Non abelian Hydrodynamics: The Dimensional Reduction and ET Approaches

(A progress report)

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Based on, in collaboration: JJ. Fernandez-M. A. Ruiperez-Vicente:

–A new Approach to Non-Abelian Hidro., JJ, Rey,Surowka, ArXiv: 1605.06080
–Non-Abelian Hidrody.: the compactification road to Quark-Gluon Plasma. JJ,AR,ET. Arxiv:1703.nnnn

–Non abelian Hidrody. and duality: the embedding tensor and tensor hierarchies. JJ,AR,ET. Arxiv:1704.nnn

INTRO: where is MURCIA...?

MURCIA/KYOTO?

MURCIA/KYOTO Correspondence?

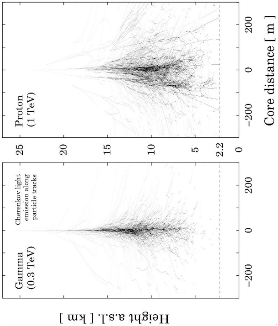
INTRO

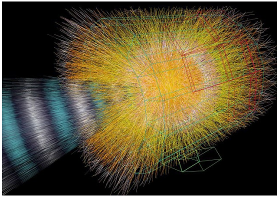
- **DEF** hydrodynamics: effective theory describing real-time dynamics of microscopic systems at large scales.
$\longrightarrow$ ADS-hydrodinamic,$\longrightarrow$ ADS/CFT: η/s bounds,
$\longrightarrow$ QCD transport coefficients: QGP, Neutron stars, Early Universe...
$\longrightarrow$ **like in effective QFT, is good strategy to explore all possibilities compatible with symmetries**

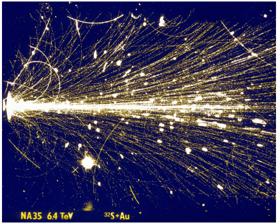
- **CONTENT:**

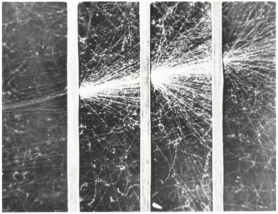
1. Introduction: Landau model. Stress tensor: "Landau Frame".
2. (Cho/Freund 75$\longrightarrow$ SS 79) D-gct$\longrightarrow$ d-gct+ n-gauge group.
   Non-abelian fluids: Dim reduc D-dim E. eqs. in n-group manifold.
   – Dim reduction of the fluid: Perfect/Dissipative
   – EX: Abelian Fluids ($n = 1, 2$). Non-abelian fluids: $n >= 3$...
3. The embedding tensor road:ET/tensor hierachies and "gauged" fluids
4. Conclusions (= summary).

INTRO: Hadronic Collisions/Landau Model

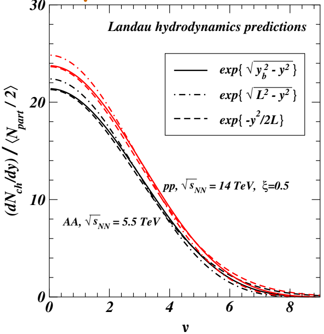


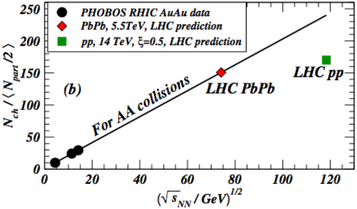






EXP/ Landau Gaussian Approximation





- Landau 1955, Kalasnikov 19544.
- Rapidity distributions. LAUNDAU Gaussian Approximation: $(\eta = \log \tan \theta)$

$$\frac{dN}{d\eta} \simeq Ce^{-\eta^2/2L},$$

$$L \simeq \frac{2c_0^2}{1 - c_0^4} \log(\sqrt{s_{NN}}/2m_p)$$

- Global Particle Multiplicity

$$N_{ch} \simeq K(\sqrt{s_{NN}})^{1/2}.$$

(scalar) fields and hydro

- Free field $\hat{\varphi}(x)$: $\mathcal{L} = 1/2(\partial_\mu \varphi)^2 - m\varphi^2$. STATE: $\hat{\rho}$.

$$\begin{aligned} \hat{\varphi} &= \varphi^+ + \varphi^- = \int dk(\hat{a}_k e^{-ikx} + \hat{a}_k^\dagger e^{ikx})\delta^+(k^2 - m^2), \\ (\Box + m^2)\hat{\varphi}^\pm &= 0. \end{aligned}$$

- Green F:

$$\begin{aligned} G(x_1, x_2) &= \mathrm{tr}\rho T \hat{\varphi}_1 \hat{\varphi}_2 = < T\hat{\varphi}_1 \varphi_2 > \\ &= \theta^+ G^+ + \theta^- G^-, \quad (\theta^\pm = \theta(\pm x)), \\ (\Box + m^2) G(x_1, x_2) &= -i\delta^4(x_1 - x_2), \\ G^+_{12} &= G^-_{21} = < \hat{\varphi}_1 \hat{\varphi}_2 > = < \hat{\varphi}^-_1 \hat{\varphi}^+_2 > \end{aligned}$$

- WIGNER DIST.: CM, rel: $x_{1,2} = x \pm r/2$, $p_{1,2} = p \pm q/2$.

$$\begin{aligned} G^\pm(x,p) &= \int d^4 r e^{ipr} G^\pm(x + r/2, x - r/2), \\ DEF: \quad f(x,p) &= \theta(p_0) G^-(x,p) = \int d^4 x e^{ipr} < \varphi_1 \varphi_2 > . \end{aligned}$$

- *FROM* $$(\Box + m^2)\hat{\varphi}^{\pm} \quad = \quad 0 \longrightarrow p^{\mu}\partial_{\mu} f(x,p) = 0, \quad (BE).$$
- DEFs:

$$J^{\mu} = nu^{\mu} \quad = \quad \int d^4 p f(x,p) p^{u} =< \hat{\varphi}^{-}\partial_{\mu}\hat{\varphi}_2^{+} - \hat{\varphi}^{+}\partial_{\mu}\hat{\varphi}_2^{-} >$$

$$T^{\mu\nu} \quad = \quad \int d^4 p f(x,p) p^{\mu} p^{\nu}.$$

- COSERVATION: FROM BE:

$$\partial_{\mu} J^{\mu} \quad = \quad \int d^4\ p \partial_{\mu} f(x,p) p^{u} = 0,$$

$$\partial_{\mu} T^{\mu\nu} \quad = \quad \int d^4 p\ \partial_{\mu} f(x,p) p^{\mu} p^{\nu} = 0.$$

Thermal properties

- Assume $\rho = \rho_{eq} + \rho^1 + ..., \longrightarrow G = G_{eq} + ..., \; f = f_{eq} + ...$
- Take $\rho_{eq} = e^{-\beta \hat{H}}/Z$:

$$\begin{aligned} G^-_{eq}(x,p) = & \quad \delta(p^2 - m^2)[\theta^+ n_{BE}(p_0) + \theta^-(1 + n_{BE}(-p_0))], \\ f_{eq}(x,p) \equiv & \quad \delta^+(p^2 - m^2) g_{eq}(x,p) = \delta^+(p^2 - m^2) n_{BE}(E_p) \end{aligned}$$

$$THEN: \quad 1/V < dN/d^3p >_V \; = \; \frac{1}{e^{\beta w_p} - 1},$$

$$< N(x) >= n(x) = \; = \; \int d^3 p n_{eq},$$

$$\epsilon(x) = \int d^3 p E_p n_{eq}, \qquad p(x) = \int d^3 p (p^2/3E_p) n_{eq}$$

- $\longrightarrow T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu - p g^{\mu\nu}$.
- ADD TERMS: A) non-eq $\rho \longrightarrow f = f_{eq} + ...$ B) Interacting fields.

