安定な相対論的散逸流体方程式の構成

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Introduction

- Relativistic hydrodynamics for a perfect fluid is widely and successfully used in the RHIC phenomenology. D. Teaney et al ...

- A growing interest in dissipative hydrodynamics. hadron corona (rarefied states); Hirano et al ...
  Generically, an analysis using dissipative hydrodynamics is needed even to show the dissipative effects are small.
  A. Muronga and D. Rischke; A. K. Chaudhuri and U. Heinz; R. Baier, P. Romatschke and U. A. Wiedemann; R. Baier and P. Romatschke (2007) and the references cited in the last paper.

However, is the theory of relativistic hydrodynamics for a viscous fluid fully established?

The answer is No! unfortunately.
Fluid dynamics = a system of balance equations

\[ \partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu N^\mu = 0. \]

\[ T^{\mu\nu} \equiv \epsilon u^\mu u^\nu - p \Delta^{\mu\nu} + \delta T^{\mu\nu}, \quad N^\mu \equiv n u^\mu + \delta N^\mu \]

Dissipative part

no dissipation in the number flow; Describing the flow of matter

\[ \delta T^{\mu\nu} = w^\mu T^\lambda \left( \frac{1}{T} \nabla^\nu T - D u^\nu \right) + u^\nu T^\lambda \left( \frac{1}{T} \nabla^\mu T - D u^\mu \right) \]
\[ + 2 \eta \frac{1}{2} \left( \nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3} \Delta^{\mu\nu} \nabla \cdot u \right) + \zeta \Delta^{\mu\nu} \nabla \cdot u \]
\[ \delta N^\mu = 0. \]

--- Involving time-like derivative ---

no dissipation in energy flow Describing the energy flow.

\[ \delta T^{\mu\nu} u_\nu = 0, \]

No dissipative energy-density nor energy-flow

\[ u_\mu \delta N^\mu = 0 \]

No dissipative particle density

with transport coefficients:

\[ \zeta; \text{Bulk viscosity,} \quad \eta; \text{Shear viscosity} \]
\[ \lambda; \text{Heat conductivity} \]
Fundamental problems with relativistic hydro-dynamical equations for viscous fluid

a. Ambiguities in the form of the equation, even in the same frame and equally derived from Boltzmann equation: Landau frame: unique, Eckart frame: Eckart eq. v.s. Grad-Marle-Stewart eq.; Muronga v.s. R. Baier et al.

b. Instability of the equilibrium state in the eq.’s in the Eckart frame, which affects even the solutions of the causal equations, say, by Israel-Stewart. W. A. Hiscock and L. Lindblom (’85, ’87); R. Baier et al (’06, ’07)

c. Usual equations are acausal as the diffusion eq. is, except for Israel-Stewart and those based on the extended thermodynamics with relaxation times, but the form of causal equations is still controversial.

---- The purpose of the present talk ---

Instead of applying an existing equation to RHIC phenomenology, we focus on the fundamental problems. For analyzing the problems a and b, we deriving hydrodynamical equations for a viscous fluid from Boltzmann equation on the basis of a mechanical reduction theory (so called the RG method) and a natural ansatz on the origin of dissipation. We also show that the new equation in the Eckart frame is found to be stable, which would affect the existing and possible solutions to the problem c.
The separation of scales in the relativistic heavy-ion collisions

Liouville $\xrightarrow{\text{Boltzmann}}$ Boltzmann $\xrightarrow{\text{Fluid dyn.}}$

pQCD $\xrightarrow{\text{finite-T field th.}}$ With phase tr. (3-cr.)

