# Nambu-Goldstone bosons in nonrelativistic systems

Haruki Watanabe
University of California, Berkeley

Tomas Brauner


Hitoshi Murayama

Ashvin Vishwanath
(Ph. D advisor)

# Plan of my talk

## 1. General theorems on NGBs (16 min)

- Low energy effective Lagrangian
- General counting rules
- Dispersion relations

## 2. Anderson Tower of States (2 min)

- Detecting SSB in finite size systems

## 3. Interactions (2 min)

- Among NGBs
- NGBs with other low-energy modes

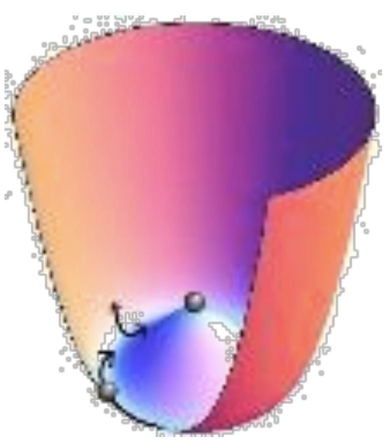
# General theorems on NGBs

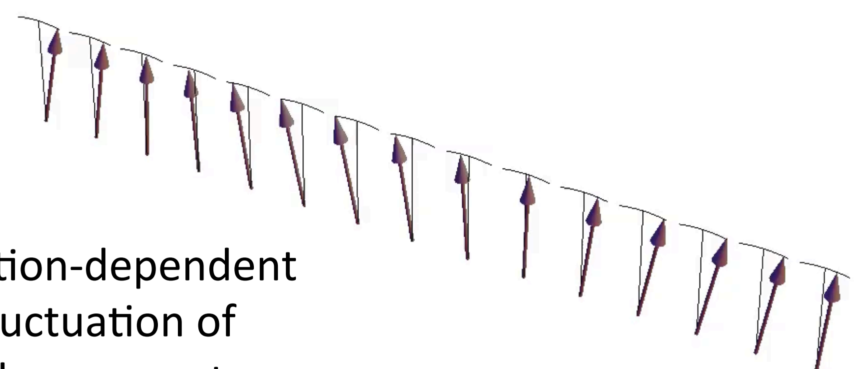
HW and H. Murayama, Phys. Rev. Lett. 108, 251602 (2012)
HW and H. Murayama, arXiv:1402.7066.

Spontaneous Symmetry Breaking (SSB)
of *global* and *internal* symmetries

Nambu-Goldstone Bosons (NGBs)
*gapless* particle-like excitation

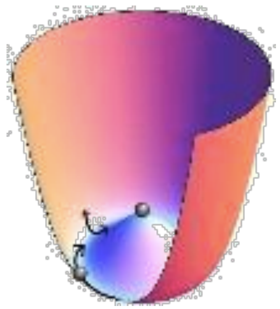
Higgs (amplitude) boson
*gapped* particle-like excitation





position-dependent fluctuation of order parameter in the flat direction

# The *definition* of NGBs



- Gapless modes

(fluctuation in the flat direction may have a gap

- Fluctuation in the flat direction of the potential

= transform *nonlinearly* under *broken* symmetries

+ transform *linearly* under *unbroken* symmetries

Superfluid

$$\theta' = \theta + \epsilon$$

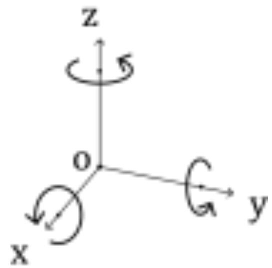
c.f. linear transformation

$$\vec{v}' = M\vec{v}$$

Magnets with unbroken $S_z$

rotation around y (broken)

$$\delta n'_x = \epsilon_y, \delta n'_y = 0$$




rotation around z (unbroken)

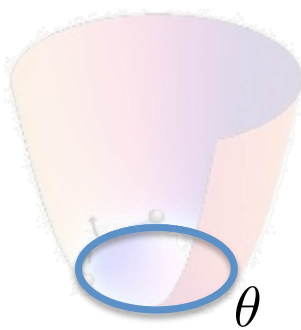
$$\delta n'_x = -\epsilon_z \delta n_y, \delta n'_y = +\epsilon_z \delta n_x$$

$$\vec{n} = \begin{pmatrix} \delta n_x \\ \delta n_y \\ 1 \end{pmatrix}$$

# Flat direction of the potential

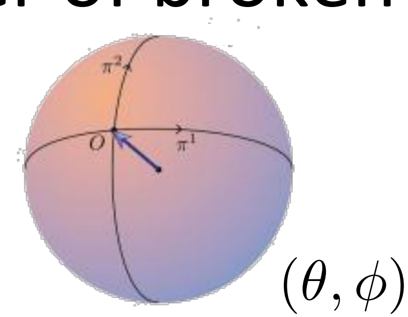
- Lie group G: symmetry of the Lagrangian
- Lie group H: symmetry of the ground state
- Coset space G/H: the manifold of degenerated ground states.
- dim(G/H) = dim(G)–dim(H)

