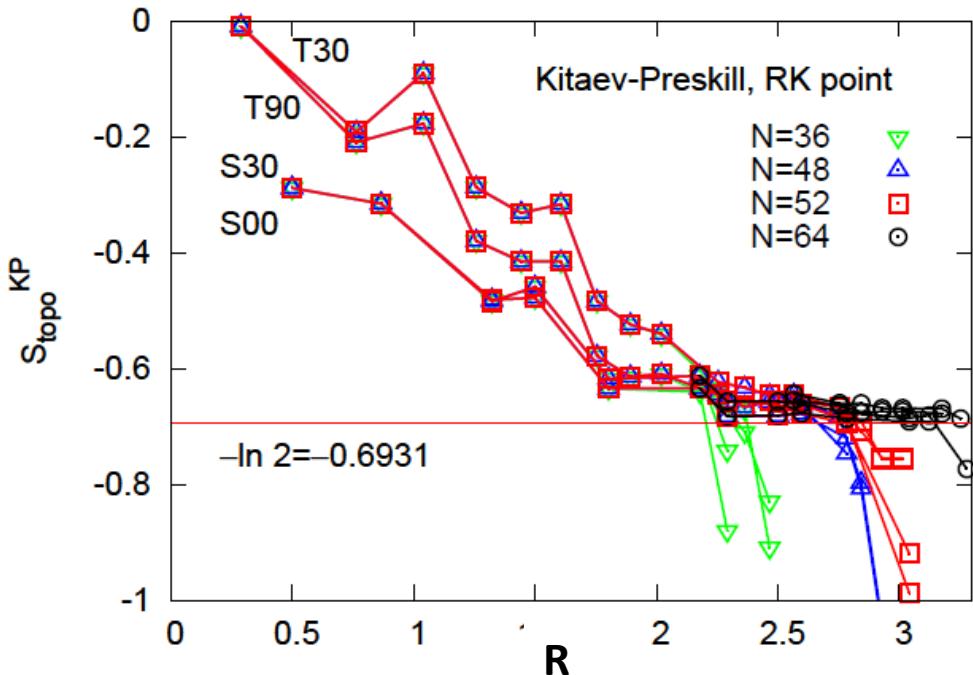
# Application: Quantum dimer model on the triangular lattice

SF, G. Misguich, Phys. Rev. B 75, 214407 (2007)

$$H = \sum [-t(|\underline{-}\rangle\langle\underline{/}| + h.c.) + v(|\underline{-}\rangle\langle\underline{-}| + |\underline{/}\rangle\langle\underline{/}|)]$$

Rokhsar-Kivelson point ( $t=v$ )  $|\text{RK}\rangle = \frac{1}{\sqrt{|\mathcal{E}|}} \sum_{c \in \mathcal{E}} |c\rangle$

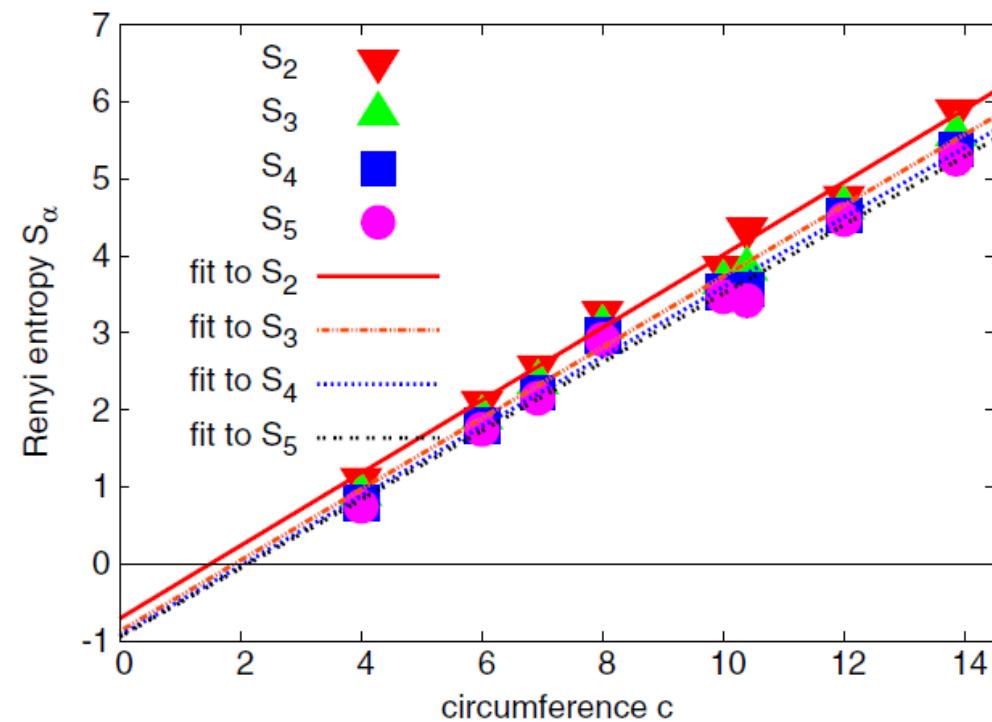
Kitaev-Preskill construction



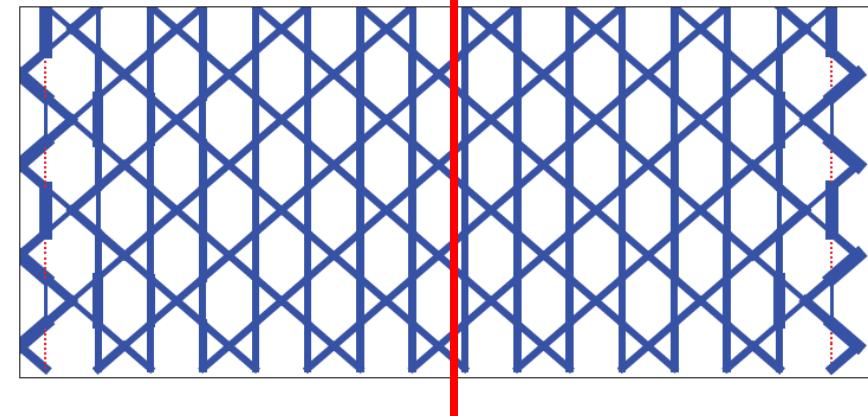
**99% agreement with  
the prediction !**