$\longrightarrow$ Transport coeffs: Kubo relations: $\eta \sim T^3/\lambda^2$

Landau Model: free pion gas: basic equations

- ASSUME: quantum scalar field $\sim$ pion with thermal density matrix.
- BASIC HIDRO EQUATIONS: $(w = e + p)$

$$
\begin{aligned}
T^{\mu\nu} &= w u^\mu u^\nu - p g^{\mu\nu}. \\
\partial_\mu T^{\mu\nu} &= 0. \\
(EULER) \quad & w u^\nu \partial_\nu u^\mu = \Pi^{\mu\rho} \partial p, \\
& w \partial_\nu u^\nu = -u^\nu \partial_\nu (w - p)
\end{aligned}
$$

- THERMO EQS: pion number: non conserved, $\mu = 0$.

$$
\begin{aligned}
\epsilon + p &= Ts, \quad dp = s dT, \quad d\epsilon = T ds, \\
\epsilon_B &= k(p - B), \; k = c_0^{-2}. \\
c_0^2 &= dp/d\epsilon = cte.
\end{aligned}
$$

- OBJECTIVE: SOLVE THE EQUATIONS...

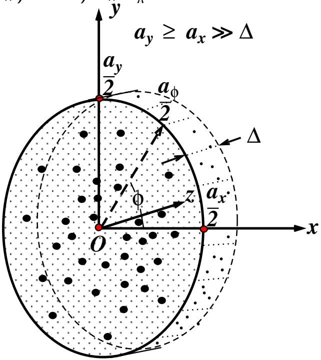
OBJECTIVE: SOLUTIONS

- A) Thermo eqs: $(V_\pi = 4/3m_\pi^{-3}, T_\pi = m_\pi, B = \text{Bag cte.})$

$$
\begin{aligned}
s &= \alpha(T/T_\pi)^\beta = \lambda(1+c_0^2)(T/m_\pi)^{c_0^{-2}} V_\pi^{-1}, \\
\epsilon &= \alpha_2(T/T_\pi)^{c_2} + c_4 = (\lambda(T/m_\pi)^{1+c_0^{-2}} + B)m_\pi V_\pi^{-1}, \\
p &= \alpha_3(T/T_\pi)^{c_2} - c_4 = (\lambda c_0^2(T/m_\pi)^{1+c_0^{-2}})m_\pi V_\pi^{-1}
\end{aligned}
$$

- **B) solve Hidrodinamics:**
$\epsilon(x,t), u(x,t)$...
$\longrightarrow N(E_{CM}), dN/d\theta, ...$

- Landau/Kalasnitkov SOLUTION:
decouple Long/transversal modes.



Initial configuration

Landau model solution: Decoupling

- $(u = (u^0, u^1, u^2) = (\cosh y, \sinh y, u^0 v_x), t_\pm = t \pm z, y\pm = \log(t_\pm/\Delta))$

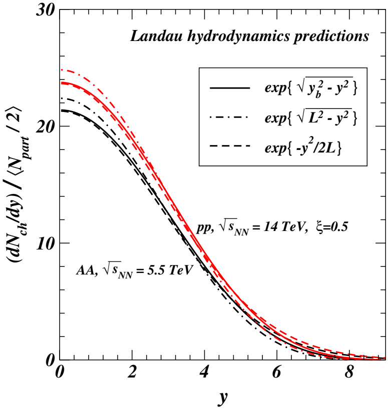
$$\begin{aligned} \frac{T^{00}}{\partial t} + \frac{T^{01}}{\partial z} &= 0, \\ \frac{T^{01}}{\partial t} + \frac{T^{11}}{\partial z} &= 0, \\ \frac{T^{02}}{\partial t} + \frac{T^{22}}{\partial x} &= 0. \end{aligned}$$

$$\begin{aligned} \frac{\partial \epsilon}{\partial t_+} + 2\frac{\partial \epsilon e^{-2y}}{\partial t_-} &= 0, \\ 2\frac{\partial \epsilon e^{2y}}{\partial t_+} + \frac{\partial \epsilon}{\partial t_-} &= 0, \\ 4/3\epsilon (u^0)^2 \frac{\partial v_x}{\partial t} &= -\frac{\partial p}{\partial x}. \end{aligned}$$

- Solutions (1st stage, 2nd stage):

$$\begin{aligned} 1st \qquad & \epsilon(t,z) = \epsilon(y_+, y_-) = \epsilon_0 e^{-4/3)(y_+ + y_- - \sqrt{y_+ y_-})}, \\ & y(t,z) = y(y_+, y_-) = ..., \\ 2snd \qquad & \epsilon(t,y) = \epsilon(t_F, y) t_F^4/t^4, \\ & u^0(t,y) = u^0(t_F, y) t/t_F. \end{aligned}$$

- Finally $\longrightarrow$ Get $N, dN/dy = F(..), \longrightarrow$ Landau Gaussian



- Gaussian aproximation:

$(y_b \simeq \log(\sqrt{s_{NN}}/m_p))$

$$\frac{dN}{dy} \simeq \exp\sqrt{y_b^2 - y^2}, \simeq \exp(-\frac{y^2}{(2y_b^2)}).$$

- ...EVEN AT LHC M€ €...!!

INTRO: what is a fluid dinamical system?. Ingredients..

- 1)stress tensor: $T_{\mu\nu} = T^f + T^{nf}$, $T_{\mu\nu}(u, w, p, \theta, \mu, s, T.., \partial u; F, \varphi))$.

From: A)Micro-theory (QFT,Kin.,..), B)Symmetry/Pheon.qualitative

- 2) Equations (abelian charged fluid) $(D + ...)$:

$$\begin{aligned} \nabla_\mu T^{\nu\mu} &= 0 \to \nabla^\mu T^f_{\nu\mu} = F^{\mu\nu} J_\mu \\ \nabla_\mu J^\mu &= c F_{\mu\nu} \tilde{F}^{\mu\nu}, \\ \nabla^\mu F_{\mu\nu} &= J_\nu \end{aligned}$$

- 3) THERMO: Functional eqs ("state eqs") + integrable Pfaffian eqs. ("thermodinamic relations")

$$\begin{aligned} w = \rho + p &= Ts + \mu_I q_I, \\ d\rho &= Tds + \mu_I dq_I, \\ dp &= sdT + q_I d\mu_i. \end{aligned}$$

- ISSUES: Stability?, causality?, hiperbolicity?, 2nd law $\partial S >= 0$?.
- What is the structure of T^f, J??

$$
\begin{aligned}
T^f_{\mu\nu} &= T_{\mu\nu}(....u, \partial u, F), \\
J_\mu &= J_\mu(u, \partial u, F...).
\end{aligned}
$$

- $T^f_{\mu\nu}$: general physical structure: Free/eq+ non-eq/interactions

$$
T^{\nu\mu} = T^{pf} + T^{diss} = wu^\nu u^\mu + pg^{\nu\mu} + \tau^{\nu\mu}
$$

$$
\tau^{\nu\mu} = \sum_{n=1,\infty} \tau^{(n),\nu\mu}(\partial^n u...)
$$

- 1st order dissipative $\longrightarrow$ A) inestabilities, B) Causality/infinite speed propagation $\longrightarrow$ 2nd order

INTRO FLUIDS: T_{AB} general algebraic structure

- $D(D+1)/2$ terms. Fix: Time-like u_A, q_A: space like.