  = the number of broken generators




G = U(1)
H = {e}

U(1)/{$e$} = $S^1$




G = SO(3)
H = SO(2)

SO(3)/SO(2) = $S^2$

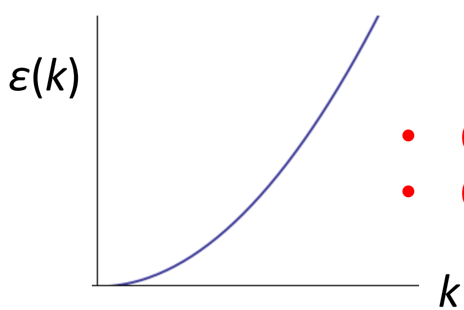
# Example of NGB (1): Magnets

Symmetry of the Heisenberg model: G = SO(3) (3 generators)

Symmetry of (anti)ferromagnetic GS : H = SO(2) (1 generator)

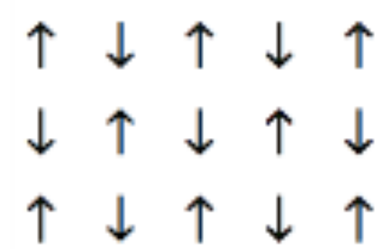
Two (3 – 1 = 2) symmetries are spontaneously broken

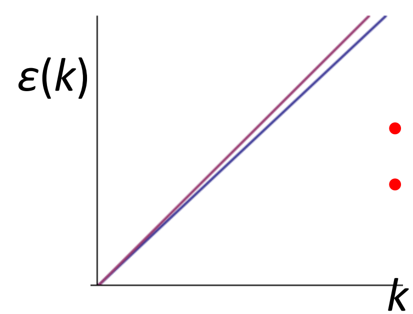
Ferromagnets




- One NGB
- Quadratic dispersion

Antiferromagnets



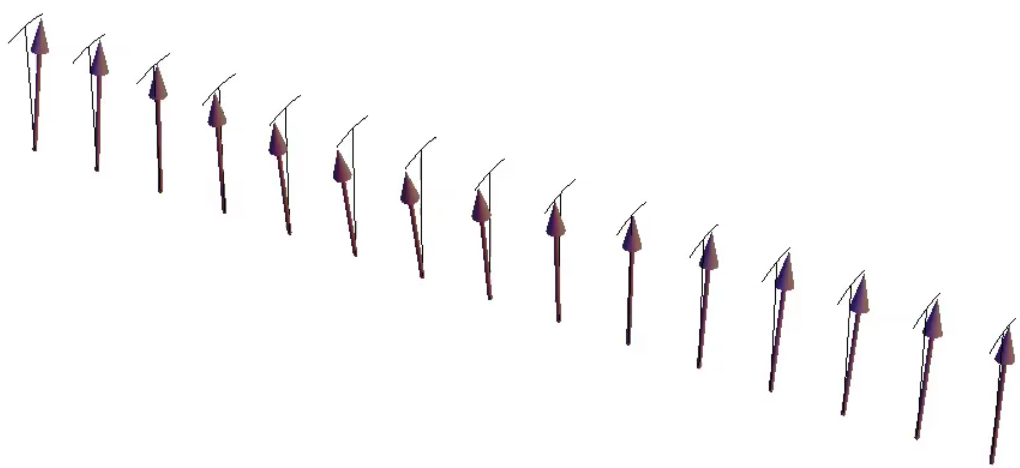



- Two NGBs
- Linear dispersion

# Antiferromagnet

↑ ↓ ↑ ↓ ↑
↓ ↑ ↓ ↑ ↓
↑ ↓ ↑ ↓ ↑

No net magnetization $\langle S_z \rangle = 0$



# Ferromagnet

Nonzero magnetization $\langle S_z \rangle \neq 0$

The time reversed motion is not a low-energy fluctuation

# Example of NGB (2): Spinor BEC

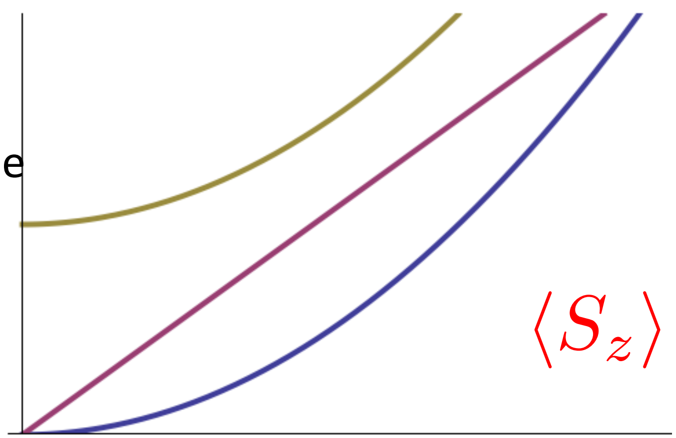
G = U(1) x SO(3) (4 generatosr)

→ H = SO(2) (1 generator)