Radius $R$	$-S_{\text{topo}}^{\text{KP}} / \ln 2$	
	$N = 52$	$N = 64$
2.18	0.9143	0.9143
2.29	0.9839	0.9835
2.50	0.9822	0.9822
2.60	0.9765	0.9760
2.78	1.0014	0.9897
3.04	1.3252	0.9967
3.12		0.9967

# Application: kagome Heisenberg antiferromagnet



DMRG on a long cylinder



Topological entropy =  $-\log 2$



Strong indication of  
a Z2 spin liquid

Depenbrock, McCulloch, & Schollwoeck, PRL, 2012  
Jiang, Wang, & Balents, Nature Physics, 2012

Use of minimally entangled state:

Zhang, Grover, Turner, Oshikawa, & Vishwanath, PRB, 2012

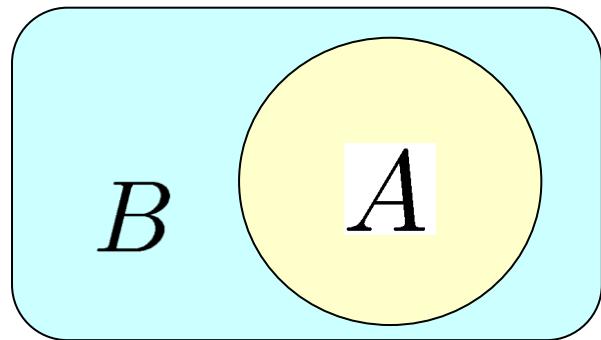
## 第一部

- 「トポロジカル相」とは？「トポロジカル秩序相」とは？
- トポロジカル秩序と普通の秩序の違いとは？  
(縮約密度行列の視点)

## 第二部

- エンタングルメント・エントロピーのトポロジカル秩序相への応用
- エンタングルメント・スペクトルのトポロジカル相への応用

# Entanglement spectrum



$$\rho_A = \text{Tr}_B |\Psi\rangle\langle\Psi|$$

Topological entropy does not provide a complete picture of topological phases.

In principle, the full spectrum of  $\rho_A$  contains more information than  $S_A$ .

$$\rho_A = e^{-H_e} \quad H_e : \text{entanglement Hamiltonian}$$

**Entanglement spectrum** = full eigenvalue spectrum of  $H_e$ :  $\{\xi_i\}$

## Entanglement entropy

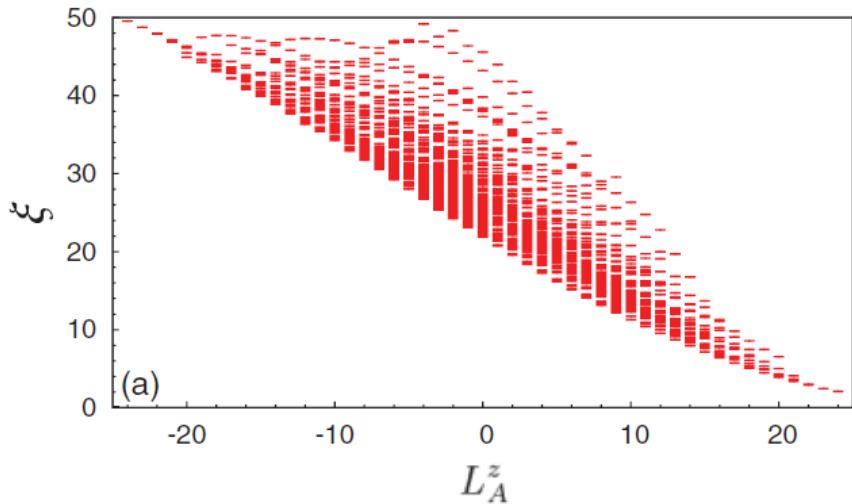
= Thermal entropy of  $H_e$  at the fictitious temperature  $T=1$

# Entanglement spectra in topological phases

46

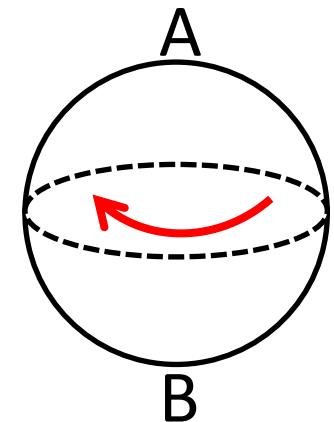
Remarkable correspondence with edge-state spectrum

Li and Haldane, PRL 101, 010504 (2008)

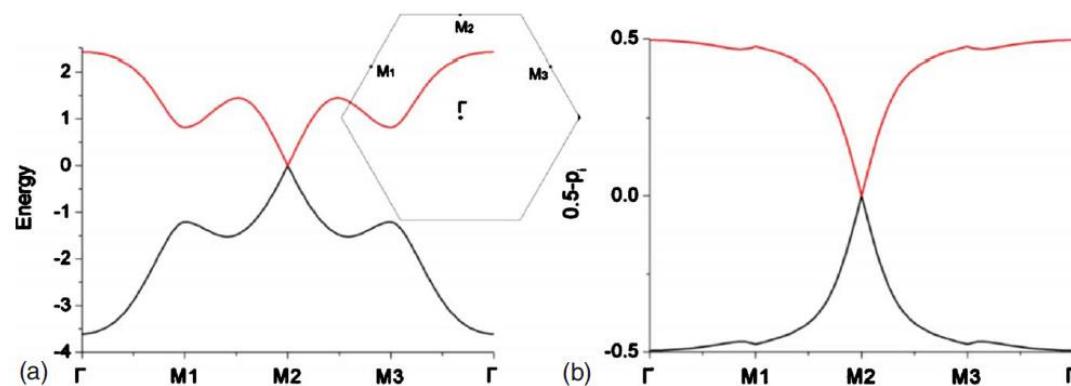


$\nu = 1/3$  Laughlin state

spectrum reminiscent of  
chiral TLL



Sterdyniak et al.,  
PRB 85, 125308 (2012)



