▨ Topological integer and edge mode in quasi-particle boson systems.

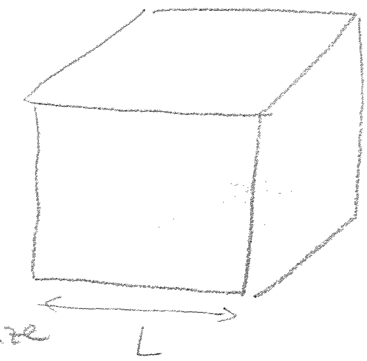
▨-1 introduction

- Topological integers (numbers) or Chern integers are defined for bulk electronic wavefunction, which is calculated with periodic boundary condition:

$$\underbrace{\Psi_{n,\mathbf{k}}(\mathbf{r})}_{\text{Bloch wavefunction}} = e^{i\mathbf{k}\cdot\mathbf{r}} \underbrace{u_{n,\mathbf{k}}(\mathbf{r})}_{\text{periodic part}}$$

n: band index

$\mathbf{k}$: crystal momentum



with

Linear dimension of system size

$$\Psi_{n,\mathbf{k}}(\mathbf{r}+L) = \Psi_{n,\mathbf{k}}(\mathbf{r})$$

Born-von Karman boundary condition

$$u_{n,\mathbf{k}}(\mathbf{r}+a) = u_{n,\mathbf{k}}(\mathbf{r})$$

translation by unit cell.

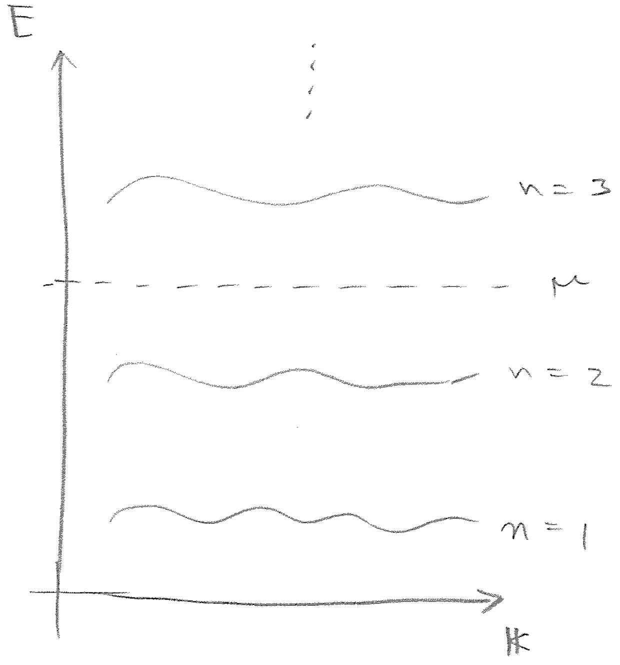
- Berry curvature and the Chern integers

$$\Omega^{\mu\nu}_{n,\mathbb{k}} \equiv i\,\epsilon_{\mu\nu} \int_{\text{unit cell}} d\mathbb{r}\ \frac{\partial u^*_{n,\mathbb{k}}(\mathbb{r})}{\partial k_\mu}\ \frac{\partial u_{n,\mathbb{k}}(\mathbb{r})}{\partial k_\nu}$$

$$(\text{1st})\ \mathrm{Ch}_n = \frac{1}{2\pi} \int_{\text{1st Brillouin zone}} dk_x\, dk_y\ \Omega^{xy}_{n,\mathbb{k}}$$

TKNN (Thouless–Kohmoto–Nightingale–Nijs) number

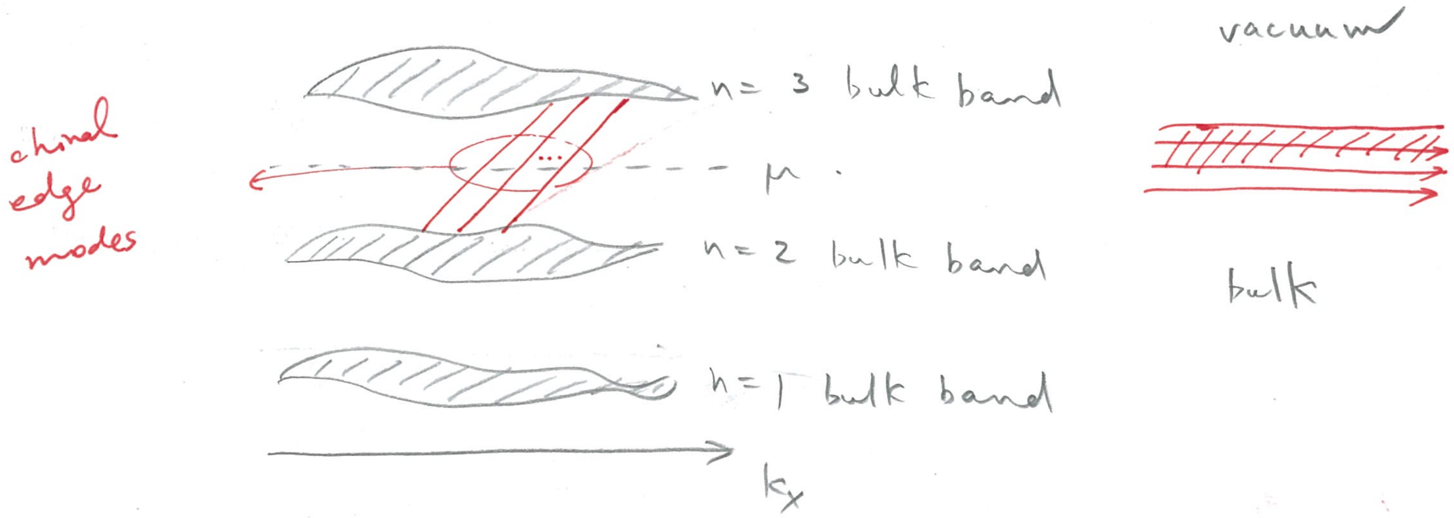
is defined from an inner product of the momentum derivative of the periodic part of the Bloch wavefunction. The Chern integer is defined for each energy band. The sum of the integer over all filled energy band times e^2/h was shown by these people to be the electric Hall conductivity.



$$\sigma_{xy} = \frac{e^2}{h} \sum_{n}^{E_{n,\mathbf{k}} < \mu} ch_n$$

TKNN, PRL <u>49</u> 405 (1982).