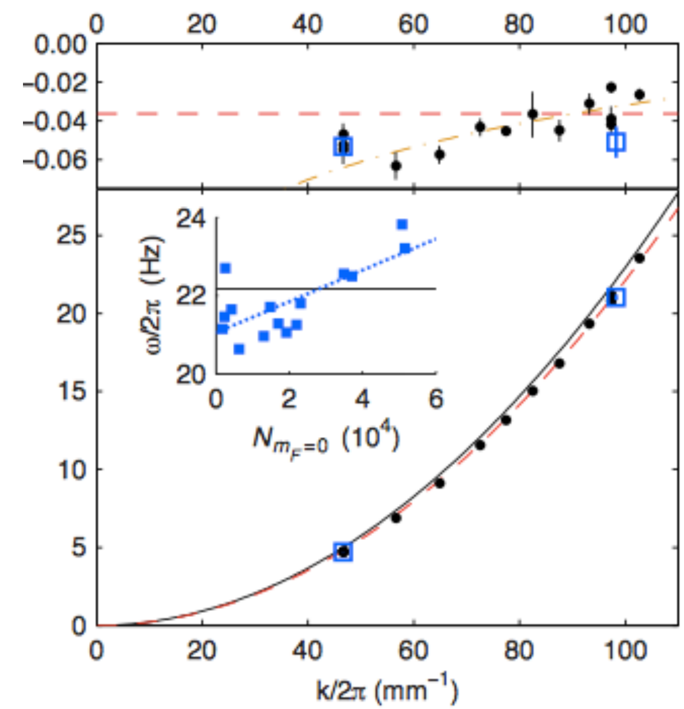
4 – 1 = 3 broken symmetries

Only 2 NGBs
- one linear mode (sound wave)
- one quadratic mode (spin wave)

amplitude mode →



$\langle S_z \rangle \neq 0$




Dan Stamper-Kurn et al
arxiv:1404.5631

# Example of NGB (3): more high-energy side example

$$\mathcal{L} = D_\mu \psi^\dagger D^\mu \psi - m^2 \psi^\dagger \psi - \frac{g}{2}(\psi^\dagger \psi)^2$$

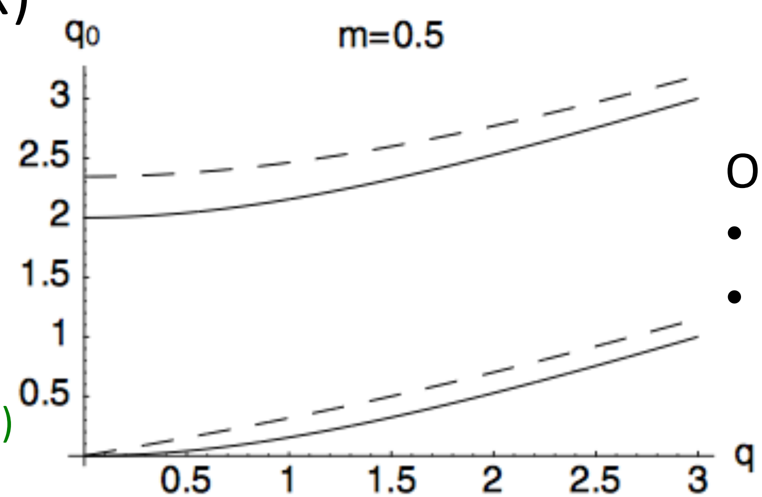
$D_\nu = \partial_0 + i\mu\delta_{\nu,0}$ ($\mu$: chemical potential) $\psi = (\psi_1, \psi_2)^T$

Symmetry of the Lagrangian: G = U(2) (4 generators)
Symmetry of the condensate : H = U(1) (1 generator)
Three (4 – 1 = 3) symmetries are spontaneously broken

$$\langle \psi \rangle = v(0,1)^T$$

$\varepsilon(k)$

amplitude mode →
a fluctuation in G/H →




Only two NGBs (gapless)
- One quadratic
- One linear

$$\langle Q_3 \rangle \neq 0$$

V. Miransky & I. Shavkovy (2002)
T. Schäfer *et al.* (2001)

# Questions

- In general, how many NGBs appear?
- When do they have quadratic dispersion?
- What is the necessary information of the ground state to predict the number and dispersion?
- What is the relation to expectation values of conserved charges (generators)?

Y. Nambu, J. Stat. Phys. 115, 7 (2004)

$\langle [Q_a, Q_b] \rangle \neq 0$ ➡ Their zero modes are conjugate. Not independent modes.