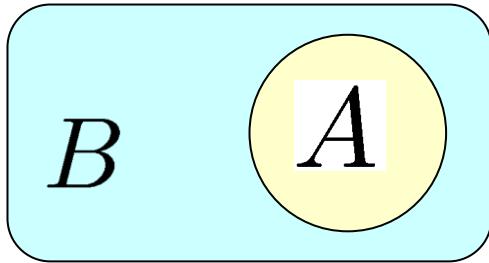
Edge spectrum

entanglement spectrum

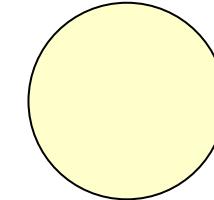
3D  $Z_2$  topological insulator  
Turner, Zhang, and Vishwanath,  
PRB 82, 241102 (2010)

$H_e \propto H_{\text{edge}}$   
at low energies

Correspondence between entanglement and edge-state spectra



property of bulk  
wave function



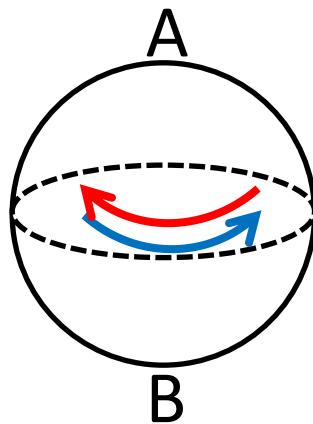
→ Manifestation of the bulk-edge correspondence

Practical aspect:

In finite-size simulations, the entanglement spectrum better reflects the bulk topological features (shows a clearer gapless structure than edge states).