- In those 2D electronic systems which have finite quantized Hall conductance, there are chiral modes localized at its two dimensional boundary, which goes across the Fermi level.



- The number of these chiral edge modes were shown by these people to be given by the same integer

$$\#(\text{chiral edge modes}) = \sum_{n}^{E_n < \mu} ch_n$$

- Halperin PRB 25 2185 (1982)
- Hatsugai PRL 71 3697 (1993).

- Based on this observation, the quantization of the Hall conductance was also explained by these people from an edge transport picture.

    - Halperin PRB (1982)
    - Büttiker PRB 38 9375 (1988)

- These are a celebrated classic knowledge in Quantum Hall physics.

- Later, these are further generalized theoretically, some of whose aspect are experimentally confirmed in actual condensed matter experiments.

- One famous generalization is a four-dimensional quantum Hall effect by these people, from which so-call. 3D Time-Reversal invariant (TR) topological insulator was derived by a dimensional reduction method :

  - Zhang-Hu, Science 294 823 (2001)
  - Qi-Hughes-Zhang, PRB 81 134508 (2010).

- Another well-known generalization is a topological superconductor, in which a particle-hole mixed eigenstate of the Bogolinbov de Gennes (BdG) Hamiltonian is regarded as Bloch wavefunction in the electronic system.

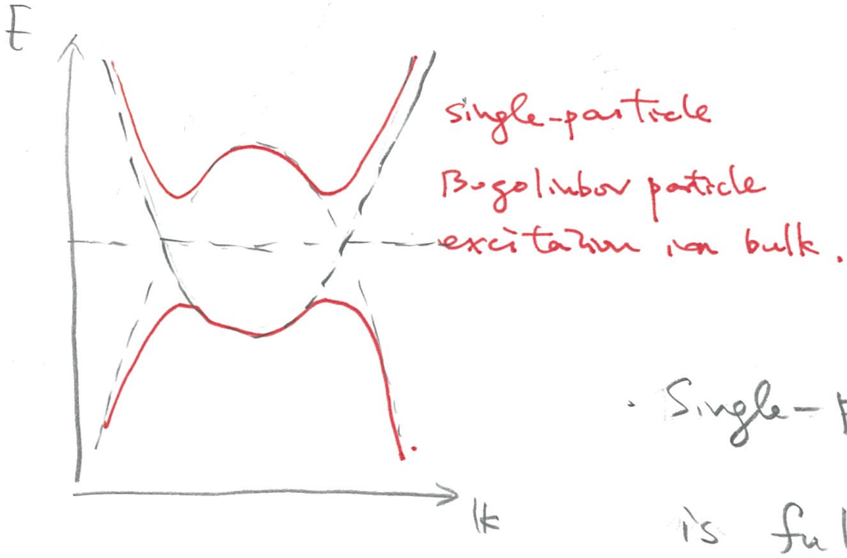
$$\mathcal{H} = \sum_{\mathbb{k}} \left(c_{\mathbb{k}}^{\dagger} \; c_{-\mathbb{k}} \right) \cdot \mathcal{H}(\mathbb{k}) \cdot \begin{pmatrix} c_{\mathbb{k}} \\ c_{-\mathbb{k}}^{\dagger} \end{pmatrix}$$

$$\mathcal{H}(\mathbb{k}) = \begin{pmatrix} \epsilon(\mathbb{k}) & \Delta(\mathbb{k}) \\ \Delta^{*}(\mathbb{k}) & -\epsilon(\mathbb{k}) \end{pmatrix}$$ Cooper pairing

- BdG Hamiltonian for (spinless) superconductor

$$\mathcal{H}(\mathbb{k}) \, | u_{\pm}(\mathbb{k}) \rangle = \pm \sqrt{\epsilon^2(\mathbb{k}) + |\Delta(\mathbb{k})|^2} \, | u_{\pm}(\mathbb{k}) \rangle$$

- particle-hole mixed eigenstate



- Single-particle Bulk excitation is fully gapped:

$$\sqrt{\epsilon_{\mathbb{k}}^2 + |\Delta(\mathbb{k})|^2} \neq 0 \quad \text{for } \forall \mathbb{k}$$

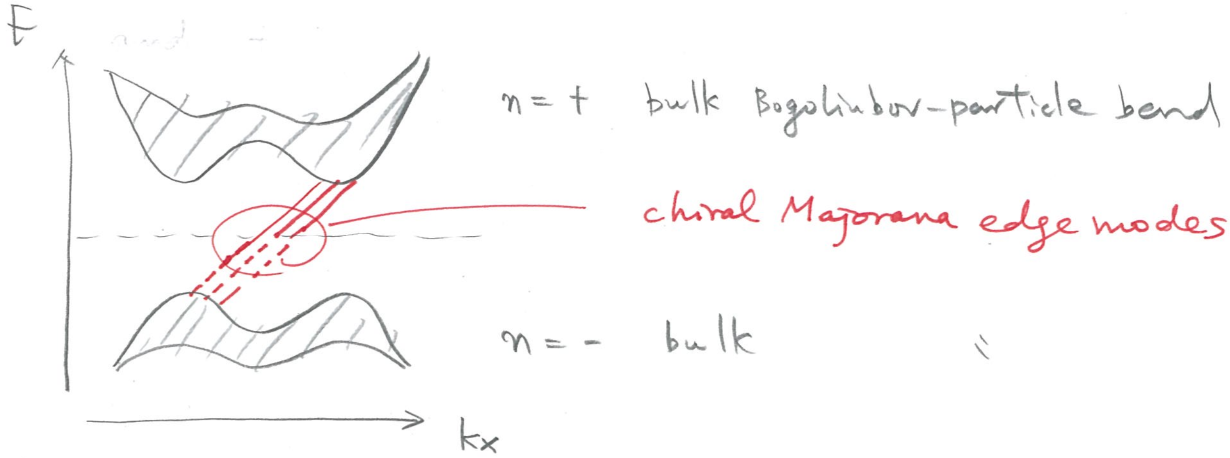
So that the Berry's curvature and Chern integers are defined from the inner product of the momentum derivative of this p-h mixed eigenstate.. ①-7

$$\Omega_{n\mathbb{k}}^{\mu\nu} = i\epsilon_{\mu\nu}\left(\frac{\partial}{\partial k_\mu}\langle u_n(\mathbb{k})|\right)\left(\frac{\partial}{\partial k_\nu}|u_n(\mathbb{k})\rangle\right)$$

$$Ch_n \equiv \frac{1}{2\pi}\int dk_x dk_y\ \Omega_{n,\mathbb{k}}^{xy}$$

- Read and Green PRB 61 10267 (2000).
- Goryo and Ishikawa Physics Letter A. 260 294 (1999).