$\Pi_{AB} = \eta_{AB} - u_A u_B$:

- Theorem: General algebraic decomposition:

$$\begin{aligned} T_{AB} &= \rho u_A u_B + q_A u_B + u_A q_B + \tau_{AB}, \\ J_A &= q u_A + \nu_A. \end{aligned}$$

where

$$\begin{aligned} u_A u^A &= \sigma, \\ u_A q^A &= 0, \quad u_A \nu^A = 0 \\ \tau_{AB} u^B &= 0, \\ \tau_{AB} &= \varpi + (\theta + p)\Pi_{AB} \end{aligned}$$

- (ϖ: traceless shear tensor, θ : dissipative pressure).
- u_A CHOICE: Arbitrary: Eckart(1940), "Landau Frame",...

"Landau Frame" Model

(A) • u^A : timelike eigenv. of T_{AB}. $\longrightarrow$ $u_\mu \propto$ energy flux density

$$\begin{aligned} T^{AB} u_B &\propto u^A \\ \nu_A u^A &= 0. \end{aligned}$$

$\longrightarrow$ This implies:

$$\begin{aligned} \rho u^A + q_A &\propto u^A \\ q_A &= 0. \end{aligned}$$

$$\longrightarrow \quad \begin{aligned} T_{AB} &= \rho u_A u_B + \tau_{AB} = w u_A u_B + p \Pi_{AB} + (\varpi_{AB} + \theta \Pi_{AB}). \\ J_A &= q u_A + \Pi_{AB} \nu_B. \end{aligned}$$

(B) • ASSUME only 1st order terms: $\tau \sim f(\nabla_\mu u_\nu, F_{\mu\nu}, ...)$

Landau Model: dissipative structure:

- $$\nabla_\mu u_\nu = \sigma_{\mu\nu} + \omega_{\mu\nu} - a_\mu u_\nu + \frac{1}{D-1}\theta\Pi_{\mu\nu}$$

where
$$\begin{aligned}\theta &\equiv \nabla_\mu u^\mu, \quad a_\mu \equiv u^\nu \nabla_\nu u_\mu, \\ \sigma_{\mu\nu} &= \nabla_{(\mu} u_{\nu)} + u_{(\mu} a_{\nu)} - \frac{1}{D-1}\theta P_{\mu\nu} \\ \omega_{\mu\nu} &= \nabla_{[\mu} u_{\nu]} + u_{[\mu} a_{\nu]}.\end{aligned}$$

- Physically (Landau): η, ζ : shear-, bulk- viscosities.

$$\tau^{\text{diss}}_{\mu\nu} = \varpi_{\mu\nu} + \theta\Pi_{\mu\nu} = -2\eta\sigma_{\mu\nu} - \zeta\Pi_{\mu\nu}\theta,$$

- A) "KIN": σ, θ, B) "DIN/EXP:" η, ζ:

$\longrightarrow$ For $\lambda\phi^4 \longrightarrow \eta \propto_{T\to\infty} 1/\lambda^2$.
$\longrightarrow$ QCD;: $\eta \propto 1/g^4 \log(1/g)$.
$\longrightarrow$ ADS/CFT: $\eta/s \sim 1/4\pi$.

- ENTROPY: from $Ts + \mu n = w$, the entropy current relation

$$s^{\mu} \quad = \quad \frac{1}{T}\left(pg^{\mu\nu} - T^{\mu\nu}\right)u_{\nu} - \frac{\mu}{T}n^{\mu}$$

- In addition, the Landau current terms:

$$\begin{aligned} J^{I}_{\mu} &= q^{I}u_{\mu} - \frac{\mu^{I}}{T}\nu^{I}_{\mu}, \\ s_{\mu} &= su_{\mu} - \frac{\mu}{T}\nu_{i} - \frac{\mu_{I}}{T}\nu^{I}_{\mu} \\ \nu^{I}_{\mu} &= \kappa^{IJ}P_{\mu\nu}\nabla^{\nu}\frac{\mu^{J}}{T}, \quad \nu_{\mu} = \kappa_{T}P_{\mu\nu}\nabla^{\nu}\frac{\mu}{T}. \end{aligned}$$

$\kappa^{IJ}, \kappa_{T} = \kappa'_{T}(\frac{nT}{w})^{2}$ charge, thermal conductivities.

Entropy

- Use Equilibrium relations: $w = Ts + \mu n + \mu^I q_I$.

$$\partial_\mu s^\mu \quad \sim \quad \varpi^2 + \theta^2 + n^2 > 0$$

- LANDAU problems: instabilites...$\longrightarrow$ Israel-stewart second order...
- **..BUT: "off-the-shelf" choice.**
- **PROBLEMATIC for fluid dimensional reduction: Consistency??**

$$\hat{T}^{AB} u_B \quad \propto \quad \hat{u}^A, \longrightarrow$$
$$T^{\mu\nu} u_\nu \quad \propto \quad u^\mu \ ??.$$

FLUID DIMENSIONAL REDUCTION

- D-dim EOMs: $$\hat{G}_{AB} = \hat{R}_{AB} - \frac{1}{2}\eta_{AB}\hat{R} = \hat{T}^{\mathrm{f}}_{AB}.$$

$\hat{T}_{AB}$: (DISSIPATIVE)-FLUID $(\hat{u}_A, \hat{\epsilon}, \hat{p}, \hat{s})+$ Thermo eqs.

- $M_D \longrightarrow M_d + X_n$. (SS Metric+Fluid ansatz)

$\longrightarrow$ D-gct $\longrightarrow$ d-gct+gauge symmetry
$\longrightarrow$ $D-$ fields $(\hat{g}, \hat{u})$ $\longrightarrow$ $(g, u, A, \varphi, axions...)$
$\longrightarrow$ D- Einsteins Eqs. $\longrightarrow$ d-Einstein Eqs+ gauge-eqs+scalar eqs.
$\longrightarrow$ D-Bianchis $\longrightarrow$ d-Bianchis+ gauge cons. laws
$\longrightarrow$ D-thermo $\longrightarrow$ fluid tensor, thermo relations.

- NOTATION: D-dim $\hat{x} = (x^\mu, z^m)$. Flat: $A = (a, \alpha)$, world $M = (\mu, m)$.
$\hat{\eta}_{AB} = (+ - \cdots -)$.

DIM reduction: Metric Ansatz

- D-dim: $\hat{g}_{MN} = E_M{}^A E_N^B \hat{\eta}_{AB}, \quad E_M{}^A E_A{}^N = \delta_M{}^N$
- Following Scherk-Schwarz (1978):

$$E_M{}^A(\hat{x}) = \begin{pmatrix} e_\mu{}^a & e^{\beta\varphi}\Phi_m{}^\alpha A_\mu{}^m \\ 0 & e^{\beta\varphi}u_n{}^m(z)\Phi_m{}^\alpha \end{pmatrix},$$

$$\Phi_s{}^\alpha \Phi_\alpha{}^p \quad = \delta_s{}^p. \quad \det \Phi_m{}^\alpha = 1$$

- We have $\sqrt{\hat{g}} = \sqrt{g}\sqrt{g_z} = \det E_M^A = u(z) \det(\Phi_m^\alpha) e^{n\beta\varphi} \det(e_\mu^a)$.