# Applications: bosonic integer quantum Hall state

48

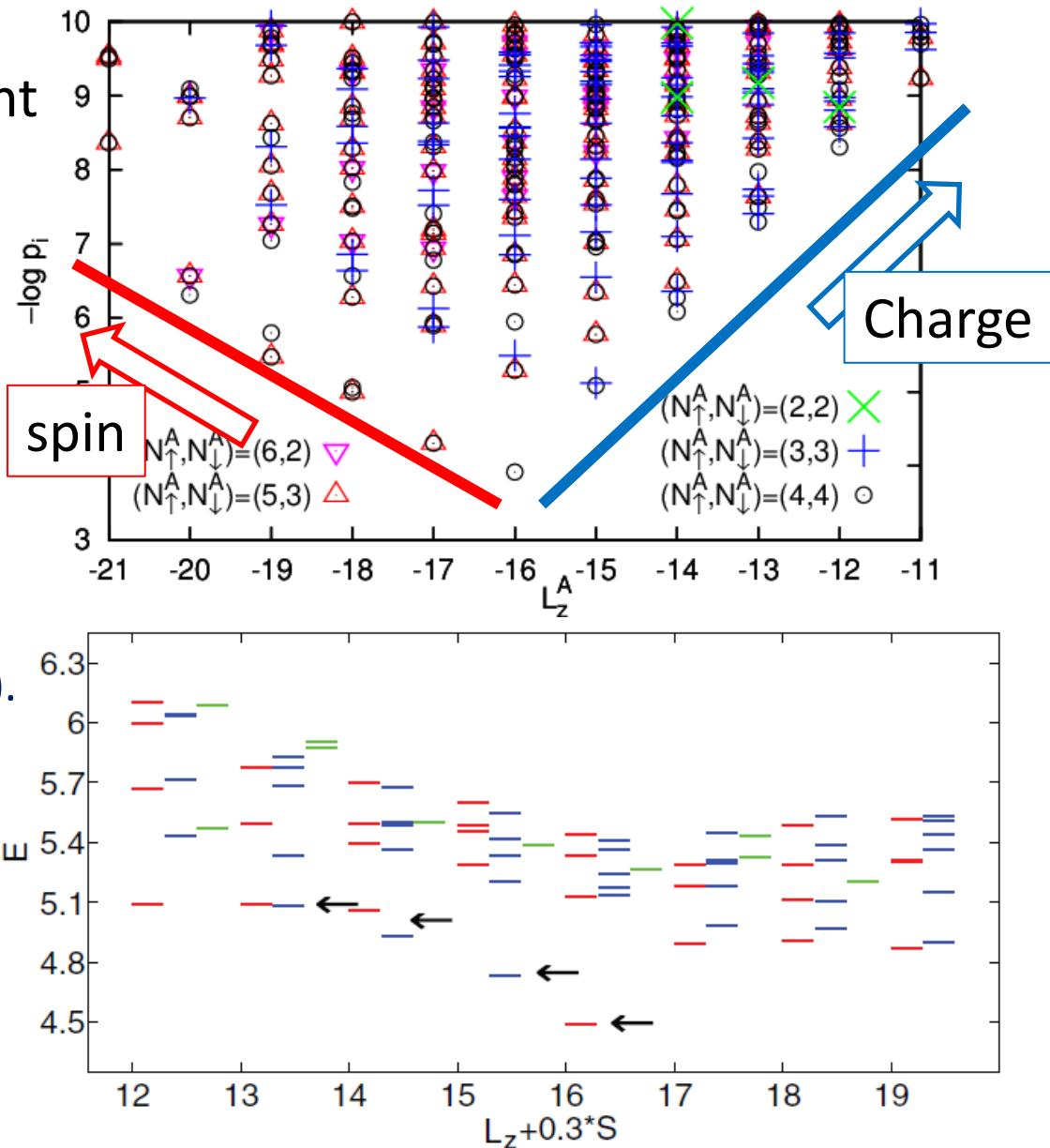


entanglement  
spectrum

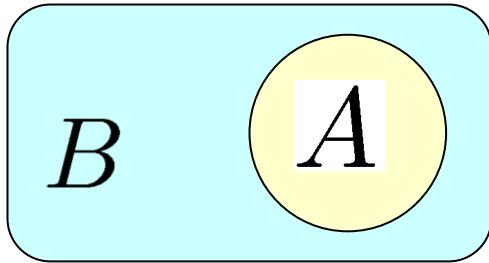
2-component Bose gas

Furukawa & Ueda,  
Phys. Rev. Lett. 111, 090401 (2013).  
Wu & Jain,  
Phys. Rev. B 87, 245123 (2013).

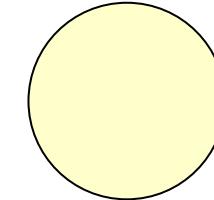
edge-state  
spectrum



## Correspondence between entanglement and edge-state spectra



property of bulk  
wave function



→ Manifestation of the bulk-edge correspondence

### Practical aspect:

In finite-size simulations, the entanglement spectrum better reflects the bulk topological features (shows a clearer gapless structure than edge states).

### Questions:

Why do we have this correspondence?

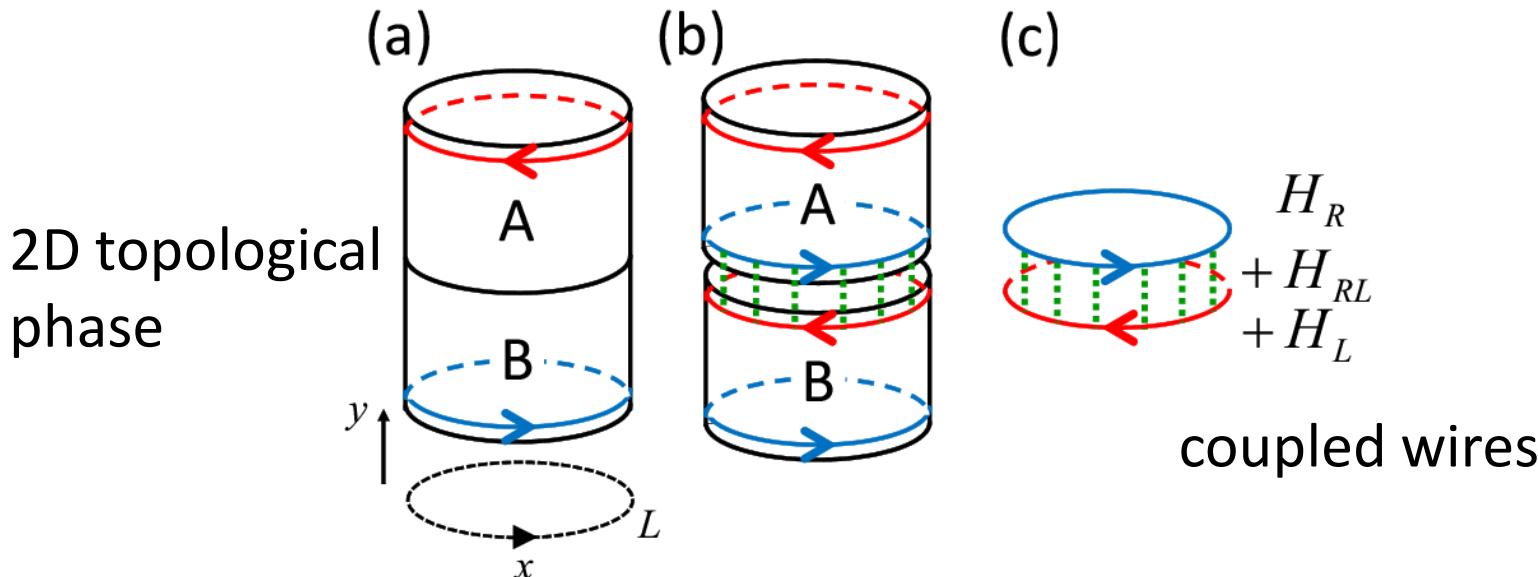
Does this correspondence apply to all topological phase?

When can it be violated?

- We consider the entanglement properties of 2D topological phases.
- We can reduce this problem to a problem of coupled Tomonaga-Luttinger liquids (TLLs).

Qi, Katsura, and Ludvig, PRL 108, 196402 (2012)

- We solve the entanglement problem of two coupled TLLs using a very simple free theory method.