In 2D spinless superconductors which takes finite topological integer, there are chiral edge modes for quasi-particle Bogoliubov excitation, which has a nature of Majorana particle: $(\psi(x) = \psi^\dagger(x))$



- The number of this chiral Majorana edge modes is equal to this topological number.

- This TR breaking (spinless) topological superconductor is a "mother" of many other topological superconductors which have been extensively studied during last 5 years.

In this lecture, I like to talk about currently not-so-famous, but (I believe) very natural another generalization of 2D Quantum Hall physics which is a quasi-particle boson's generalization.

- Quasi-particle boson is quite ubiqitous in nature, such as photon, phonon, magnon, exciton, polariton, ----.
- Like electron, they have been well-studied in a lot of different research communities.
- Unlike electron, these systems have fewer symmetries in general; their quadratic boson Hamiltonian usually don't even have the U(1) symmetry.
- Namely, it has "Cooper pairing" term from the begining.

$$\mathcal{H}_b = \sum_{\mathbb{k}} \left(b^\dagger_{\mathbb{k}} \;\; b_{-\mathbb{k}} \right) \cdot \mathcal{H}(\mathbb{k}) \cdot \begin{pmatrix} b_{\mathbb{k}} \\ b^\dagger_{-\mathbb{k}} \end{pmatrix}.$$

$$\mathcal{H}(\mathbb{k}) = \begin{pmatrix} \epsilon(\mathbb{k}) & \Delta(\mathbb{k}) \\ \Delta^*(-\mathbb{k}) & \epsilon^*(-\mathbb{k}) \end{pmatrix}$$ "Cooper pairing"

- BdG Hamiltonian. for quasi-particle boson systems.

- The most important difference between bosons and fermions is commutation relation. between creation and annihilation.

- For boson case, it is a commutator. while an anticommutator for fermion

$$b_i b_j^\dagger - b_j^\dagger b_i = \delta_{ij} \qquad (B.)$$

$$c_i c_j^\dagger + c_j^\dagger c_i = \delta_{ij} \qquad (F).$$

- As a result of this, the inner product used in the Berry's curvature's definition is taken with a Pauli matrix $\hat{\sigma}_3$, which takes $+1$ for particle space and -1 for the hole space :

particle - hole mixed eigenstate of BdG Hamiltonian

$$\tilde{\Omega}^{\mu\nu}_{n,\mathbb{k}} = i\epsilon_{\mu\nu}\left(\frac{\partial}{\partial k_\mu}\langle u_n(\mathbb{k})|\right)\cdot\hat{\sigma}_3\cdot\left(\frac{\partial}{\partial k_\nu}|u_n(\mathbb{k})\rangle\right)$$

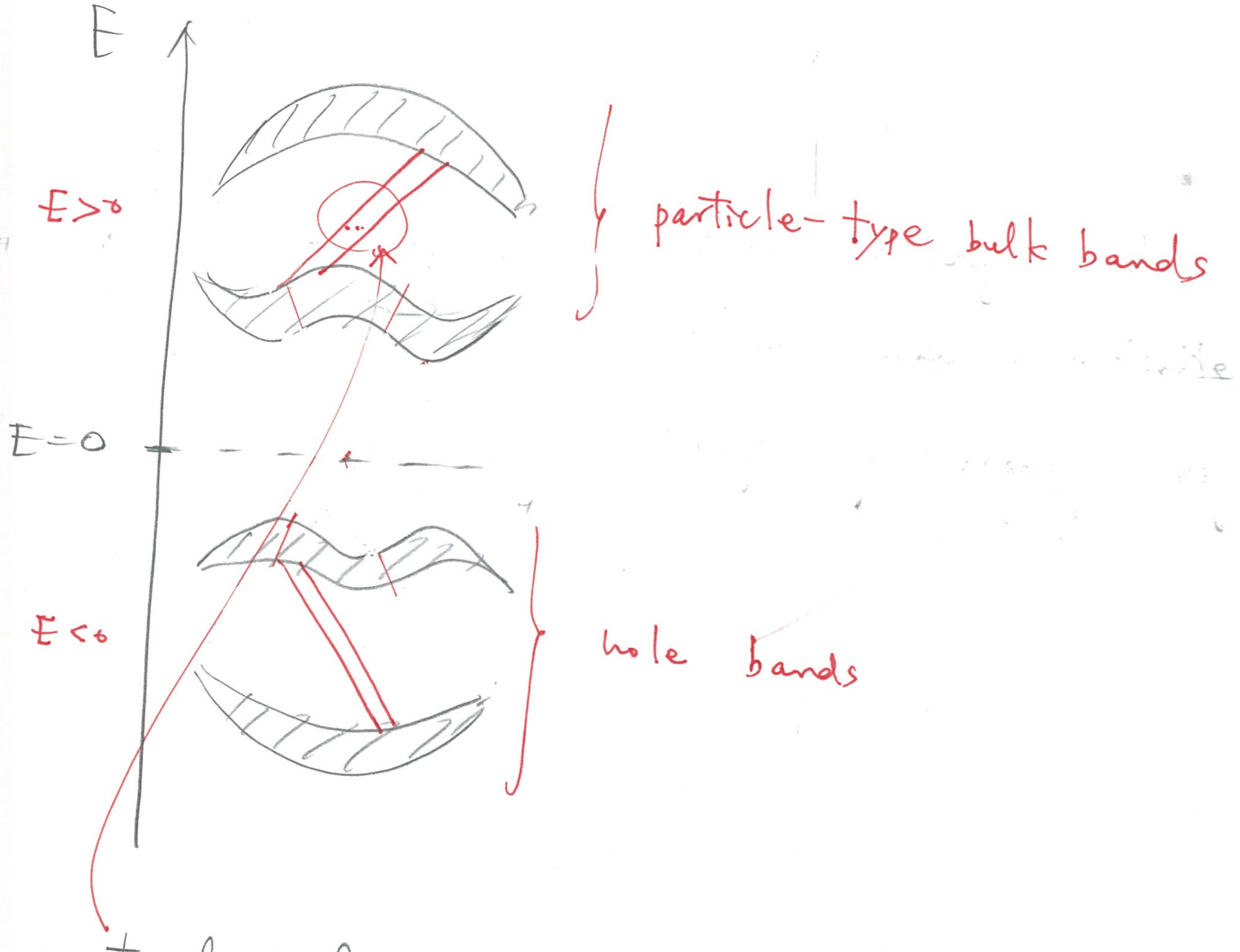
$$ch_n = \frac{1}{2\pi}\int dk_x\, dk_y\, (\sigma_3)_{nn}\, \tilde{\Omega}^{\mu\nu}_{n,\mathbb{k}}$$