# Our approach

H. Leutwyler, Phys. Rev. D 49, 3033 (1994)

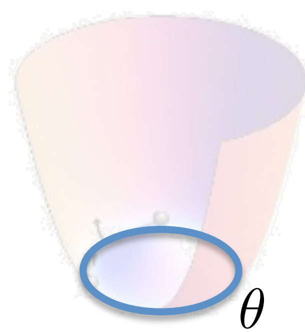
Low energy effective Lagrangian

= Non-Linear sigma model with the target space *G/H*

\+ derivative expansion

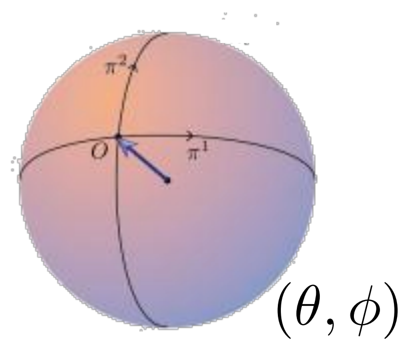
- *G/H* : the manifold of degenerated ground states
- Effective theory after integrating out all fields with a mass term

i.e., those going out of *G/H* (amplitude fields)




U(1)/{*e*} = $S^1$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\theta\partial^\mu\theta$$




SO(3)/SO(2) = $S^2$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{n}\cdot\partial^\mu\vec{n}$$

# How to get effective Lagrangian?

- 1. From a microscopic model

$$\psi = \sqrt{n_0 + \delta n}\, e^{-i\theta}$$

$$\begin{aligned}
\mathcal{L}_{\rm SF} &= i\psi^\dagger \dot{\psi} - \frac{\vec{\nabla}\psi^\dagger \cdot \vec{\nabla}\psi}{2m} - \frac{g}{2}(\psi^\dagger\psi - n_0)^2 \\
&\simeq \delta n \dot{\theta} - \frac{n_0}{2m}\vec{\nabla}\theta \cdot \vec{\nabla}\theta - \frac{g}{2}(\delta n)^2 \\
&= \frac{1}{2g}\dot{\theta}^2 - \frac{n_0}{2m}\vec{\nabla}\theta \cdot \vec{\nabla}\theta - \frac{g}{2}(\delta n - \dot{\theta}/g)^2
\end{aligned}$$

make n and θ canonically conjugate

- 2. Simply write down all terms

allowed by symmetry (+ derivative expansion)

For example:
the mass term is prohibited by symmetry

~~$\frac{1}{2}m^2\theta^2$~~

# General form of effective Lagrangian

- In the presence of Lorentz symmetry

$$\mathcal{L} = \frac{1}{2} g_{ab}(\pi) \partial_\mu \pi^a \partial^\mu \pi^b$$

- In the absence of Lorentz symmetry

$$\mathcal{L} = \underline{c_a(\pi) \dot{\pi}^a} + \frac{1}{2} \bar{g}_{ab}(\pi) \dot{\pi}^a \dot{\pi}^b - \frac{1}{2} g_{ab}(\pi) \nabla \pi^a \cdot \nabla \pi^b$$

dominant at low-energy

Taylor expand …

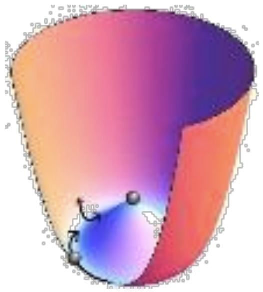
$$c_a(\pi)\dot{\pi}^a = -\frac{1}{2}\rho_{ab}\pi^a\dot{\pi}^b + O(\pi^3)$$

Canonical conjugate relation between $\pi^a$ and $\pi^b$*!!*

$$p_b = \frac{\partial \mathcal{L}}{\partial \dot{\pi}^b} = -\frac{1}{2}\rho_{ab}\pi^a$$

c.f. canonical conjugate between Goldstone mode and Amplitude

$$\mathcal{L}_{\mathrm{SF}} \ni i\psi^\dagger \dot{\psi} = -n\dot{\theta}$$



# General counting rule

Using the symmetry G of the effective Lagrangian, we can prove antisymmetric matrix $\rho_{ab}$ is related to commutator of generator!!

$$c_a(\pi)\dot{\pi}^a = -\frac{1}{2}\rho_{ab}\pi^a\dot{\pi}^b + O(\pi^3)$$

$$i\rho_{ab} = \langle [Q_a, j_b^0(\vec{x}, t)] \rangle = \lim_{\Omega\to\infty}\frac{1}{\Omega}\langle [Q_a, Q_b] \rangle$$

$\Omega$ : volume of the system

$m$ = rank $\rho$

$(\pi^1, \pi^2), (\pi^3, \pi^4), \ldots, (\pi^{2m-1}, \pi^{2m})$

→ Canonically conjugate pairs!

$$\rho = \left(\begin{array}{cccc|c} \begin{matrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{matrix} & & O & & \\ & \begin{matrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{matrix} & & & O \\ & & \ddots & & \\ O & & & \begin{matrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{matrix} & \\ \hline & O & & & O \end{array}\right)$$

c.f. $\mathcal{L}_{\rm SF} \ni n\dot{\theta}$ term in superfluid

# General counting rule

- type-A (unpaired) NGBs

  $n_A = \dim(G/H) - \mathrm{rank}\,\rho$

- type-B (paired) NGBs

  $n_B = (1/2)\mathrm{rank}\,\rho$

- The total number of NGBs

  $n_A + n_B = \dim(G/H) - (1/2)\mathrm{rank}\,\rho$

$$\rho = \left(\begin{array}{cccc|c} \begin{bmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{bmatrix} & & & O & \\ & \begin{bmatrix} 0 & \lambda_2 \\ -\lambda_2 & 0 \end{bmatrix} & & & O \\ & & \ddots & & \\ O & & & \begin{bmatrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{bmatrix} & \\ \hline & O & & & O \end{array}\right)$$

$$i\rho_{ab} = \langle [Q_a, j_b^0(\vec{x}, t)] \rangle = \lim_{\Omega \to \infty} \frac{1}{\Omega} \langle [Q_a, Q_b] \rangle$$