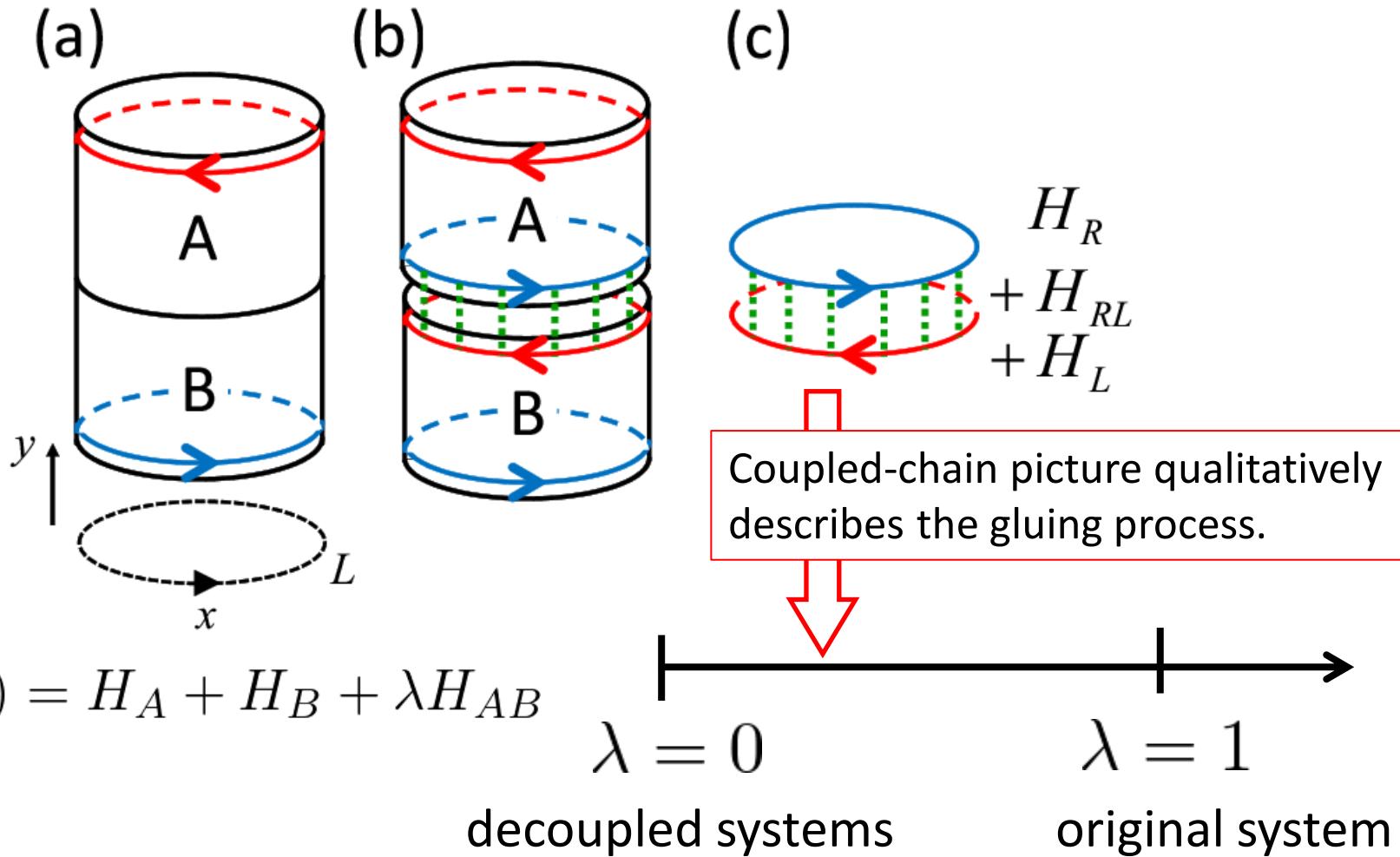


# "Cut and glue" approach

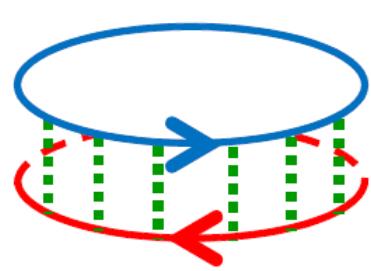
51

Qi, Katsura, and Ludvig, PRL 108, 196402 (2012)

$\nu = 1/q$  FQH state ( $q$ =even for bosons, odd for fermions)



# Two coupled chiral TLLs



$H_R$       Two edges of  $\nu = 1/q$  FQH state

$$+ H_{RL} \\ + H_L$$

$$H_{R/L} = \int_0^L dx \frac{qv_0}{4\pi} (\partial_x \phi_{R/L})^2,$$

$$\rho_{R/L} = \frac{1}{2\pi} \partial_x \phi_{R/L}$$

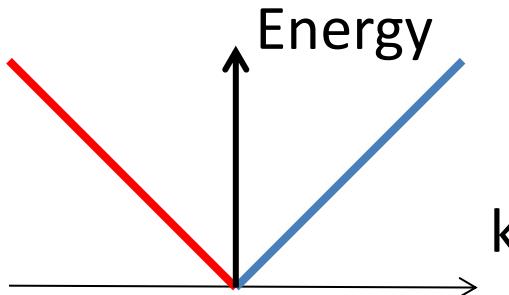
$H_{RL}$  : particle tunneling and density-density interactions

$$\phi = \frac{1}{\sqrt{4\pi}} (\phi_L + \phi_R), \quad \theta = \frac{q}{\sqrt{4\pi}} (\phi_L - \phi_R) \quad \text{Non-chiral bosonic fields}$$

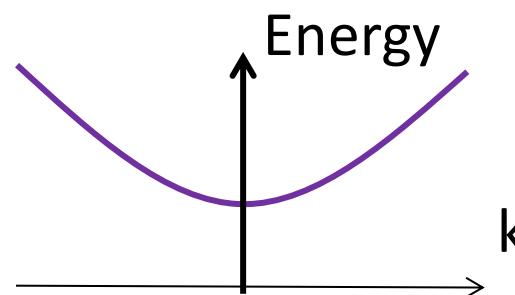
$$H \equiv H_L + H_R + H_{RL}$$

$$= \int_0^L dx \left[ \frac{v}{2} \left( K(\partial_x \theta)^2 + \frac{1}{K} (\partial_x \phi)^2 \right) + \frac{g}{\pi} \cos \left( \sqrt{4\pi} q \phi \right) \right]$$

sine-Gordon Hamiltonian



$g$  decays to zero under RG.



$g$  grows under RG.

→ Locking of  $\phi$   
Energy gap opens up.