(• Berry's curvature and chern integer for the quasi-particle boson system)

where

$$\hat{\sigma}_3 \overset{\text{def}}{\equiv} \begin{array}{c} \begin{array}{cc} \overbrace{\quad\quad}^{\text{particle}} & \overbrace{\quad\quad}^{\text{hole}} \end{array} \\ \left(\begin{array}{c|c} \mathbb{1} & \\ \hline & -\mathbb{1} \end{array}\right) \end{array}$$

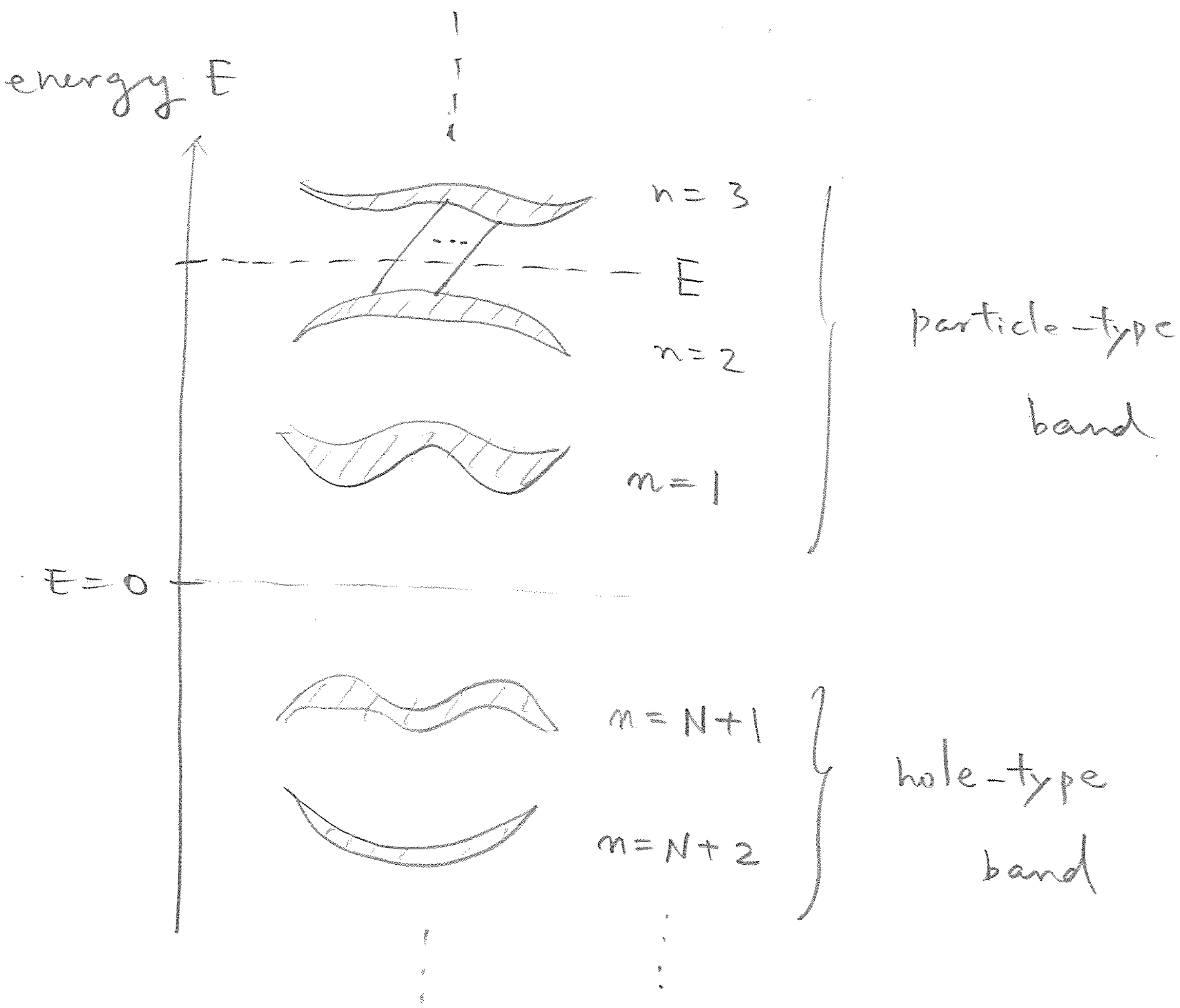
- I will explain that, because of this, there is no possibility of having gapless topological edge modes in usual quasi-particle boson system, which connect the particle bands and the hole bands:



topological edge modes appear only in a finite frequency (energy) regime.

# of edge modes which go across at a given energy E (>) is given by sum of the Chern integers of those particle-type bulk bands whose energy is below E.

$$\#(\text{chiral edge modes at } E) = \sum_{n}^{0<E_n<E} ch_n$$



- In the following, I will first explain these general properties, material realization of the idea, ~~and~~

- In the next class, I will explain how the thermal Hall conductivity is given by this Berry's curvature, from a picture of bulk heat transport.

- Then, I will further explain how the ~~thermal~~ Hall conductivity is characterized from a picture of edge heat transport along these chiral edge modes.

- Besides, I will discuss about a localization effect in quantum magnon Hall systems.

▨ − 2. topological integer and edge modes in quasi-particle boson systems.