D-gct ANSATZ

- D-Eqs. are invariant under D-gct transformations:

$$\delta\hat{x}^M = -\hat{\xi}^M(\hat{x}), \rightarrow \delta E_M{}^A = \hat{\xi}^N\partial_N E_M{}^A + \partial_M\hat{\xi}^N E_N{}^A,$$
$$\delta\hat{g}_{MN} = \hat{\xi}^P\partial_P\hat{g}_{MN} + \hat{g}_{RN}\partial_M\hat{\xi}^R + \hat{g}_{MR}\partial_N\hat{\xi}^R$$

- Subgroup PRESERVING reduction ansatz: $\delta\hat{x}^M = -\xi^M(x,z)$.

Clousure constraint: $[\xi_2,\xi_1] = \xi_2\partial\xi_1 - \xi_1\partial\xi_2 = \xi_3$.

- $\longrightarrow$ Standard possibility:

$$\xi^M : \quad \hat{\xi^\mu}(x,z) = \xi^\mu(x),$$
$$\hat{\xi^m}(x,z) = (u^{-1}(z))_n{}^m\xi^n(x) \quad (+\Lambda^m_n z^n)$$

- Then $\quad [\xi_2,\xi_1] = \xi_3, \rightarrow \xi_3^P = f_{MN}{}^P\xi^P{}_1(x)\xi^N{}_2(x),$

$$f_{MN}{}^P \equiv (u^{-1}u^{-1}(\partial u - \partial u) = cte.$$

- $\longrightarrow$ z^m coordinates of a Lie group manifold. $\sigma^m = u^m{}_n dz^n$.

Interpretation of transformations?

- ASSUME:

$$\begin{aligned} \xi^{\hat{\mu}} &= \xi^{\mu}(x), \\ \xi^{\hat{m}} &= u^{-1}(z)_n{}^m \xi^n(x) + \Lambda^m_n z^n \end{aligned}$$

- for $\xi^{\hat{\mu}} = \xi^{\mu} \longrightarrow$ d-gct.
- for $\xi^{\hat{m}} = (u^{-1}(z))_n{}^m \xi^n$:

$$\begin{aligned} \delta g &= 0, \\ \delta A^m_\mu &= \partial_\mu \xi^m + f_{np}{}^m \xi^n A^p_\mu, \\ \delta h_{mn} &= f_{mp}{}^q \xi^p h_{qn} + f^q_{np} \xi^p h_{mq}. \end{aligned}$$

$\longrightarrow$ Non abelian gauge transf., n gauge vectors, $f_{mn}{}^p$.

- for $\xi^{\hat{m}} = \Lambda^m_n z^n$: $\Lambda \in Aut(f)$. $\longrightarrow$ GL global scale symm.

$$\begin{aligned} \delta g &= k g_{\mu\nu}, \\ \delta A^m_\mu &= \Lambda^m_n A^p_\mu, \\ \delta \phi &= k \end{aligned}$$

Additional constraints on TWIST MATRIX $u(z)_n{}^m$

- D-gct scalars $\phi(\hat{x}^\mu) = \phi(x^\mu) \longrightarrow$ d-gct scalars.
- D-gct scalar densities $\sqrt{\hat{g}}\phi(\hat{x}^\mu) \longrightarrow$ are d-gct densities?. $(S = \int ...)$

$$\begin{aligned}
\delta(\sqrt{\hat{g}}\phi) &= \partial_M(\sqrt{\hat{g}}\phi\hat{\xi}^M) \\
&= \partial_\mu(\sqrt{g}\sqrt{g_z}\phi\hat{\xi}^\mu) + \partial_m(\sqrt{g}\sqrt{g_z}\phi(u^{-1})_n{}^m\xi^n) \\
&= u\partial_\mu(\sqrt{g}\ e^{n\beta\varphi}\phi\xi^\mu) + \sqrt{g}\phi\xi^n\partial_m(\sqrt{g_z}(u^{-1})_n{}^m).
\end{aligned}$$

- Two possibilities:

A) $\partial_m(\sqrt{g_z}(u^{-1})_n{}^m) = 0, = \partial_m(u(u^{-1})_n{}^m)$.

$\longrightarrow f_{mn}{}^n = 0 \sim tr(T_m^{ad})$: unimod groups (compact or non-compact)

$\longrightarrow$ action invariant.

B) $\longrightarrow f_{mn}{}^n \neq 0$: non-compact groups

$\longrightarrow$ action not invariant (EOMS invariant).

Spin-connection

- 1st STEP:

$$
\begin{aligned}
\hat{\omega}_{cab} &= \omega_{cab}, \\
\hat{\omega}_{\gamma ab} &= -\tfrac{1}{2} e^{\beta\varphi} F^{m}{}_{ab} \Phi_{m\gamma}, \\
\hat{\omega}_{ac\beta} &= \tfrac{1}{2} e^{\beta\varphi} F^{m}{}_{ac} \Phi_{m\beta}, \\
\hat{\omega}_{\gamma\alpha b} &= \mathbb{P}_{a\beta\gamma} + \beta \partial_a \varphi \delta_{\beta\gamma} \\
\hat{\omega}_{c\alpha\beta} &= \mathbb{Q}_{c\alpha\beta}, \\
\hat{\omega}_{\gamma\alpha\beta} &= \tfrac{\epsilon_V}{2} e^{-\beta\varphi} f_{mn}{}^{p} \left[\Phi_{\gamma}{}^{n} \Phi_{\alpha}{}^{m} \Phi_{p\beta} + \Phi_{\alpha}{}^{m} \Phi_{\beta}{}^{n} \Phi_{p\gamma} - \Phi_{\beta}{}^{m} \Phi_{\gamma}{}^{n} \Phi_{p\alpha} \right],
\end{aligned}
$$

where $\longrightarrow$ the spin-connection only depends on the external coordinates.

$$
\begin{aligned}
F^{m}{}_{ab} &= \partial_{[a} A^{m}{}_{b]} - f_{np}{}^{m} A^{n}{}_{a} A^{p}{}_{b}, \\
\Phi_{m\alpha} &\equiv \Phi_{m}{}^{\beta} \delta_{\alpha\beta}, \ \mathbb{P}, \mathbb{Q} = \ldots \Phi \partial \Phi \ldots
\end{aligned}
$$

- SECOND STEP (EFRAME): Redefine: $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = e^{2\alpha\varphi} g_{\mu\nu}$

where $\alpha = -\frac{n}{d-2}\beta, \quad \alpha^2 = \frac{n}{2(d+n-2)(d-2)}$.

2ND STEP: The Einstein frame

$$\hat{R}_{ab} = e^{-2\alpha\varphi}\left[R_{ab} + \frac{1}{2}\partial_a\varphi\partial_b\varphi + \alpha\eta_{ab}\nabla^2\varphi + \frac{1}{2}e^{\kappa_1\varphi}F^m{}_{ad}F^{nd}{}_b h_{mn} + \mathbb{P}^2_{ab}\right],$$

$$\hat{R}_{\alpha b} = \frac{1}{2}e^{\kappa_2\varphi}e_b{}^\mu\left\{\nabla_\rho\left[e^{\kappa_2\varphi}F^m{}_\mu{}^\rho\Phi_{m\alpha}\right] + e^{\kappa_2\varphi}F^m{}_\mu{}^\rho\Phi_m{}^\gamma\Phi_\alpha{}^n D_\rho\Phi_{n\gamma} + 2\mathbb{P}_{\mu\gamma\beta}f_{nm}{}^p\Phi_\alpha\right.$$

$$\hat{R}_{\alpha\beta} = e^{-2\alpha\varphi}\left[-\mathfrak{D}_\rho\mathbb{P}^\rho{}_{\alpha\beta} + \delta_{\alpha\beta}\nabla^2\varphi + \frac{1}{4}e^{\kappa_1\varphi}F^n{}_{\rho\sigma}F^{n\sigma\rho}\Phi_{m\alpha}\Phi_{n\beta} + \frac{1}{4}e^{-\kappa_1\varphi}V_{\alpha\beta}\right],$$

where $D = ..., \mathfrak{D}_\mu = D_\mu + \mathbb{Q}....$
In addition

$$\hat{R} = R + \frac{1}{2}(\partial\varphi)^2 + \frac{1}{4}e^{2(\beta-\alpha)\varphi}F^2(\Phi) + \mathbb{P}^2 - \frac{1}{4}e^{-2(\beta-\alpha)\varphi}V(\Phi).$$

with $(h^{nm} \equiv \delta^{\alpha\beta}\Phi_\alpha{}^n\Phi_\beta{}^m)$

$$V(h) = f_{nm}{}^p f_{rs}{}^t h^{nr}h^{ms}h_{pt} + 2f_{nm}{}^r f_{rs}{}^n h^{ms}. \quad \partial_\nu V = 2V_{\alpha\beta}P^{\alpha\beta}_\nu.$$