Very simple description of the gapped phase:

$$\frac{g}{\pi} \cos(\sqrt{4\pi q}\phi) \approx \text{const.} + \frac{vm^2}{2K} (\phi - \bar{\phi}_0)^2 + \dots,$$

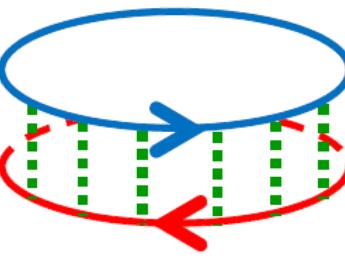
→ H : massive Klein-Gordon model with a mass gap  $vm$

Just with methods for free theories, we can discuss the entanglement spectrum!

cf. Qi, Katsura, and Ludvig used boundary CFT to describe the gapped phase.

# Calculation of entanglement Hamiltonian

54


$$H_R + H_{RL} + H_L \quad \text{Mode expansions}$$
$$\phi_R = \sum_{k>0} \sqrt{\frac{2\pi}{qL|k|}} (a_k e^{ikx} + a_k^\dagger e^{-ikx}) + \text{zero modes}$$
$$\phi_L = \sum_{k<0} \sqrt{\frac{2\pi}{qL|k|}} (a_k e^{ikx} + a_k^\dagger e^{-ikx}) + \text{zero modes}$$

(ignored  
in this talk)

$H$  : certain quadratic form of bosonic operators

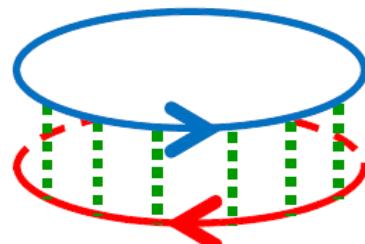
→ Obtain the ground state. Calculate correlation functions in the 1<sup>st</sup> chain:  $\langle 0 | a_k^\dagger a_k | 0 \rangle \quad k > 0$

→ Ansatz:  $\rho_A^{\text{osc}} = \frac{1}{Z_e^{\text{osc}}} e^{-H_e^{\text{osc}}} \quad H_e^{\text{osc}} = \sum_{k>0} w_k \left( a_k^\dagger a_k + \frac{1}{2} \right)$

Determine  $w_k$  so that the ground-state correlations are reproduced (Peschel's method, 2003).

# Result: entanglement Hamiltonian

55


$$H_e = v_e \left[ \frac{\pi q}{L} N_R^2 + \sum_{k>0} k a_k^\dagger a_k - \frac{\pi}{12L} \right] \propto H_R$$

with  $v_e = 4qK/m$

(at low energies)

→ Simple physical proof of entanglement-edge correspondence in quantum Hall states!

By calculating the thermal entropy of  $H_e$ , we can also obtain the topological entanglement entropy.

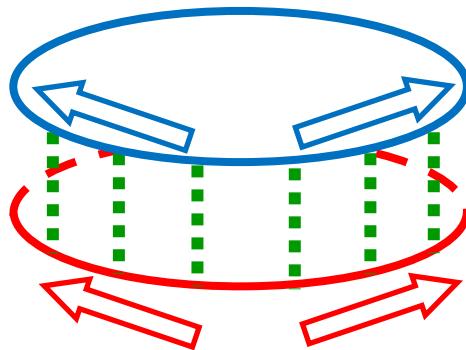
$$Z_e(\beta) = \text{Tr } e^{-\beta H_e}$$

$$S = \frac{\partial(T \ln Z_e(\beta))}{\partial T} \Big|_{T=1} \approx -\ln \sqrt{q}$$

Consistent with  
Kitaev-Preskill-Levin-Wen  
result

No boundary-law term:  
artifact of a theory without a high-energy cutoff

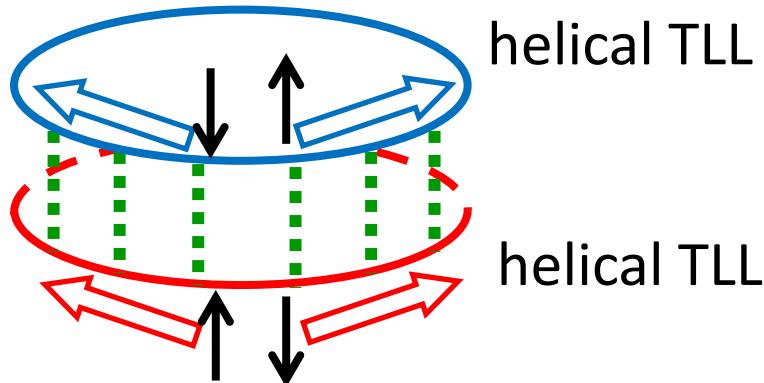
# Two coupled non-chiral TLLs



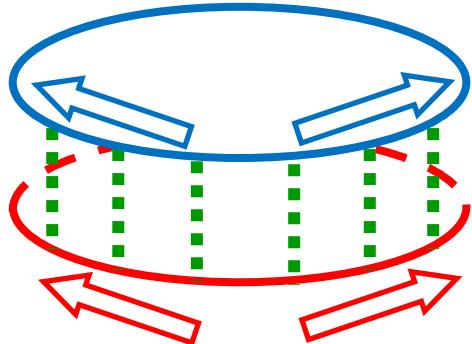
Direct applications to ladder systems  
and Hubbard chains

cf. Numerical study on spin ladders  
Entanglement spectrum remarkably resembles  
the single-chain spectrum.