Consider the BdG Hamiltonian for boson in the momentum space :

$$\mathcal{H}_b = \frac{1}{2}\sum_{\mathbb{k}} \left(\mathbb{b}^{\dagger}_{\mathbb{k}} \;\; \mathbb{b}_{-\mathbb{k}} \right) \cdot \mathcal{H}(\mathbb{k}) \cdot \begin{pmatrix} \mathbb{b}_{\mathbb{k}} \\ \\ \mathbb{b}^{\dagger}_{-\mathbb{k}} \end{pmatrix} \qquad (A-1)$$

$\sum_{\mathbb{k}}$: over the Brillouin zone

$$\mathbb{b}^{\dagger}_{\mathbb{k}} = \left(b^{\dagger}_{1,\mathbb{k}},\; b^{\dagger}_{2,\mathbb{k}},\; \cdots,\; b^{\dagger}_{N,\mathbb{k}} \right)$$

N : # distinct boson degree of freedom within the unit cell

$$\mathcal{H}(\mathbb{k}) = \left(\begin{array}{c|c} \mathbb{h}_{N\times N}(\mathbb{k}) & \Delta_{N\times N}(\mathbb{k}) \\ \hline \Delta^{*}_{N\times N}(-\mathbb{k}) & \mathbb{h}^{*}_{N\times N}(-\mathbb{k}) \end{array} \right) \begin{array}{l} \} \text{ particle} \\ \} \text{ hole} \end{array}$$

$2N \times 2N$ BdG Hamiltonian — (A-2)

Since a new basis which diagonalizes this BdG Hamiltonian should also obeys the boson commutation relation, this is diagonalizes by a (so-called) paraunitary transformation instead of unitary trans.

$$\mathcal{H}_b = \frac{1}{2}\sum_{\mathbb{k}} \left(\gamma_{\mathbb{k}}^{\dagger} \quad \gamma_{-\mathbb{k}} \right) \underbrace{\begin{pmatrix} \epsilon_1(\mathbb{k}) & & & & & \\ & \ddots & & & & \\ & & \epsilon_N(\mathbb{k}) & & & \\ & & & \epsilon_1(-\mathbb{k}) & & \\ & & & & \ddots & \\ & & & & & \epsilon_N(-\mathbb{k}) \end{pmatrix}}_{\equiv\ \hat{E}_{d,\mathbb{k}}} \begin{pmatrix} \gamma_{\mathbb{k}} \\ \\ \gamma_{-\mathbb{k}}^{\dagger} \end{pmatrix} \quad \text{(A-3)}$$

$$= \sum_{j=1}^{N} \sum_{\mathbb{k}} \epsilon_j(\mathbb{k})\, \gamma_{j,\mathbb{k}}^{\dagger}\, \gamma_{j,\mathbb{k}}$$

$$\gamma_{\mathbb{k}}^{\dagger} \equiv \left(\gamma_{1,\mathbb{k}},\ \gamma_{2,\mathbb{k}},\ \cdots,\ \gamma_{N,\mathbb{k}} \right)$$

where the new basis ($\gamma_{j,\mathbb{k}}$) is related with the old basis ($b_{j,\mathbb{k}}$) by the transformation:

$$\begin{pmatrix} \mathbb{b}_{\mathbb{k}} \\ \mathbb{b}^{\dagger}_{-\mathbb{k}} \end{pmatrix} = \hat{T}_{\mathbb{k}} \begin{pmatrix} \gamma_{\mathbb{k}} \\ \gamma^{\dagger}_{-\mathbb{k}} \end{pmatrix} \quad \text{— (A-4)}$$

with

$$\hat{T}^{\dagger}_{\mathbb{k}} \cdot \hat{H}(\mathbb{k}) \cdot \hat{T}_{\mathbb{k}} = \hat{E}_{d,\mathbb{k}} \quad \text{— (A-5)}$$

and

$$\hat{T}^{\dagger}_{\mathbb{k}} \cdot \hat{\sigma}_3 \cdot \hat{T}_{\mathbb{k}} = \hat{\sigma}_3 \quad \text{— (A-6)}$$

paraunitary condition.

where

$$\hat{\sigma}_3 \equiv \left(\begin{array}{c|c} \mathbb{1}_{N\times N} & \\ \hline & - \mathbb{1}_{N\times N} \end{array}\right) \quad \text{— (A-7)}$$

This minus sign is due to the commutation relation of boson fields

$$\begin{cases} b_{i,\mathbb{k}}\, b^{\dagger}_{j,\mathbb{k}} - b^{\dagger}_{j,\mathbb{k}}\, b_{i,\mathbb{k}} = \delta_{ij} \\ \gamma_{i,\mathbb{k}}\, \gamma^{\dagger}_{j,\mathbb{k}} - \gamma^{\dagger}_{j,\mathbb{k}}\, \gamma_{i,\mathbb{k}} = \delta_{ij} \end{cases} \quad \text{— (A-8)}$$

$$\begin{pmatrix} \mathbf{b}_{\mathbf{k}} \\ \mathbf{b}^{\dagger}_{-\mathbf{k}} \end{pmatrix} \left(\mathbf{b}^{\dagger}_{\mathbf{k}} \ \ \mathbf{b}_{-\mathbf{k}} \right) - \left(\begin{pmatrix} \mathbf{b}^{\dagger}_{\mathbf{k}} \\ \mathbf{b}_{-\mathbf{k}} \end{pmatrix} \left(\mathbf{b}_{\mathbf{k}} \ \ \mathbf{b}^{\dagger}_{-\mathbf{k}} \right) \right)^{T} = \sigma_3 \qquad \text{—(A-9)}$$

because the (i, j) component is given as.

- $b_{\mathbf{k},i}\, b^{\dagger}_{\mathbf{k},j} - b^{\dagger}_{\mathbf{k},j}\, b_{\mathbf{k},i} = \delta_{ij} \qquad (1 \leq i,j \leq N)$

(i+N, j+N) component

- $b^{\dagger}_{-\mathbf{k},i}\, b_{-\mathbf{k},j} - b_{-\mathbf{k},j}\, b^{\dagger}_{-\mathbf{k},i} = -\delta_{ij} \qquad (\cancel{N+1 \leq i,j \leq 2N})$

(i, j+N) -component

- $b_{\mathbf{k},i}\, b_{-\mathbf{k},j} - b_{-\mathbf{k},j}\, b_{\mathbf{k},i} = 0. \qquad \left(\begin{matrix} \cancel{1 \leq i \leq N} \\ \cancel{N+1 \leq j \leq 2N} \end{matrix} \right)$

Likewise, we have

$$\begin{pmatrix} \gamma_{\mathbf{k}} \\ \gamma^{\dagger}_{-\mathbf{k}} \end{pmatrix} \left(\gamma^{\dagger}_{\mathbf{k}} \ \ \gamma_{-\mathbf{k}} \right) - \left(\begin{pmatrix} \gamma^{\dagger}_{\mathbf{k}} \\ \gamma_{-\mathbf{k}} \end{pmatrix} \left(\gamma_{\mathbf{k}} \ \ \gamma^{\dagger}_{-\mathbf{k}} \right) \right)^{T} = \sigma_3 \qquad \text{—(A-10)}$$