Slower dynamics

Possible role of relativistic dissipative fluid dynamics, Consistently connected kinetic (Boltzmann) equation

on the basis of the RG method; Chen-Goldenfeld-Oono('95), T.K. ('95)

C.f. Y. Hatta and T.K. ('02)
K.Tsumura and TK ('05)
Geometrical image of reduction of dynamics

\[ \frac{dX}{dt} = F(X) \]

\[ \begin{align*} 
\frac{ds}{dt} &= G(s) \\
\mathcal{M} &= \{X | X = X(s)\} 
\end{align*} \]

eg.

\[ X = f(r,p) \] ; distribution function in the phase space (infinite dimensions)

\[ s = \{u^\mu, T, n\} \] ; the hydrodynamic quantities (5 dimensions), conserved quantities in collision
Derivation of the relativistic fluid dynamical equation from the rel. Boltzmann eq. --- an RG-reduction of the dynamics


**Ansatz of the origin of the dissipation** = the spatial inhomogeneity, leading to Navier-Stokes in the non-rel. case.

\[ \mathbf{a}_p^\mu \] would become a macro flow-velocity

Coarse graining of space-time

\[ \tau \equiv \mathbf{a}_p^\mu x_\mu, \quad \sigma^\mu \equiv \left(g^{\mu\nu} - \frac{\mathbf{a}_p^\mu \mathbf{a}_p^\nu}{a_p^2}\right)x_\nu \equiv \Delta_p^{\mu\nu} x_\nu \quad \mathbf{x}_\mu \quad \tau \quad \sigma^\mu \]

\[ \frac{\partial}{\partial \tau} = \frac{1}{a_p^2} \mathbf{a}_p^\mu \partial_\mu \equiv D, \quad \text{time-like derivative} \]

\[ \Delta_p^{\mu\nu} \frac{\partial}{\partial \sigma^\nu} = \Delta_p^{\mu\nu} \partial_\nu \equiv \nabla^\mu \quad \text{space-like derivative} \]

Rewrite the Boltzmann equation as,

\[ \frac{\partial}{\partial \tau} f_p(\tau, \sigma) = \frac{1}{p \cdot a_p} C[f]_p(\tau, \sigma) - \frac{1}{p \cdot a_p} p \cdot \nabla f_p(\tau, \sigma) \]

**1st order**

\[ \frac{\partial}{\partial \tau} \tilde{f}_p^{(1)} = \sum_q A_{pq} \tilde{f}_q^{(1)} + F_p \]

\[ A_{pq} \equiv \frac{1}{p \cdot a_p} \frac{\partial}{\partial f_q} C[f]_p \bigg|_{f=f^{eq}} \]

\[ F_p \equiv -\frac{1}{p \cdot a_p} p \cdot \nabla f_p^{eq} \]

Only spatial inhomogeneity leads to dissipation.

RG gives a resummed distribution function, from which \( T^{\mu\nu} \) and \( N^\mu \) are obtained.
Eckart (particle-flow) frame:

Setting  \[ a_p^\mu = \frac{m}{p \cdot u} u^\mu \]

\[ T^{\mu \nu} = (\epsilon + 3 \zeta \tilde{X}) u^\mu u^\nu - (p + \zeta \tilde{X}) \Delta^{\mu \nu} + \lambda T u^\mu \tilde{X}^\nu + \lambda T u^\nu \tilde{X}^\mu + 2 \eta X^{\mu \nu} \]

\[ N^\mu = \frac{m m u^\mu}{\delta N^\mu} = 0. \]

(i) This satisfies the GMS constraints but not the Eckart’s.
(ii) Notice that only the space-like derivative is incorporated.
(iii) This form is different from Eckart’s and Grad-Marle-Stewart’s, both of which involve the time-like derivative.

\[ \tilde{X} \equiv -\{1/3 (4/3 - \gamma)^{-1}\}^2 \nabla \cdot u \]

\[ \tilde{X}^\mu \equiv \nabla^\mu \ln T. \]

c.f. Grad-Marle-Stewart equation;

\[ \delta T^{\mu \nu} = -3 (3 T^{-1} C_T + 1)^{-1} \zeta u^\mu u^\nu \nabla \cdot u + u^\mu T \lambda \left( \frac{1}{T} \nabla^{\nu} T - D u^\nu \right) + u^{\nu} T \lambda \left( \frac{1}{T} \nabla^{\mu} T - D u^\mu \right) \]

\[ + 2 \eta \frac{1}{2} \left( \nabla^{\mu} u^\nu + \nabla^{\nu} u^\mu - \frac{2}{3} \Delta^{\mu \nu} \nabla \cdot u \right) + (3 T^{-1} C_T + 1)^{-1} \zeta \Delta^{\mu \nu} \nabla \cdot u, \]

\[ \delta N^\mu = 0. \]
Compatibility with the underlying kinetic equations?