# Dispersion relations

$$\mathcal{L} = -\frac{1}{2}\rho_{ab}\pi^a\dot{\pi}^b + \frac{1}{2}\bar{g}_{ab}(0)\dot{\pi}^a\dot{\pi}^b - \frac{1}{2}g_{ab}(0)\nabla\pi^a\cdot\nabla\pi^b + \cdots$$

$\omega$ $\omega^2$ $k^2$

- Type-A NGBs: linear dispersion (Type-I NGBs )
- Type-B NGBs: quadratic dispersion (Type-II NGBs )

Nielsen-Chadha's counting rule

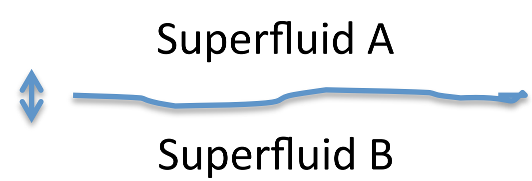
$n_{\mathrm{I}} + 2\, n_{\mathrm{II}} \geq \dim(G/H)$

H. B. Nielsen and S. Chadha (1976)

We proved the equality!

$n_{\mathrm{A}} + 2n_{\mathrm{B}} = \dim(G/H)$

c.f. Ripple motion of a domain wall
= Goldstone mode of translation




$\omega^2 = k^{3/2}$

# Effective Lagrangian for magnets

Ferromagnets

$$\mathcal{L} = -m\frac{n^x\dot{n}^y - n^y\dot{n}^x}{1+n^z} + \frac{\bar{g}}{2}\dot{\vec{n}}\cdot\dot{\vec{n}} - \frac{g}{2}\partial_i\vec{n}\cdot\partial_i\vec{n}$$

$n_A$ = dim($G/H$) - rank $\rho$ = 2 – 2 = 0

$n_B$ = (1/2)rank $\rho$ = 1

Antiferromagnets

$$\mathcal{L} = -m\frac{n^x\dot{n}^y - n^y\dot{n}^x}{1+n^z} + \frac{\bar{g}}{2}\dot{\vec{n}}\cdot\dot{\vec{n}} - \frac{g}{2}\partial_i\vec{n}\cdot\partial_i\vec{n}$$

$n_A$ = dim($G/H$) - rank $\rho$ = 2 – 0 = 2

$n_B$ = (1/2)rank $\rho$ = 0

$$m = \frac{\langle[S_x, S_y]\rangle}{i\Omega} = \frac{\langle S_z\rangle}{\Omega}$$ : magnetization density

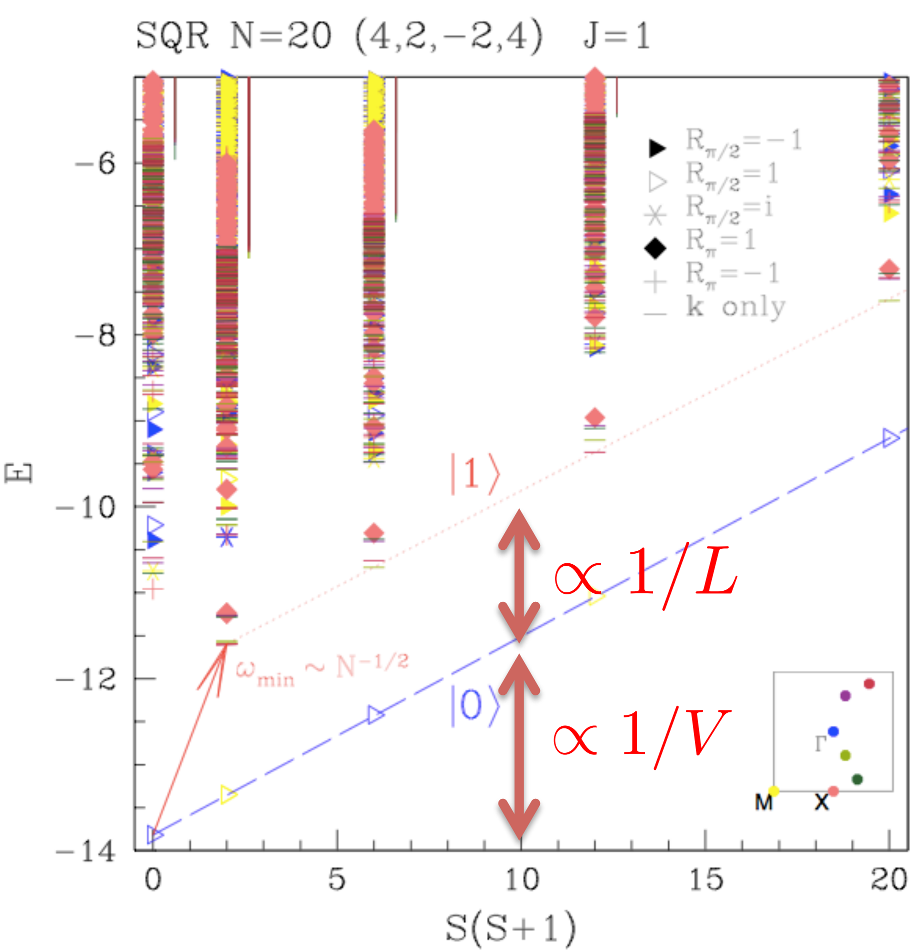
$$\rho_{ij} = \frac{\langle[S_i, S_j]\rangle}{i\Omega} = \begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix}$$ rank ρ = 2 or 0