Substituting eq. (A-4) into eq. (A-9), and using eq (A-10), we obtain

$$\hat{T}_{\mathbf{k}} \begin{pmatrix} \gamma_{\mathbf{k}} \\ \gamma^{\dagger}_{-\mathbf{k}} \end{pmatrix} \left(\gamma^{\dagger}_{\mathbf{k}} \ \ \gamma_{-\mathbf{k}} \right) \cdot \hat{T}^{\dagger}_{\mathbf{k}}$$

$$- \left(\hat{T}^{*}_{\mathbf{k}} \cdot \begin{pmatrix} \gamma^{\dagger}_{\mathbf{k}} \\ \gamma_{-\mathbf{k}} \end{pmatrix} \left(\gamma_{\mathbf{k}} \ \ \gamma^{\dagger}_{-\mathbf{k}} \right) \cdot T_{\mathbf{k}}^{T} \right)^{T} = \sigma_3$$

$$\hat{T}_k \cdot (\text{r.h.s. of eq.(A-10)}) \cdot \hat{T}_k^{\dagger} = \sigma_3$$

- So that we first obtain a completeness relation:

$$\hat{T}_k \cdot \sigma_3 \cdot \hat{T}_k^{\dagger} = \sigma_3 \qquad \text{— (A-6)}'$$

- Applying $\hat{T}_k^{-1}$ from left, $\hat{\sigma}_3$ from right then $\hat{T}_k$ from right, and finally $\hat{\sigma}_3$ from left, we finally have an orthnormal condition;

$$\hat{T}_k^{\dagger} \cdot \sigma_3 \cdot \hat{T}_k = \sigma_3 \qquad \text{— (A-6)}$$

(Note that $\hat{T}_k^{-1}$ always exists provided that $\hat{E}_{d,k}$ is positive definite)

- From (A-5), we have

$$\hat{H}(k) \cdot \hat{T}_k = \sigma_3 \cdot \hat{T}_k \begin{pmatrix} \begin{matrix} \epsilon_1(k) & & \\ & \ddots & \\ & & \epsilon_N(k) \end{matrix} & \\ & \begin{matrix} -\epsilon_1(-k) & & \\ & \ddots & \\ & & -\epsilon_N(-k) \end{matrix} \end{pmatrix} \qquad \text{— (A-7)}$$

which takes a form of (generalized) eigenvalue problem of $\hat{H}(\mathbb{k})$.

- Thus, we regard a column of $T_{\mathbb{k}}$ as (a periodic part of) Bloch wave function, and the Berry's curvature and Chern integer will be defined from $T_{\mathbb{k}}$.

- To this end, we note that the topological integer for a given band measures a global topology of a manifold associated with the band.

- As such, it is generally defined by a projection operator into the energy band $P_{\mathbb{k},n}$

$$Ch_n \equiv i\frac{\epsilon_{\mu\nu}}{2\pi}\int_{B.Z.} dk_x dk_y \, \mathrm{Tr}\big[(\hat{1}-\hat{P}_{\mathbb{k},n})$$

$$\times(\partial_{k_\mu}\hat{P}_{\mathbb{k},n})(\partial_{k_\nu}\hat{P}_{\mathbb{k},n})\big] \quad -(A\text{-}8)$$

Avron, Seiler, Simon PRL 51, 51 (1983)

where $\hat{P}_{\mathbb{k},n}$ as well as $\hat{H}(\mathbb{k})$ is an operator which acts on the 2N-dimensional vector space.

- $\hat{P}_{\mathbb{k},n}$ must satisfy the following natures as the projection operator:

$$\begin{cases} \displaystyle\sum_{n=1}^{2N} \hat{P}_{\mathbb{k},n} = \mathbb{1} \\ \hat{P}_{\mathbb{k},i}\,\hat{P}_{\mathbb{k},j} = \delta_{i,j}\,\hat{P}_{\mathbb{k},i} \end{cases} \qquad \text{— (A-9)}$$

- From Eq (A-6) and (A-6)', we can define such projection operator as follows;

$$\hat{P}_{\mathbb{k},n} \equiv \hat{T}_{\mathbb{k}} \cdot \hat{\Gamma}_n \cdot \hat{\sigma}_3\, \hat{T}^{\dagger}_{\mathbb{k}} \cdot \sigma_3 \qquad \text{— (A-10)}$$

where $\hat{\Gamma}_n$ is a diagonal matrix which takes +1 for (n, n) - component, while zero otherwise.

Using this projection operator, the Chern integer in (A-8) can be given by the integral over the Berry's curvature;

$$\mathrm{Ch}_n = \frac{1}{2\pi} \int_{BZ} dk_x dk_y \; \Omega^{xy}_{n,\mathbf{k}} \qquad \text{— (A11)}$$

---

where

$$\Omega^{xy}_{n,\mathbf{k}} = i \,\mathrm{Tr}\left[\Gamma_n \hat{\sigma}_3 (\partial_{k_x} T^{\dagger}_{\mathbf{k}}) \hat{\sigma}_3 (\partial_{k_y} T_{\mathbf{k}}) \right] - (k_x \leftrightarrow k_y) \qquad \text{— (A-12)}$$

(integrand of eq. (A-8))

$$= \mathrm{Tr}\Big[(1 - \hat{P}_{\mathbf{k},n}) \cdot \{ (\partial_\mu T_k) \Gamma_n \sigma_3 T_k^{\dagger} \sigma_3 + T_k \cancel{\Gamma_n \sigma_3 (\partial_\mu T_k^{\dagger}) \sigma_3} \} \cdot \{ (\partial_\nu T_k) \cancel{\Gamma_n \sigma_3 T_k^{\dagger} \sigma_3} + T_k \Gamma_n \sigma_3 (\partial_\nu T_k^{\dagger}) \sigma_3 \} \Big]$$

$$\left\{ \begin{array}{l} (1 - P_n) T_k \Gamma_n \sigma_3 = 0 \\ \Gamma_n \sigma_3 T_k^{\dagger} \sigma_3 (1 - P_n) = 0 . \end{array} \right.$$

$$= \mathrm{Tr}\Big[(\partial_\mu T_k) \underbrace{\cancel{\Gamma_n \sigma_3 T_k^{\dagger} \sigma_3 T_k \Gamma_n \sigma_3}}_{= \Gamma_n \sigma_3} (\partial_\nu T_k^{\dagger}) \sigma_3 \Big]$$

$$- \mathrm{Tr}\left[\underbrace{T_k \Gamma_n \sigma_3 T_k^\dagger \sigma_3}_{P_{k,n}} \partial_\mu T_k \Gamma_n \sigma_3 (\partial_\nu T_k^\dagger) \sigma_3 \right]$$

$P_{k,n}$.