Poilblanc, PRL 105, 077202 (2010)



Application to time-reversal-invariant  
topological insulators



$$H_\nu = \int dx \frac{v_0}{2} \left[ K (\partial_x \theta_\nu)^2 + \frac{1}{K} (\partial_x \phi_\nu)^2 \right], \quad \nu = 1, 2.$$

K: TLL parameter

K<1: repulsive interaction  
K>1: attractive interaction

Coupled system: separation into symmetric and antisymmetric channels

$$\phi_{\pm} = \frac{1}{\sqrt{2}}(\phi_1 \pm \phi_2), \quad \theta_{\pm} = \frac{1}{\sqrt{2}}(\theta_1 \pm \theta_2)$$

$$H_+ = \int dx \left\{ \frac{v_+}{2} \left[ K_+ (\partial_x \theta_+)^2 + \frac{1}{K_+} (\partial_x \phi_+)^2 \right] + g_+ \cos \left( \sqrt{2} \phi_+ / r \right) \right\}$$

$$H_- = \int dx \left\{ \frac{v_-}{2} \left[ K_- (\partial_x \theta_-)^2 + \frac{1}{K_-} (\partial_x \phi_-)^2 \right] + g_- \cos \left( \sqrt{2} \theta_- / \tilde{r} \right) \right\}$$

g+ and g- grow under RG

$\rightarrow$  { gapped phases of spin ladders  
topological insulators }

# Result for a fully gapped phase

## Entanglement Hamiltonian

$$H_e = \int dx \frac{v_e}{2} \left[ K_e (\partial_x \theta_1)^2 + \frac{1}{K_e} (\partial_x \phi_1)^2 \right]$$

$$v_e = 4 \sqrt{\frac{K_+}{K_- m_+ m_-}}, \quad K_e = \sqrt{\frac{K_+ K_- m_-}{m_+}}$$

$v_+ m_+, v_- m_-$ :  
mass gap in  
"+" and "-" channels

- This resembles  $H_1$ , but has **a modified TLL parameter**.  
The entanglement-edge correspondence is slightly violated!
- In some special cases, symmetry enforces  $K_e = K$ .

Non-interacting topological insulators:  $K_e = K = 1$

SU(2)-symmetric spin ladders:  $K_e = K = 1/2$

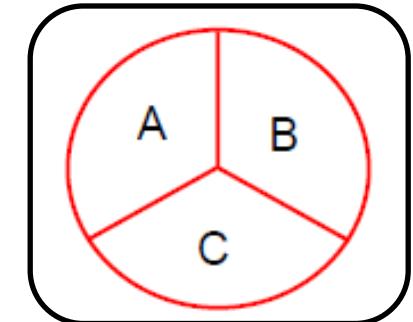
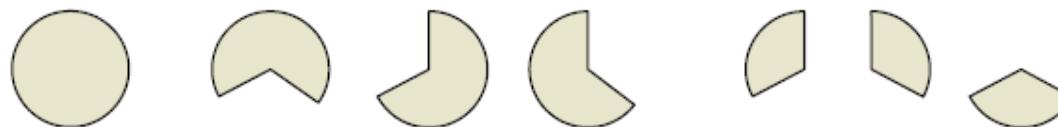
cf. In XXZ spin ladders,  $H_e$  has been found to be given by an XXZ chain with an anisotropy modified from the physical chain.

Peschel and Chung, EPL, 2011; Lauchli and Schliemann, PRB, 2012

➤ エンタングルメント・エントロピー → トポロジカル秩序を検出

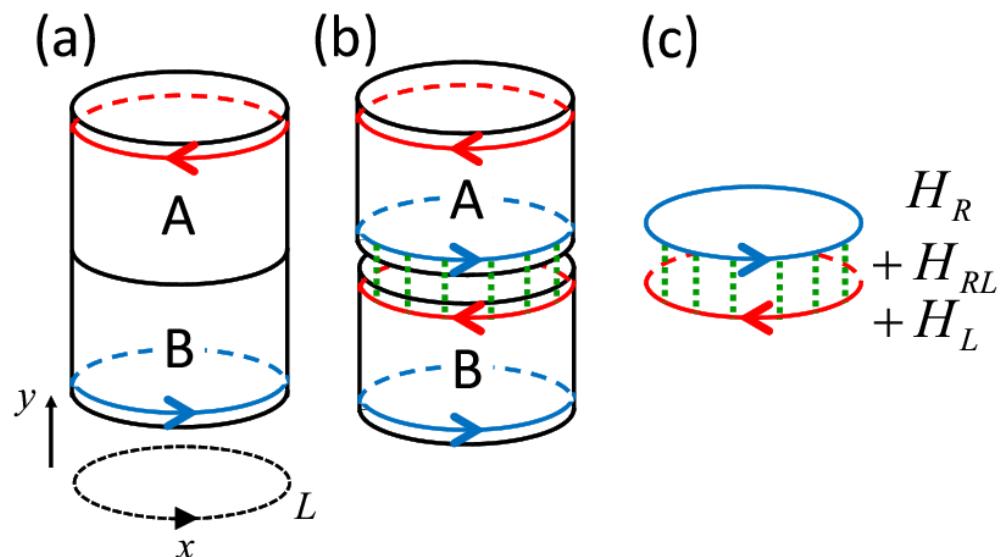
$$S_{\Omega} = \alpha L - \gamma \quad \gamma = \ln D_{\text{topo}}$$

$$S_{ABC} - (S_{AB} + S_{BC} + S_{AC}) + (S_A + S_B + S_C) \rightarrow -\gamma$$



➤ エンタングルメント・スペクトル → トポロジカル相の端状態の情報

"Cut and glue" approach  
による理解



# Results for gapless phases

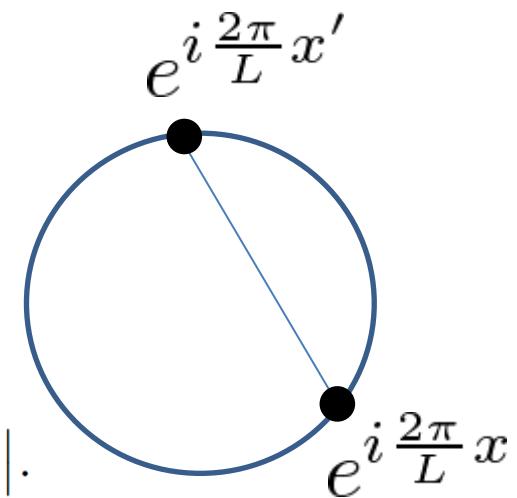
60

More remarkable violations of the entanglement-edge correspondence are found in gapless phases of ladder systems