# Anderson Tower of States

Ref: (textbooks) Sachdev, Xiao-Gang Wen, P.W. Anderson

# Antiferromagnet on a squrelattice

Simultanous diagonalization of H and $S^2$=S(S+1) (in the sector $S_z$=0)
N = 20 is the total number of sites




The exact ground state is a |S = 0, $S_z$=0 > (Marshall-Lieb-Mattis theorem)
However, this state does not have a Neel order

$$[\vec{S}^2, \text{Néel order}] \neq 0$$

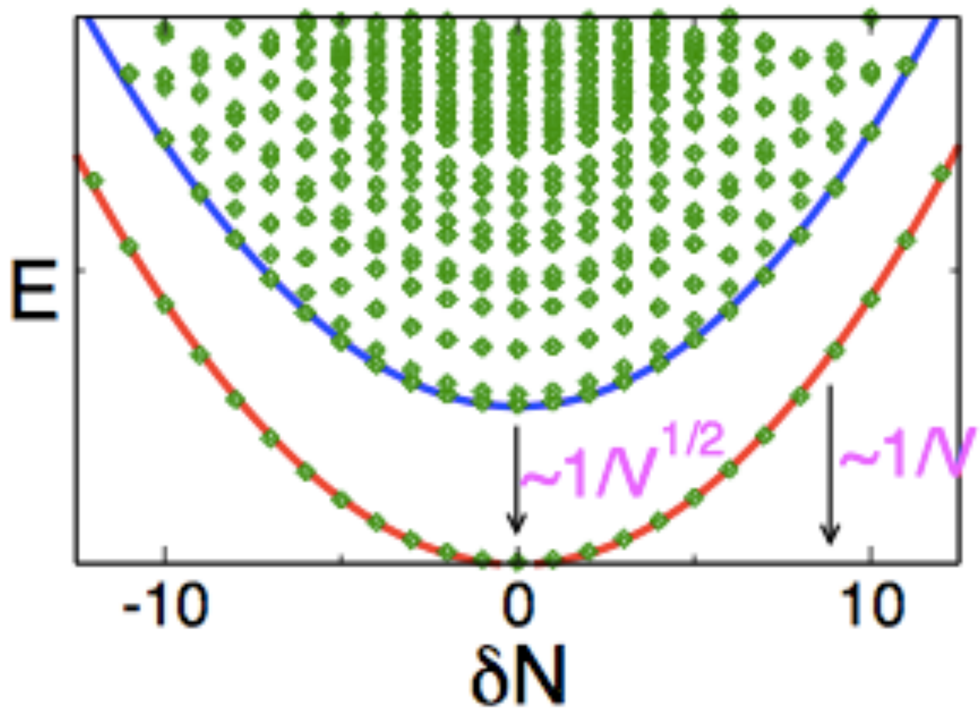
A symmetry breaking state with a well-defined order parameter is a superposition of low-lying excited state with energy S(S+1)/V = $1/L^d$

On the top of it, there is a Goldstone excitation with the excitation energy 1/L.

Well-separation of two energy scales in dimensions d > 1

Claire Lhuillier, arXiv: cond-mat/0502464

# Bose Hubbard model on a lattice for t>>U




$$[N, \theta] \neq 0$$

V. Alba et al.

http://www.mpipks-dresden.mpg.de/~esicqw12/Talks_pdf/Alba.pdf

# Tower of States from the effective Lagrangian

Nonlinear sigma model

$$\mathcal{L} = \frac{\rho}{2v^2}\dot{\vec{n}} \cdot \dot{\vec{n}} - \frac{\rho}{2}\partial_i \vec{n} \cdot \partial_i \vec{n}$$

$$\mathcal{H} = \frac{v^2}{2\rho}\vec{s} \cdot \vec{s} + \frac{\rho}{2}\partial_i \vec{n} \cdot \partial_i \vec{n}, \quad \vec{s} = (\rho/v^2)\dot{\vec{n}} \times \vec{n}$$