(1) − 23

The second term vanishes under the summation over μ and ν with $\epsilon_{\mu\nu}$, because

$$\mathrm{Tr}\left[T_k \Gamma_n \sigma_3 T_k^\dagger \sigma_3 \underline{(\partial_\mu T_k) \Gamma_n \sigma_3} (\partial_\nu T_k^\dagger) \sigma_3 \right]$$

$$\neq \mathrm{Tr}\left[T_k \Gamma_n \sigma_3 T_k^\dagger \sigma_3 (\partial_\nu T_k) \Gamma_n \sigma_3 (\partial_\mu T_k^\dagger) \sigma_3 \right].$$

$$(\partial_\nu T_k^\dagger) \sigma_3 T_k = - T_k^\dagger \sigma_3 (\partial_\nu T_k)$$

$$= \mathrm{Tr}\left[\Gamma_n \sigma_3 (\partial_\nu T_k^\dagger) \sigma_3 (\partial_\mu T_k) \right].$$

Since $\hat{\Gamma}_n$ is a diagonal matrix which takes $+1$ only for (n,n)-component, Berry's curvature in (A-12) essentially takes a same-form as in electronic case, with its inner product in the particle-hole mixed space being modified properly:

$$\Omega^{xy}_{n,k} = i\, (\sigma_3)_{nn} \epsilon_{xy} \left(\frac{\partial}{\partial k_x} \langle t_n(k) | \right) \cdot \sigma_3 \cdot \left(\frac{\partial}{\partial k_y} | t_n(k) \rangle \right)$$

where $| t_n(\mathbb{k}) \rangle$ is n-th column vector encoded in $T_{\mathbb{k}}$ and plays role of a Bloch wave function for the n-th bulk band.

$$(\sigma_3)_{nn} = +1 \quad \text{for the particle-type band} \quad (n = 1, \cdots, N).$$

$$= -1 \quad \text{for the hole-type band} \quad (n = N+1, \cdots, 2N)$$

- Since an integrand in eq. (A11) can be given by a rotation of a vector potential, the integral over a closed surface (BZ) can be shown to be integer × 2π.

  - TKNN PRL 49 405 (1982)
  - Kohmoto Ann. Phys. (NY) 160 343 (1985)

- From Eq (A-1) & (A-2), one can see that BdG Hamiltonian for boson systems have a generic particle-hole symmetry:

$$\hat{\sigma}_1 \cdot H(\mathbb{k}) \cdot \hat{\sigma}_1 = \mathcal{H}^*(-\mathbb{k}). \quad \text{— (A-13)}$$

(where

$$\hat{\sigma}_1 \equiv \begin{array}{cc} \begin{array}{cc} \overbrace{\quad\quad}^{p} & \overbrace{\quad\quad}^{h} \end{array} & \\ \left(\begin{array}{c|c} & \mathbb{1}_{N\times N} \\ \hline \mathbb{1}_{N\times N} & \end{array}\right) & \begin{array}{l} \}\,p \\ \}\,h \end{array} \end{array}$$

)

- Thus, when the eigenvalues for the hole space are arranged as in eq.(A-3), the paraunitary transformation given in eq (A-4) has a following symmetry too:

$$\sigma_1 \hat{T}_{\mathbb{k}} \sigma_1 = \hat{T}^*_{-\mathbb{k}} \quad \text{— (A-14)}$$

(apart from U(1) phase D.O.F. for each column vectors (eigenvectors) in $\hat{T}_{\mathbb{k}}$)

- Since the Berry curvature is free from this $U(1)$ phase degree of freedom, the relation between $T_{\mathbb{k}}$ and $T_{-\mathbb{k}}$ leads to the following;

$$\Omega^{xy}_{n,\mathbb{k}} = -\Omega^{xy}_{n+N,-\mathbb{k}} \quad (n = 1, 2, \cdots, N) \qquad \text{(A-15)}$$

$$\left(\text{with} \quad (E_{d,\mathbb{k}})_{nn} = (E_{d,-\mathbb{k}})_{n+N,\,n+N}\right)$$

$\because$

$$\Omega^{xy}_{n,\mathbb{k}} = i\,\mathrm{Tr}\left[\Gamma_n \hat{\sigma}_3\, \partial_x \left(\hat{\sigma}_1 T^{T}_{-\mathbb{k}} \hat{\sigma}_1\right) \hat{\sigma}_3\, \partial_y \left(\hat{\sigma}_1 T^{*}_{-\mathbb{k}} \hat{\sigma}_1\right)\right] - (x \Leftrightarrow y)$$

$$= i\,\mathrm{Tr}\left[\Gamma_{n+N}\, \hat{\sigma}_3 \left(\partial_x T^{T}_{-\mathbb{k}}\right) \hat{\sigma}_3 \left(\partial_y T^{*}_{-\mathbb{k}}\right)\right] - \cdots$$

$$= i\,\mathrm{Tr}\left[\Gamma_{n+N}\, \hat{\sigma}_3 \left(\partial_y T^{\dagger}_{-\mathbb{k}}\right) \hat{\sigma}_3 \left(\partial_x T_{-\mathbb{k}}\right)\right] - \cdots$$

$$= -\Omega^{xy}_{n+N,-\mathbb{k}}.$$

Thus, we have

$$\mathrm{Ch}_n = -\mathrm{ch}_{n+N} \qquad (n = 1, \cdots, N) \qquad \text{(A-16)}$$

$$\left(\text{with} \quad (\hat{E}_{d,\mathbb{k}})_{nn} = (\hat{E}_{d,-\mathbb{k}})_{n+N,\,n+N}.\right)$$

and

$$\sum_{n=1}^{2N} \mathrm{Ch}_n = 0 \quad \text{— (A-17)}$$

- In addition to this, the sum of Ch_n over only particle-type band ($n = 1, \cdots, N$) turns out to be zero, provided that

  $\epsilon_n(\mathbb{k}) > 0$ for $\forall n$ and $\mathbb{k}$.