- Partially gapless case (gap only in "-" channel)

$$H_e^{\text{osc}} = \sum_{k \neq 0} w_k \left( b_{k,1}^\dagger b_{k,1} + \frac{1}{2} \right) \quad w_k \approx 2\sqrt{v_e |k|}$$

$$H_e = \int dx \frac{v_e}{2} \left[ K_+ (\partial_x \theta_1)^2 + \frac{1}{K_+} (\partial_x \phi_1)^2 \right] - \frac{2K_+}{\pi} \iint dx dx' \partial_x \theta_1(x) \partial_{x'} \theta_1(x') \ln |e^{i \frac{2\pi}{L} x} - e^{i \frac{2\pi}{L} x'}|.$$



Emergence of a long-range interaction <-> critical correlations

- Fully gapless case

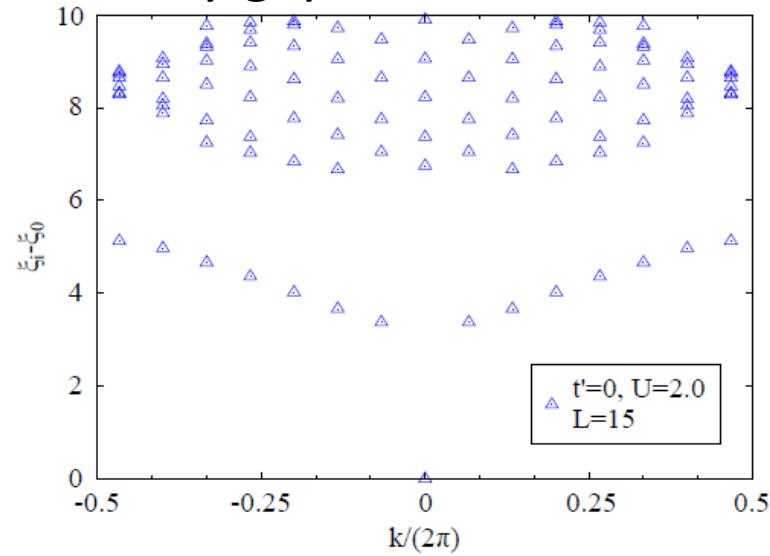
$$H_e = w \sum_{k \neq 0} \left( b_{k,1}^\dagger b_{k,1} + \frac{1}{2} \right)$$

w is determined by  
K+ and K-

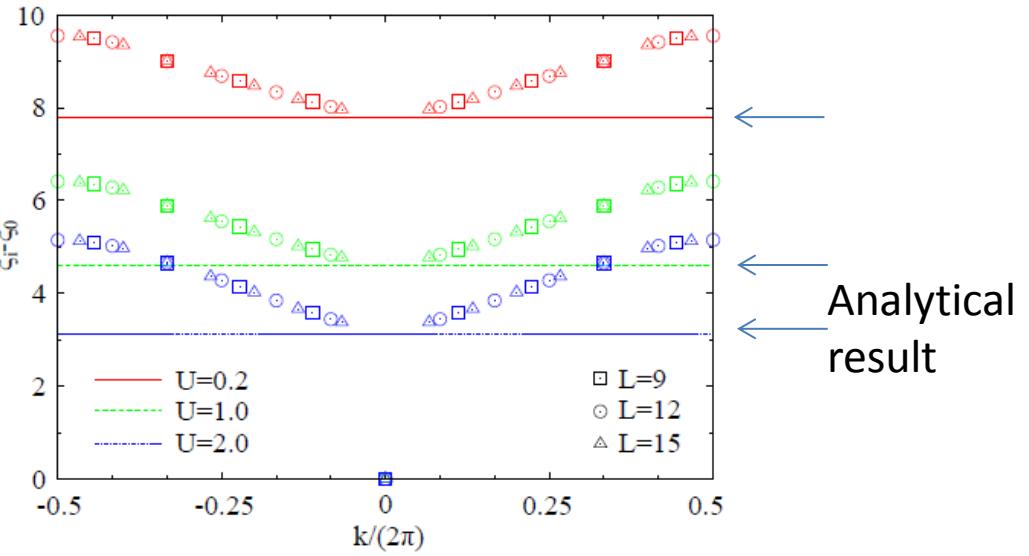
Bosonic modes with a flat dispersion

cf. Chen & Fradkin, PRB, 2013

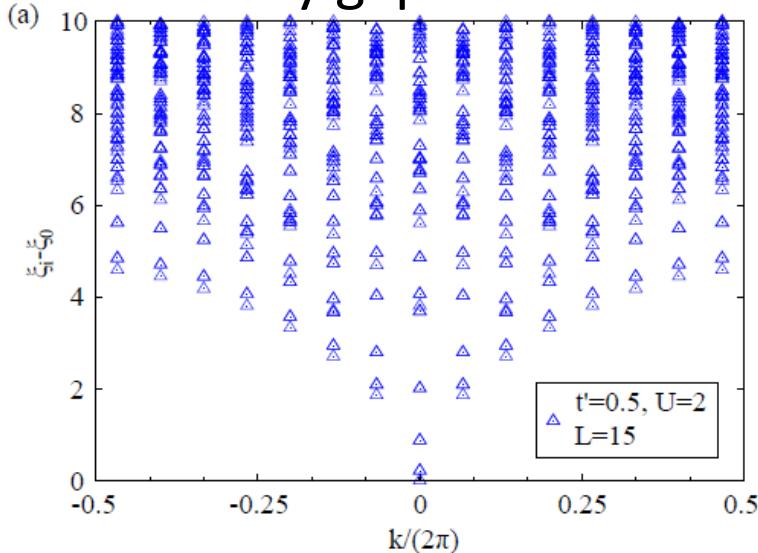
## (a) Fully gapless case



(b)



## (a) Partially gapless case



(b)