Fourier transform:

$$H = \frac{v^2}{2\rho V}\vec{S}^2 + \sum_{\vec{k}} \left[ \frac{v^2}{2\rho}\vec{s}_{-\vec{k}} \cdot \vec{s}_{\vec{k}} + \frac{\rho k^2}{2}\vec{n}_{-\vec{k}} \cdot \vec{n}_{\vec{k}} \right]$$

$$\vec{S} = \int \mathrm{d}^d x \, \vec{s}(\vec{x}, t)$$

Superfluid $$H_{\mathrm{TOS}} \propto \frac{(N-N_0)^2}{V}$$

Antiferromagnet $$H_{\mathrm{TOS}} \propto \frac{\vec{S}^2}{V}$$

Crystals $$H_{\mathrm{TOS}} \propto \frac{\vec{P}^2}{m n_0 V}$$

---

Ferromagnet $$H_{\mathrm{TOS}} = 0$$

Symmetry can be broken even in a finite size system / 1+1 dimension

From this argument, softer dispersion $E = p^{n>2}$ is impossible!

What happens for when both type-A and type-B present?

# Interactions

# Scaling of interactions among NGBs

- Quadratic part (free) part of action

$$S_{\text{free}}^{\text{type-A}} = \int \mathrm{d}^d x \mathrm{d}t \left( \frac{\bar{g}_{ab}(0)}{2} \dot{\pi}^a \dot{\pi}^b - \frac{g_{ab}(0)}{2} \vec{\nabla}\pi^a \cdot \vec{\nabla}\pi^b \right)$$

$$S_{\text{free}}^{\text{type-B}} = \int \mathrm{d}^d x \mathrm{d}t \left( -\frac{\rho_{ab}}{2} \pi^a \dot{\pi}^b - \frac{g_{ab}(0)}{2} \vec{\nabla}\pi^a \cdot \vec{\nabla}\pi^b \right)$$

- Scaling of fields to keep the free part

$$\pi'^a(\alpha\vec{x}, \alpha t) = \alpha^{\frac{1-d}{2}} \pi^a(\vec{x}, t)$$

$$\pi'^a(\alpha\vec{x}, \alpha^2 t) = \alpha^{-\frac{d}{2}} \pi^a(\vec{x}, t)$$

- Most relevant interactions

$$\mathrm{d}^d x \mathrm{d}t \nabla^2 \pi^3 , \quad \mathrm{d}^d x \mathrm{d}t \partial_t^2 \pi^3$$

$$\mathrm{d}^d x \mathrm{d}t \nabla^2 \pi^3 , \quad \mathrm{d}^d x \mathrm{d}t \partial_t \pi^3$$

- Their scaling raw and *condition for the free fixed point*

$$\alpha^{-\frac{1-d}{2}} \quad \Rightarrow \quad d > 1$$

$$\alpha^{-\frac{d}{2}} \quad \Rightarrow \quad d > 0$$

Symmetries will be restored
in 1+1 dimensions (Coleman's theorem)

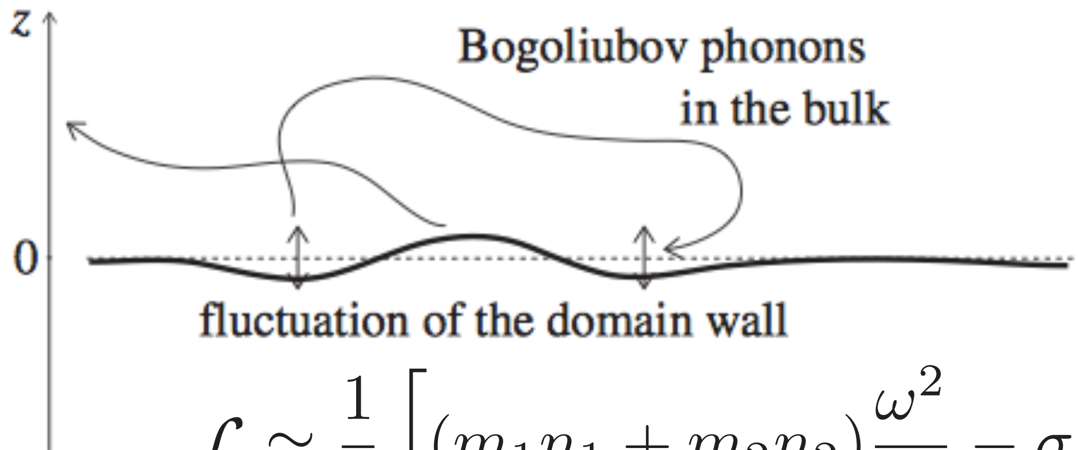
- SSB in 1+1 dimensions is OK!
- Order parameters commute with H

→ GS is one of their simultaneous eigenstates
→ No quantum fluctuation

# Ripplons

HW and H. Murayama, PRD (2014)
H. Takeuchi and K. Kasamatsu PRA (2013).