EOMs

- D-dim EOMs: $$\hat{R}_{AB} - \frac{1}{2}\eta_{AB}\hat{R} = \hat{T}^{\mathrm{f}}_{AB}.$$
- Split:
$$\begin{aligned}
\hat{R}_{ab} - \frac{1}{2}\eta_{ab}\hat{R} &= \hat{T}^{\mathrm{f}}_{ab}, \\
\hat{R}_{a\beta} &= \hat{T}^{\mathrm{f}}_{a\beta}, \\
\hat{R}_{\alpha\beta} &= \hat{T}^{\mathrm{f}}_{\alpha\beta} + \delta_{\alpha\beta}\frac{\hat{T}^{\mathrm{f}}}{d+n-2}.
\end{aligned}$$

where $$\hat{T}^{\mathrm{f}} \equiv \hat{\eta}^{AB}\hat{T}^{f}{}_{AB} = \eta^{ab}\hat{T}^{f}{}_{ab} - \delta^{\alpha\beta}\hat{T}^{f}{}_{\alpha\beta}.$$

Matter EOMS, Einstein frame.

- Gauge Eq. where $Q_c = e^{\kappa/2\varphi(x)}. \kappa = \frac{\alpha(d+n-2)}{2n}, F^2 = ...$

$$\begin{aligned} D^\rho \left[Q_c^{-2} h_{mn} F^m{}_{\mu\rho}\right] &= Q_c^{-2} j_{\mu n}, \\ Q_c^{-2} j_{\mu n} &= 2e^{-\kappa/2\varphi} e_\mu{}^b \Phi_n{}^\alpha \hat{T}^f_{\alpha b} + 2\mathbb{P}_{\mu\gamma\beta} f_{nm}{}^p \Phi^{\gamma m} \Phi_p{}^\beta. \end{aligned}$$

- Dilaton field:

$$\nabla^2\varphi = \kappa(-Q_c^{-2}F^2 - Q_c^2 V + 4e^{2\alpha\varphi}\hat{T}_{eff})$$

- Axion fields: with $\mathfrak{D}_\rho = ...$

$$\begin{aligned} \mathfrak{D}_\rho \mathbb{P}^\rho{}_{\alpha\beta} = &-\frac{1}{4} Q_c^{-2}\left(F^m{}_{\rho\sigma}F^{n\rho\sigma}\Phi_{m\alpha}\Phi_{n\beta} - \frac{1}{n}F^2(\Phi)\right) + \\ &\frac{1}{4}Q_c^2\left[V_{\alpha\beta} - \frac{1}{n}V\right] - e^{2\alpha\varphi}\left(\hat{T}^f_{\alpha\beta} - \frac{1}{n}\delta^{\alpha\beta}\hat{T}^f_{\alpha\beta}\right). \end{aligned}$$

d-Einstein Equations (Einstein Frame)

- $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$

$$
\begin{aligned}
T_{\mu\nu} \quad = \quad & T^{\mathrm{f}}_{\mu\nu} - \frac{1}{2} Q_c^{-2} h_{mn} \left[F^{m}{}_{\mu\rho} F^{n\rho}{}_{\nu} - \frac{1}{4} g_{\mu\nu} F^{m\mu\nu} F^{n}_{\mu\nu} \right] - \\
& \frac{1}{2} \left[\partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} (\partial\varphi)^2 \right] - \left(\mathbb{P}^2_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathbb{P}^2 \right) - \frac{1}{8} g_{\mu\nu} Q_c^2 V(\Phi).
\end{aligned}
$$

- where $$T^{\mathrm{f}}{}_{\mu\nu} \quad = \quad e^{2\alpha\varphi} e_\mu{}^a e_\nu{}^b \hat{T}^{\mathrm{f}}{}_{ab},$$

conservation laws

- From gauge Bianchi/Ricci: $(DF^n)_{\mu\nu\rho} = 0$.

$$\mathrm{j}^{m\mu} = h^{mn}\left[j_n{}^{\mu} - Q_c^2 F^{r\,\mu\rho} D_\rho\left(Q_c^{-2} h_{nr}\right)\right]$$
$$D_\mu \mathrm{j}^{m\mu} = 0.$$

- From Einstein Bianchis $\longrightarrow$ LORENTZ FORCE EQUATION

$$\nabla^\mu T_{\mu\nu} = 0 = \nabla^\mu (T^f_{\mu\nu} + \ldots) \longrightarrow$$
$$\nabla^\mu T^f{}_{\mu\nu} = \frac{1}{2} F^n{}_{\nu\rho} j_n{}^\rho + \tilde{\mathfrak{D}}_{\alpha\beta} \mathbb{P}_\nu{}^{\alpha\beta} + \alpha \tilde{D} \partial_\nu \varphi$$

with $$\tilde{D} = \tilde{D}(T^f, \varphi), \quad \tilde{\mathfrak{D}}_{\alpha\beta} = \tilde{\mathfrak{D}}_{\alpha\beta}(F, \Phi, T^f).$$

Matter reduction: $\hat{T}^f_{AB}$ Motivation

- Metric tensor, vielbein, reduction ansatz $\hat{x} = (x^\mu, z^m)$

$$\hat{g}_{MN}(\hat{x}) = \hat{g}_{MN}(x,z) \quad \sim \quad \begin{pmatrix} A(x) & B(x,z) \\ B^t(x,z) & C(x,z). \end{pmatrix}$$

$$E_M{}^A(x,z) \quad \sim \quad U_M{}^N(z)\tilde{E}_N{}^A(x) \sim \begin{pmatrix} \delta_\mu{}^\nu & 0 \\ 0 & u_n{}^m(z) \end{pmatrix} \begin{pmatrix} \dots & \dots \\ 0 & \dots \end{pmatrix}$$

- D-Einstein Tensor:

$$\hat{G}_{AB}(\hat{x}) \quad \equiv \quad \hat{R}_{AB} - \frac{1}{2}\hat{g}_{AB}\hat{R} \sim \begin{pmatrix} A(x) & B(x) \\ B(x) & C(x) \end{pmatrix},$$

$$\begin{aligned} \hat{G}_{MN}(x,z) \quad &= \quad E_M{}^A(x,z)E^B_N(x,z)\hat{G}_{AB}(x) \\ &\equiv \quad U_M{}^P(z)U_N{}^S(z)\tilde{G}_{PS}(x) \end{aligned}$$

where $\tilde{G}_{PS} \sim \tilde{E}_P{}^A\tilde{E}_S{}^B\hat{G}_{AB}$.