Eckart constraints are not compatible with the Boltzmann equation. (Ch.G. van Weert (’87), Proved in K.Tsumura, T.K. and K.Ohnishi (’06))

The five collision invariants forms the fluid dynamical variables and spans the invariant manifold.
Which equation is better?

The linear stability analysis around the thermal equilibrium state.

c.f. Landau equation is stable. (Hiscock and Lindblom (’85))
Linear Stability Analysis

Def. \( T(x) = T_0 + \delta T(x) \), \( \mu(x) = \mu_0 + \delta \mu(x) \) and \( u^\mu(x) = u_0^\mu + \delta u^\mu(x) \) with \( u_0 \cdot \delta u = 0 \) and \( u \cdot u = 1 \).

Actually, we will put \( u_0 = 0 \).

Equation of Motion:

\[
0 = \partial_\mu T^{\mu\nu} = \partial_\mu \delta T^{\mu\nu} \quad \text{and} \quad 0 = \partial_\mu N^\mu = \partial_\mu \delta N^\mu
\]

Ansatz for the solution; plane-wave solution

\[
(\delta u^\mu, \delta T, \delta \mu) = (\delta \tilde{u}^\mu, \delta \tilde{T}, \delta \tilde{\mu}) e^{-i k \cdot x}
\]

\[
\sum_{\beta=1}^{5} M_{\alpha\beta} \Phi_\beta = 0,
\]

where \( M_{\alpha\beta} = M_{\alpha\beta}(k^0, \bar{k}) \)

\[
\det M_{\alpha\beta} = 0
\]

Dispersion relation: \( \omega \equiv k^0 = k^0(k) \) (generically complex.)

The stability condition: \( \text{Im}(k^0(k)) \leq 0 \quad \forall k \)
Dispersion relations

Transverse mode: \( k^0 = -i \eta |k|^2 / (\epsilon + p) \)

Longitudinal modes: \( a \omega^3 + b \omega^2 + c \omega + d = 0 \), with \( \omega = -ik^0 \)

The condition for having all the roots in the left half plane of \( \omega \) (Routh-Hurwitz theorem)

\[ a > 0, \ b > 0, \ d > 0 \text{ and } bc - ad \geq 0. \]

However,

\[ d \equiv |k|^4 \n \dot{\lambda} \frac{\partial p}{\partial \mu} T > 0 \]
The stability of the equations in the “Eckart(particle)” frame:


(i) The Eckart and Grad-Marle-Stewart equations show an instability, which has been
known, and is now found to be attributed to the fluctuation-induced dissipation,
proportional to $Du''$.

(ii) Our equation (TKO equation) can be stable, being dependent on the values of
the transport coefficients and the EOS: The numerical evaluation demonstrate that
the stability condition is satisfied for rarefied relativistic gas.
Summary and concluding remarks

• Eckart equation, which and a causal extension of which are widely used, is not compatible with the underlying Relativistic Boltsmann equation.

• The RG method gives a consistent fluid dynamical equation for the particle (Eckart) frame as well as other frames, which is new and has no time-like derivative for thermal forces.

• The linear analysis shows that the new equation in the Eckart (particle) frame can be stable in contrast to the Eckart and (Grad)-Marle-Stewart equations which involve dissipative terms proportional to $D_{\mu}u^\mu$.

• The RG method is a mechanical way for the construction of the invariant manifold of the dynamics and can be applied to derive a causal fluid dynamics, a la Grad 14-moment method. (c.f. A. Gorban and I. Karlin)

• According to the present analysis, even the causal (Israel-Stewart) equation which is an extension of Eckart equation should be modified.

• There are many fundamental isseus to clarify for establishing the relativistic fluid dynamics for a viscous fluid.