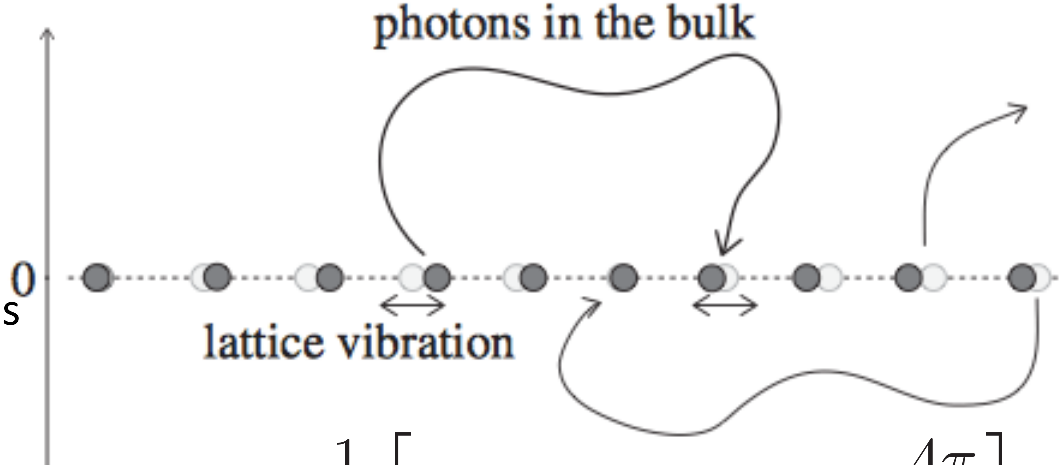
Superfluid-Superfluid interface




$$\omega \propto k^{3/2}$$

$$\mathcal{L} \simeq \frac{1}{2}\left[(m_1 n_1 + m_2 n_2)\frac{\omega^2}{k} - \sigma k^2\right] u_{-\vec{k}} u_{\vec{k}}$$

2D Crystal of electrons in 3+1 dimenisons




$$\omega \propto k^{1/2}$$

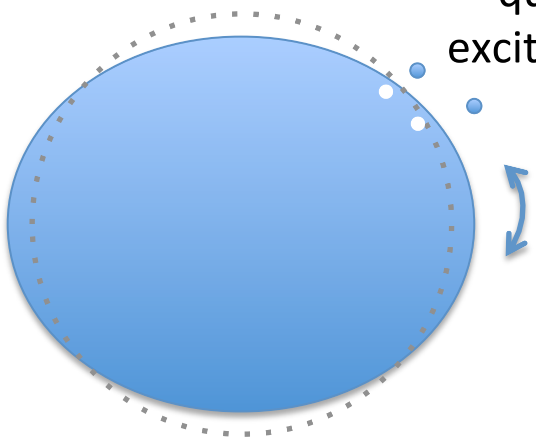
$$\mathcal{L} = \frac{1}{2}\left[m n_0 \omega^2 - (n_0 e)^2 k^2 \frac{4\pi}{k}\right] u_{-\vec{k}} u_{\vec{k}}$$

# Non-Fermi liquid through NGBs

- Usually, interaction between NGBs with other fields are derivative coupling $\psi^\dagger \vec{\nabla} \psi \cdot \vec{\nabla} \theta$

interaction vanishes in the low-energy, long wavelenghth limit

- However, there is an exception



quasi-particle excitations near FS → non-Fermi liquid behavior

Goldstone mode (orientational mode) → Landau damping

V. Oganesyan, S. A. Kivelson, and E. Fradkin, Phys. Rev. B 64, 195109 (2001).

- I pinned down the condition for NFL:

$$[Q, \vec{P}] = 0$$

HW and Ashvin Vishwanath, arXiv:1404.3728

# Conclusion

- Clarified number and dispersion of Nambu-Goldstone bosons
- Looking at the zero mode part of the effective Lagrangian, we can derive the tower of state structure
- Interactions among low-energy modes result in nontrivial outcome

NGBs with fractional-power dispersion

Non-Fermi liquid behavior of electrons

# How do you define Higgs?

In general, if no fine tuning or other special reasons,

(the number of gapped modes)

= (the total number of modes) – (the number of NGBs)

We can easily change the total number of modes without changing the symmetry of the system.

example 1: add a completely decoupled field (+ infinitesimal interaction)

$$\mathcal{L} = \mathcal{L}_{\text{Higgs}}(\psi, A) + \frac{1}{2}(\dot{\phi}^2 - m^2\phi^2) + \mathcal{L}_{\text{int}}(\phi, \psi, A)$$

example 2: copy and mix [$\Theta_1$, $\Theta_2$]

$$\mathcal{L} = \mathcal{L}_{\text{SF1}} + \mathcal{L}_{\text{SF2}} + \mu(\psi_1^\dagger\psi_2 + c.c.) - g(\psi_1^\dagger\psi_1^\dagger\psi_2\psi_2 + c.c.)$$

Is it necessary for us to look at the behavior near the critical point? (i.e., cannot define Higgs in only one phase)