- A generic stress tensor:

$$\hat{T}_{AB} \to \hat{T}_{MN}(x,z) = E_M{}^A(x,z)E_N^B(x,z)\hat{T}_{AB}(x,z)$$
$$\equiv U_M{}^P(z)U_N{}^S(z)\tilde{T}_{PS}(x,z).$$

- Einstein eqs.: in flat indices: $\hat{G}_{AB} = \hat{T}_{AB}$, $\longrightarrow$ In curved:

$$U_M{}^P(z)U_N{}^S(z)\tilde{G}_{PS}(x) = U_M{}^P(z)U_N{}^S(z)\tilde{T}_{PS},$$

implies

$$\tilde{G}_{PS}(x) = \tilde{T}_{PS}, \longrightarrow \tilde{T}_{PS} = \tilde{T}_{PS}(x)$$
$$\hat{G}_{AB}(x) = \hat{T}_{AB}.$$

- CONSEQUENCE: Any consistent ansatz: $\hat{T}_{AB}(x^\mu)$.

Fluid compactification ansatz:

- $\hat{T}^f_{AB} = \hat{T}^f_{AB}(\hat{u}, \hat{w}, \hat{p}, \partial_A u_B ...)$. $\hat{u}_A$: time like
- Constraint: $\hat{u}_A = (\hat{u}_a, \hat{u}_\alpha)$ $\hat{u}_A \hat{u}^A = \sigma (= +1)$.
- FLUID ANSATZ:

A) TO KEEP normalization ($\longrightarrow$ $u_\mu u^\mu = 1$):

$$\begin{aligned} \hat{u}^a(x) &= u^a(x) \cosh \xi(x), \\ \hat{u}^\alpha(x) &= n^\alpha(x) \sinh \xi(x), \end{aligned}$$

imply $u^a u^b \eta_{ab} = 1, \quad n^\alpha n^\beta \delta_{\alpha\beta} = 1,$

B) Any other quantity, generically $\hat{\rho}_i = f(x)$:

- DEF:

$$\begin{aligned} n_m(x) &= \Phi_m{}^\alpha n_\alpha, \qquad n_m n^m \equiv \Phi_m{}^\alpha \Phi_\beta{}^m n_\alpha n_\beta = \delta_{\alpha\beta} \; n_\alpha n_\beta = 1, \\ u_m &= \Phi_m{}^\alpha \hat{u}_\alpha = n_m \sinh \xi \end{aligned}$$

Structure color currents/charges . Wong Equation

- Gauge equation:

$$D^{\rho} F^{m}{}_{\mu\rho} = \mathrm{j}_{\mu}{}^{m}. \quad D^{\mu}\mathrm{j}_{\mu}{}^{q} = 0.$$

$$\mathrm{j}_{\mu}{}^{q} = h^{qn}(2e^{2\alpha\varphi} Q_c \; e_{\mu}{}^{b}\Phi_{n}{}^{\alpha}\hat{T}_{\alpha b} + Q_c^2 A_{\mu}{}^{m}_{p} f_{nm}{}^{p}) - \kappa_p^{\rho q} F^{q}_{\mu\rho}.$$

- Fluid Reduction ansatz+ T_{AB} general structure:

$$\begin{aligned} e_{\mu}{}^{b}\Phi_{n}{}^{\alpha}\hat{T}_{\alpha b} &= e_{\mu}{}^{b}\Phi_{n}{}^{\alpha}(\rho\hat{u}_{\alpha}\hat{u}_{b} + \hat{q}_{\alpha}\hat{u}_{b} + \hat{q}_{b}\hat{u}_{\alpha} + \hat{\tau}_{\alpha b}) \\ &= \rho\cosh\xi\sinh\xi \; n_n u_{\mu} + \cosh\xi \; q_n u_{\mu} + \sinh\xi \; q_{\mu} n_n + \tau_{n\mu} \\ &= (\ldots)u_{\mu} + (\ldots)q_{\mu} + (\ldots)\tau_{\mu} \end{aligned}$$

where

$$\begin{aligned} n_n &\equiv \Phi_{n}{}^{\alpha} n_{\alpha}, \quad n^{n} \equiv \Phi^{n}{}_{\alpha} n\alpha, \\ q_n &= \Phi_{n}{}^{\alpha} q_{\alpha}, \; \tau_{n\mu} = \Phi_{n}{}^{\alpha}\tau_{\alpha\mu} \\ n_n n^n &= \Phi_{n}{}^{\alpha}\Phi^{n}{}_{\beta} n_{\alpha} n_{\beta} = n_{\alpha} n_{\beta}\delta^{\alpha\beta} = 1 \end{aligned}$$

- We redefine

$$\mathfrak{n}^q \equiv 2Q_c e^{2\alpha\varphi} \tanh\xi\, h^{qn} n_n,$$
$$\mathfrak{q}^q \equiv 2Q_c e^{2\alpha\varphi} \cosh\xi\, h^{qn} q_n,$$
$$\tau^q \equiv 2Q_c e^{2\alpha\varphi} h^{qn} \tau_{n\mu},$$
$$\mathfrak{A}^q_\mu = Q_c^2 A_{\mu}{}^{m}{}_{p} h^{qn} f_{nm}{}^{p}$$

- Then:

$$\mathfrak{j}_\mu{}^q = (\mathfrak{n}^q \rho \cosh^2\xi + \mathfrak{q}^q) u_\mu + \cosh\xi\ \mathfrak{n}^q\ q_\mu + \tau^q_\mu + \mathfrak{A}_\mu{}^p - \kappa_p^{\rho q} F^q_{\mu\rho}.$$

- **COLOR CHARGE**: long/transv.projections along u_μ ($E^q_\rho \equiv F^q_{\mu\rho} u^\mu$)

$$\mathfrak{Q}^q \equiv u^\mu \mathfrak{j}_\mu{}^q = \mathfrak{n}^q \rho \cosh^2\xi + \mathfrak{q}^q + + u^\mu \mathfrak{A}_\mu{}^p - \kappa_p^{\rho q} E^q_\rho,$$
$$\nu^q_\nu \equiv \Pi_\nu{}^\mu \mathfrak{j}_\mu{}^q = \cosh\xi\ \mathfrak{n}^q\ q_\mu + \tau^q_\mu + \Pi_\nu{}^\mu \mathfrak{A}_\mu{}^p - \kappa_p^{\rho q} \Pi_\nu{}^\mu F^q_{\mu\rho}.$$

- **COLOR CURRENT**: $\mathfrak{j}_\mu{}^q = \mathfrak{Q}^q u_\mu + \nu^q_\mu$

WONG Equation

- Appliying a covariant derivative

$$
\begin{aligned}
D^{\mu}\mathfrak{j}_{\mu}{}^{q} &= D^{\mu}(\mathfrak{Q}^{q}u_{\mu}) + D^{\mu}\nu^{q}_{\mu} = 0, \\
&= D^{\mu}(\mathfrak{Q}^{q})u_{\mu} + (\mathfrak{Q}^{q})D^{\mu}u_{\mu} + D^{\mu}\nu^{q}_{\mu}.
\end{aligned}
$$

- **Wong Equation**: $(D^{\mu}u_{\mu} \equiv \theta)$

$$
u_{\mu}D^{\mu}(\mathfrak{Q}^{q}) = -(\mathfrak{Q}^{q})\theta - D^{\mu}\nu^{q}_{\mu}.
$$

CLASSIFICATION: D=d+n.