- The condition here is usually the case for any quasi-particle boson system, which is derived from a (locally) stable classical state.

- Meanwhile, there are some exceptional cases, such as highly frustrated spin system (magnon) and isostatic lattice (phonon)

- $$\sum_{n=1}^{N} \mathrm{ch}_n = 0. \quad (\text{sum rule}) \quad \text{(A-18)}$$

To see this, let us interpolate between $\hat{H}(\mathbb{k})$ and $2N \times 2N$ unit matrix by λ

$$\hat{H}(\mathbb{k}, \lambda) = (1-\lambda)\, \hat{\mathbb{1}}_{2N\times 2N} + \lambda \hat{H}(\mathbb{k}) \quad \text{— (A-19)}$$

By the condition, $\hat{H}(\mathbb{k})$ is equivalent to a positive definite diagonal matrix by a paraunitary matrix (Eqs A-5, A-3):

$$\hat{T}_{\mathbb{k}}^{\dagger} \cdot H(\mathbb{k}) \cdot T_{\mathbb{k}} = \hat{E}_{d,\mathbb{k}}$$

all the diagonal elements are positive.

Sylvester's law of inertia dictates that $\hat{H}(\mathbb{k})$ (a hermitian matrix) is also equivalent to another positive definite diagonal matrix by a unitary matrix

$$U_{\mathbb{k}}^{\dagger} \cdot H(\mathbb{k})\, U_{\mathbb{k}} = \hat{E}'_{d,\mathbb{k}} \quad \text{— (A-19)}$$

all the diagonal elements are positive.

- Therefore, by the same unitary matrix, $\hat{H}(\mathbb{k},\lambda)$ is also equivalent to a positive definite diagonal matrix for $1 < \lambda < 0$;

$$U_{\mathbb{k}}^{\dagger} \cdot H(\mathbb{k},\lambda) \cdot U_{\mathbb{k}} = (1-\lambda)\, \mathbb{1}_{2N\times 2N} + \lambda\, \hat{E}'_{d,\mathbb{k}}$$

— (A-20)

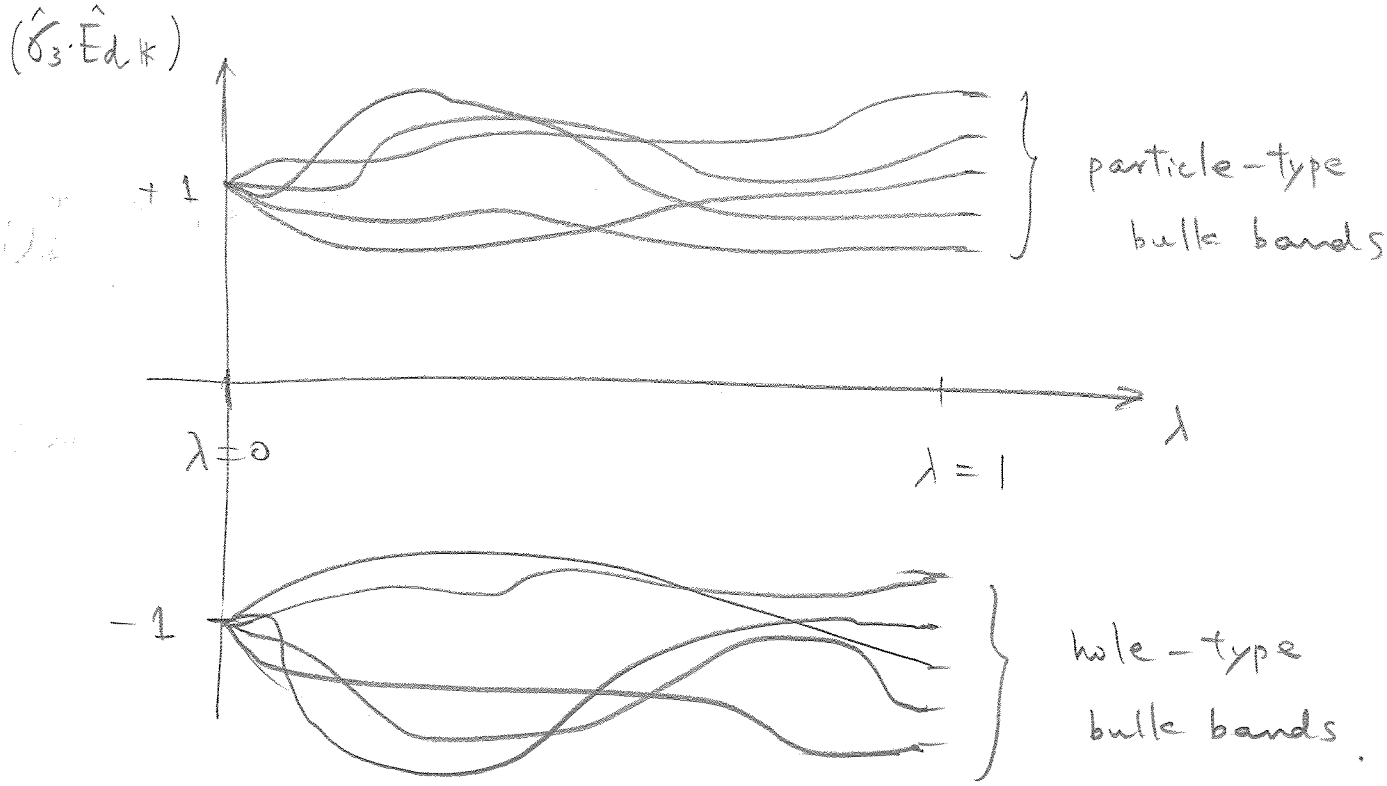
- Sylvester's law dictates that $H(\mathbb{k},\lambda)$ is also equivalent to a positive definite diagonal matrix for $1 < \lambda < 0$ by a paraunitary matrix

(A-21)

$$T_{\mathbb{k}}^{\dagger}(\lambda) \cdot H(\mathbb{k},\lambda) \cdot T_{\mathbb{k}}(\lambda) = \hat{E}_{d,\mathbb{k}}(\lambda).$$

all the diagonal elements are positive.

- This means that, during the interpolation from $\lambda = 1$ to $\lambda = 0$ (trivial limit), the N-number of particle-type bulk bands have always positive energies (those of hole-type bands have negative energies).



- The sum of the Chern integers over a group of bands does not change, unless some band in the group forms a band touching with bands outside the group,