$\longrightarrow$ CLASSIFICATION DIM REDUCTION (UNIMODULAR) FLUIDS:

- $n = 1$: Unimodular: Abelian $U(1)$. Matter: φ.
- $n = 2$: Unimodular: Only Abelian $U(1)^2$. Matter: φ, χ
- $n = 3$: $\longrightarrow$ Bianchi classification: Unimodular:

Compact: Ab. $U(1)^3$, $+ ....so(3)/su(2).. +$

Non compact: $SO(2, 1)$.

Matter: φ, $\varphi_{1,2}, \xi_{1,2,3}$.

- $n = 8$: SU(3)...
- others...

$\longrightarrow$ NON UNIMODULAR FLUIDS?

$d = 4, n = 1$: abelian dilaton fluid.

- $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu}$

$$T_{\mu\nu} = T^f_{\mu\nu} + \left(\frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{4} g_{\mu\nu} (\partial\varphi)^2 \right) + \frac{1}{2} Q_e^{-2} \left(F^2_{\mu\nu} - \frac{1}{4} F^2 g_{\mu\nu} \right). \tag{1}$$

$$T^f_{\mu\nu} \equiv e^{2\alpha\phi} e_\mu{}^a e_\nu{}^b \hat{T}^f_{ab} \, . \tag{2}$$

- DEF: $Q_e \equiv e^{3\alpha\varphi(x)}$.

$$\boldsymbol{\nabla_\rho \left(Q_e^{-2} F_\mu{}^\rho \right) = Q_e^{-2} J^e_\mu, \quad J^e_\mu \equiv 2 e^{5\alpha\phi} e_\mu{}^a \hat{T}^f_{az}.}$$

- DILATON, $(\hat{T}_{eff} = e^{2\alpha\varphi}(\hat{T}^f + 3\hat{T}^{f\,z}_z))$.

$$\boldsymbol{\nabla^2 \varphi = \rho^e_\varphi + \rho^f_\varphi \equiv \frac{3}{2} \alpha Q_e^{-2} F^2 + 2\alpha \hat{T}_{eff},}$$

$d = 4, n = 1$: abelian dilaton fluid. Conservation laws:

- CC1: $$\nabla^\mu j_\mu \equiv \nabla^\mu (Q_e^{-2} J_\mu) \quad = \quad 0.$$
- STRESS TENSOR: $\nabla^\mu T^{total}_{\mu\nu} = 0$. $F_\nu \equiv \nabla_\nu \varphi$. On-shell:

$$\begin{aligned}
\nabla^\mu T^f{}_{\mu\nu} &= \frac{1}{2} F_{\nu\rho} j^\rho + \frac{1}{4} F^2 \nabla_\nu Q_e^{-2} + \frac{1}{2} \nabla^2 \varphi \partial_\nu \varphi ...... \\
&\equiv \frac{1}{2} F_{\nu\rho} j^\rho + j_\varphi F_\nu . \\
&= \frac{1}{2} F_{\nu\rho} j^\rho + \alpha(-\frac{3}{4} F^2 Q_e^{-2} + \hat{T}_{eff}) \partial_\nu \varphi .
\end{aligned}$$

Abelian perfect fluid: reduction Ansatz

- ASSUME $\hat{T}_{AB} = \hat{w}(x)\hat{u}_A\hat{u}_B - \sigma\hat{\eta}_{AB}\hat{p}(x)$
- ANSATZ:
$$\begin{aligned} \hat{u}_a &= u_a \cosh\phi(x), \quad \eta^{ab}u_a u_b = 1. \\ \hat{u}_z &= \sinh\phi(x), \end{aligned}$$

- d-Stress tensor, J:

$$\begin{aligned} T^p_{\mu\nu} &= e^{2\alpha\varphi} e_\mu{}^a e_\nu{}^b \hat{T}_{ab} = e^{2\alpha\varphi}\hat{T}_{\mu\nu} = e^{2\alpha\varphi}(\hat{w}\cosh^2\phi u_\mu u_\nu - \sigma\hat{p}g_{\mu\nu}) \\ &\equiv w\, u_\mu u_\nu - \sigma p\, g_{\mu\nu}, \\ j_\mu &= 2Q_e^{-2} e^{(d+1)\alpha\phi} e_\mu{}^a \hat{T}^f_{az} = Q_e^{-2} e^{5\alpha\phi}\tanh(\phi) w u_\mu \equiv q_e w u_\mu, \end{aligned}$$

- THEN
$$\begin{aligned} p &= e^{2\alpha\varphi}\hat{p}, \\ \epsilon &= e^{2\alpha\varphi}\left(\hat{\rho}\cosh^2\phi + \hat{p}\sinh^2\phi\right). \\ w &= e^{2\alpha\varphi}\cosh^2\phi\ \hat{w}. \end{aligned}$$

- $\hat{\epsilon}, \hat{p}$: ADDitional EQS : $\nabla\hat{T}_{AB} = 0 +$ State equation.

summary abelian fluid

- where $e^{2\alpha\varphi}\cosh^2\phi\;\hat{w} = w$. Also

$$\begin{aligned} T^p_{\mu\nu} &= (\epsilon + p)\,u_\mu u_\nu - \sigma p\, g_{\mu\nu}, \\ j_\mu &= q_e w u_\mu \end{aligned}$$

- State equation: $\epsilon = a(x)p + b(x)$,
- Dynamic equations :

$$\begin{aligned} \nabla^\mu T^f{}_{\mu\nu} &= \frac{1}{2}F_{\nu\rho}j^\rho + \frac{1}{2}(-\rho^e_\varphi + \rho^f_\varphi)F_\nu.... \\ D_\rho Q_e^{-2}F_\mu{}^\rho &= Q_e^{-2}j_\mu, \\ \nabla^2\varphi &= \rho^e_\varphi + \rho^f_\varphi. \end{aligned}$$

Effective state equation. Speed of sound

- Equation of state:

$$\epsilon(x) = a(x)p(x) + b(x)...$$

- From this, we find the speed of sound, c_s

$$c_s^2 \equiv \frac{\partial p}{\partial \epsilon} = \frac{1}{\cosh^2 \varphi(\hat{c}_s^{-2} - 1) + 1}, \qquad \text{where} \qquad \hat{c}_s^2 = \frac{\partial \hat{p}}{\partial \hat{\epsilon}}. \qquad (3)$$

$\longrightarrow$ if $\hat{c}_s \to 1$ then $c_s \to 1$
Limits: For $\hat{c}_s^2 \to 0$, $c_s^2 \to \hat{c}_s^2 \sinh^2 \varphi$

Abelian perfect fluid: thermodynamics, entropy

- D-dim ASSUME ($\hat{\mu} = 0$)

$$\begin{aligned} \hat{\epsilon} + \hat{p} &= \hat{T}\hat{s}. \\ \hat{\mathfrak{J}}^{\hat{s}}{}_A &= \hat{s}\hat{u}_A, \ \hat{\nabla}^M \hat{\mathfrak{J}}^{\hat{s}}{}_M = 0 \end{aligned}$$

- d-dim: $\longrightarrow$ *d*- charged fluid: $\boldsymbol{\mu}^q$, associated to $\mathfrak{Q}$

$$\begin{aligned} \epsilon + p &= Ts + \mathfrak{Q}\boldsymbol{\mu} \\ \mathfrak{J}^s_\mu &= s\, u_\mu . \\ s &= e^{2\alpha\phi}\, \hat{s}\, \cosh\phi, \ quad \nabla^\mu \mathfrak{J}^s{}_\mu = 0, \end{aligned}$$

- We identity: where

$$\begin{aligned} &T = \hat{T}\, \frac{1}{\cosh\phi}\ , \\ &\boldsymbol{\mu} = \tanh\phi \longrightarrow \phi = f(\mu).. \end{aligned} \tag{4}$$

Comparison with standard Maxwell-DIlaton theory

$n = 2$: Abelian $U(1)^2$

-
- two-dimensional $SL(2,\mathbb{R})/SO(2)$ scalar coset.

parameterised by the dilaton φ and the axion χ via the $SO(2)$ invariant scalar matrix

$$\Phi_m{}^\alpha = \begin{pmatrix} e^{-\varphi/2} & \chi e^{\varphi/2} \\ 0 & e^{\varphi/2} \end{pmatrix},$$

$$h_{mn} = e^{\varphi} \begin{pmatrix} \chi^2 + e^{-2\varphi} & \chi \\ \chi & 1 \end{pmatrix}.$$

Comparison with standard Maxwell-DIlaton-axiontheory

$n = 3$: Bianchi classification

- for n=3, $[T_m, T_n] = f_{mn}{}^p T_p$, $f_{[mn}{}^q f_{p]q}{}^r = 0$.

$$f_{mn}{}^p = \varepsilon_{mnq} Q^{pq} + 2\delta_{[m}{}^p a_{n]} \,, \qquad Q^{pq} a_q = 0 \,. \tag{5}$$

$\longrightarrow$ class A,B: vanishing ($a_q = 0$, $f_{mn}{}^n = 0$)and non-vanishing trace.

- If $T_m \to R_m{}^n T_n$ with $R_m{}^n \in GL(3, \mathbb{R})$. Then

$$f_{mn}{}^p \to f'_{mn}{}^p = R_m{}^q R_n{}^r (R^{-1})_s{}^p f_{qr}{}^s \;: \quad Q^{mn} \to Q', a_m \to a' \,. \tag{6}$$

- two classes: a) $R \in Aut(f)$, $f_{mn}{}^p = f'_{mn}{}^p$. b) $f \neq f'$.
- Most general 3-Lie algebra:

$Q^{mn} = \text{diag}(q_1, q_2, q_3) \,, \qquad a_m = (a, 0, 0)$.

- Unimodular,compact$\longrightarrow$ BIANCHI IX, $SO(3)$.
- OTHER FLUIDS TO EXPLORE?

$n = 3$: $SO(3)$

- Vielbein ansatz:

$$E_M{}^A = \begin{pmatrix} e^{-\varphi/6} e_\mu{}^a & e^{\varphi/3} \Phi_m{}^\alpha A^m{}_\mu \\ 0 & e^{\varphi/3} \Phi_n{}^\alpha\, u_m{}^n \end{pmatrix} , \qquad (7)$$

- $\Phi_m{}^\alpha$: internal space $SL(3,\mathbb{R})/SO(3)$ scalar coset of the internal space. Global $SL(3,\mathbb{R})$ (left), local $SO(3)$ (right).
- Explicit representative: two dilatons ϕ, σ and three axions χ_1, χ_2, χ_3.

$$\Phi_m{}^\alpha = \begin{pmatrix} e^{-\sigma/\sqrt{3}} & e^{-\phi/2+\sigma/2\sqrt{3}}\chi_1 & e^{\phi/2+\sigma/2\sqrt{3}}\chi_2 \\ 0 & e^{-\phi/2+\sigma/2\sqrt{3}} & e^{\phi/2+\sigma/2\sqrt{3}}\chi_3 \\ 0 & 0 & e^{\phi/2+\sigma/2\sqrt{3}} \end{pmatrix} , \qquad (8)$$

- DEF: $SO(3)$ invariant scalar matrix

$$h_{mn} = \Phi_m{}^\alpha \Phi_n{}^\beta \delta_{\alpha\beta} \, , \qquad (9)$$

REY: su(2) case:Perfect colored fluid

- d-dimensional perfect colored fluid:

$$T^{pf}_{ab}(x) = [\epsilon(x) + p(x)]u_a(x)u_b(x) + p(x)\eta_{ab}, \tag{10}$$

where

$$\begin{aligned} p(x) &= e^{2\alpha\phi(x)}\hat{p}(x)\,, \\ \epsilon(x) &= e^{2\alpha\phi(x)}\left[\cosh^2\varphi(x)\hat{\epsilon}(x) + \sinh^2\varphi(x)\hat{p}(x)\right]. \end{aligned} \tag{11}$$

- speed of sound, c_s:

$$c_s^2 \equiv \frac{\partial p}{\partial \epsilon} = \frac{1}{\cosh^2\varphi(x)(\hat{c}_s^{-2} - 1) + 1}\,, \qquad \text{where} \qquad \hat{c}_s^{\,2} = \frac{\partial \hat{p}}{\partial \hat{\epsilon}}\,. \tag{12}$$

- Conserved color current.

$$\mathbf{J}^{\rm color}_{ma}(x) = Q_c(x)\mathfrak{Q}_m(x)\,u_a(x)\,. \tag{13}$$

- color charge density:

$$\mathfrak{Q}_m(x) = 2w\Phi_m{}^{\alpha}(x)\mathbf{n}_\alpha \tanh\phi(x) \tag{14}$$

$$= 2w\mathbf{n}_m \tanh\phi\,. \tag{15}$$

Entropy current

- D dimensions fluid satisfies the thermodynamic relation

$$\hat{\epsilon} + \hat{p} = \hat{T}\hat{s}.$$

$$\hat{\mathfrak{J}}^{\hat{s}}{}_A = \hat{s}\hat{u}_A, \quad \hat{\nabla}^M \hat{\mathfrak{J}}^{\hat{s}}{}_M = 0$$

- d -conserved entropy current:

$$\begin{aligned} s &= e^{2\alpha\varphi}\,\hat{s}\,\cosh\phi, \\ \mathfrak{J}^s_\mu &= s\,u_\mu,\;\; s = e^{2\alpha\varphi}\,\hat{s}\,\cosh\phi, \\ \nabla^\mu \mathfrak{J}^s{}_\mu &= 0\,. \end{aligned}$$

- d: $\boldsymbol{\mu}_m^{\mathrm{color}}$ associated $\mathfrak{Q}_m$:

$$\epsilon + p = Ts + \mathfrak{Q}^m \boldsymbol{\mu}_m^{\mathrm{color}}\,. \tag{17}$$

$\longrightarrow$ It can be identified:

$$\begin{aligned} T &= \hat{T}/\cosh\phi, \\ \boldsymbol{\mu}_m^{\mathrm{color}} &= \mathbf{n}_m \tanh\phi. \end{aligned}$$

SU(3)

CONCLUSIONS

- DIm Red Einstein Eqs+ Fluid Matter in a group manifold.

$\longrightarrow$ A) perfect fluid.

$\longrightarrow$ B) dissipative fluid : "Landau Frame" constraint condition.

...second order? $\longrightarrow$ transport coefficients

- Non-abelian$\longrightarrow$ Many Extra fields: Problem? Opportunity?

$\longrightarrow$ scalar fields: Superfluid unbroken phase order

$\longrightarrow$ In hadronic matter: plenty of chiral currents,..bosonic fields(mesons,etc)

- Fluids are effective theories: good strategy is set all the possible terms allowed by symmetries:

– new ways of coupling einstein-dilaton/axion theories to a fluid: important for (early universe) cosmology (Dark Energy late behaviour).

MURCIA/KYOTO correspondence?

HANAMI/BANDO DE LA HUERTA correspondence..

HAPPY HANAMI!

WELCOME TO BANDO DE LA HUERTA??