Lecture I: Synthetic Spin-Orbit Coupling for Ultracold Atoms and Majorana fermions

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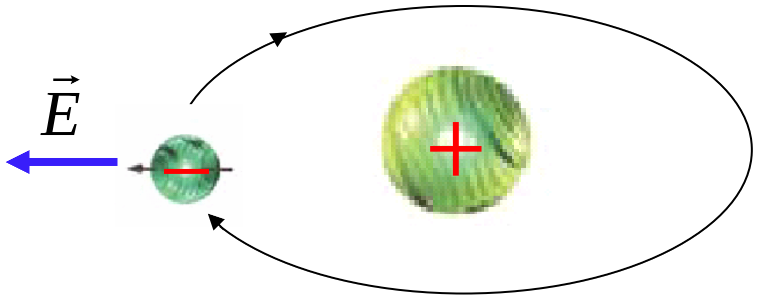
Outline

- Fundamentals of Spin-orbit coupling (SOC)
- Non-Abelian adiabatic gauge potentials
- Synthetic SOC for ultracold atoms: 1D & 2D SOCs
- Effect of synthetic SOC: Bosons
- Effect of synthetic SOC: Fermions
- Other issues

1. Fundamentals of spin-orbit coupling

1.1 Spin-orbit coupling for electrons



Consider an electron moving in the electric field of an ion in the laboratory frame. The electric field experienced by the electron is

$$\vec{E} = -\frac{1}{e}\nabla V(\vec{r})$$

where $V(\vec{r})$ is the potential of the of the ion.

Then, according to the special relativity theory, when transforming to the rest frame of the electron, we obtain an effective magnetic field by

$$\vec{B}_{\text{eff}} = \frac{\vec{\mathrm{V}}_e}{c^2} \times \vec{E}$$

where $\vec{\mathrm{V}}_e = \vec{P}/m_0$ is the velocity of the electron, c is the light speed, and m_0 is the electron mass in the vacuum. Furthermore, the electron magnetic dipole moment $\vec{\mu}_B$ couples to the magnetic field in the rest frame, yielding a spin-orbit coupling term:

$$H_{so} = -\vec{\mu}_B \cdot \vec{B}_{eff}$$

This is the Larmor spin-orbit term:

$$H_{so} = \frac{\hbar}{2m_0c^2}\vec{\sigma} \cdot (\boldsymbol{\nabla} V \times \frac{\vec{P}}{m_0})$$

$\vec{\sigma}$: the Pauli matrix on spin space.

Including the Thomas precession correction, we have finally

$$H_{so} = \frac{\hbar}{4m_0c^2}\vec{\sigma}\cdot\left(\boldsymbol{\nabla}V \times \frac{\vec{P}}{m_0}\right),$$

$\vec{\sigma}$: spin

$\vec{P}$: momentum of electron in free space

$\boldsymbol{\nabla}V$: external field

which can be derived directly from the Dirac equation. We can see a few properties from the above formula:

1) the spin-orbit term is inversely proportional to $2m_0c^2 \sim 2.0$MeV, which represents the large energy gap between electron band and positron band in the vacuum. Therefore, typically the spin-orbit interaction for electrons moving in free space is very small.
2) The spin-orbit coupling exists with the presence of a potential gradient $\boldsymbol{\nabla}V \neq 0$.
3) The components of the spin, momentum, and electric field which couple to each other are perpendicular to each other:

$$\vec{\sigma} \perp \vec{P} \perp \boldsymbol{\nabla}V$$

4) The spin-orbit term keeps time-reversal symmetry, and it can break the inversion symmetry if $V(\boldsymbol{r}) \neq V(-\boldsymbol{r})$. Namely, under inversion transformation

$$\vec{\sigma} \to \vec{\sigma}, \qquad \vec{P} \to -\vec{P}, \qquad \boldsymbol{\nabla}V(\boldsymbol{r}) \to -\boldsymbol{\nabla}V(-\boldsymbol{r}).$$

For quasi 2D electron gas, the 2D spin-orbit coupling can be induced by the inversion symmetry breaking (Rashba and Dresselhaus terms).

1.2 Spin-orbit coupling in semiconductors

In free space, the electron Hamiltonian with spin-orbit coupling

$$H = \frac{P^2}{2m_0} + \lambda_{so} \boldsymbol{\nabla} V \cdot \left(\vec{P} \times \vec{\sigma}\right), \qquad \lambda_{so} \propto 1/(m_0 c^2)$$

In semiconductors, the spin-orbit coupling can be studied with $\mathbf{k} \cdot \vec{p}$ theory. Electrons moving in the periodic lattice potential can be described as Bloch bands. For semiconductors, the Fermi energy is close to the band bottom, and the dispersion relation of the Bloch bands is parabolic, similar as the electron in the vacuum but with a different effective mass:

$$m_0 \rightarrow m_{\text{eff}}$$

Just like that the mass, m_0, in vacuum describes the gap between electron and positron bands, the effective mass m_{eff} of electron in a semiconductor is proportional to the corresponding Bloch band gap, and can be much less than the mass in vacuum: $m_{\text{eff}} \ll m_0$, which can lead to a large enhancement of spin-orbit coupling.

For quasi-2D electron gas, the Rashba spin-orbit coupling is induced by structure inversion asymmetry (SIA) (along *z* axis), which can be tuned by gate.

$$H = \frac{\hbar^2 k^2}{2m_{\text{eff}}} + \alpha_{so} \vec{e}_z \cdot (\boldsymbol{k} \times \vec{\sigma}), \qquad \alpha_{so} \propto \langle \nabla_z V(\boldsymbol{r}) \rangle \neq 0.$$

The Dresselhaus spin-orbit coupling is due to bulk inversion asymmetry (BIA):

$$H = \frac{\hbar^2 k^2}{2m_{\text{eff}}} + \beta_{so}\left(k_x \sigma_x - k_y \sigma_y\right),$$

For a system with both the Rashba and Dresselhaus spin-orbit couplings,

$$H = \frac{\hbar^2 k^2}{2m_{\text{eff}}} + \alpha_{so}\vec{e}_z \cdot (\boldsymbol{k} \times \vec{\sigma}) + \beta_{so}\left(k_x\sigma_x - k_y\sigma_y\right),$$

When $\alpha_{so} = \beta_{so}$, we have

$$H = \frac{\hbar^2 k^2}{2m_{\text{eff}}} + \alpha_{so}\left(k_x\sigma_y - k_y\sigma_x\right) + \beta_{so}\left(k_x\sigma_x - k_y\sigma_y\right),$$

$$= \frac{\hbar^2 k^2}{2m_{\text{eff}}} + \alpha_{so}k_x(\sigma_x + \sigma_y) - k_y(\sigma_x + \sigma_y),$$

$$= \frac{\hbar^2 k^2}{2m_{\text{eff}}} + \alpha_{so}(k_x - k_y)(\sigma_x + \sigma_y),$$

$$= \frac{\hbar^2 k^2}{2m_{\text{eff}}} + \alpha_{so}k_-\sigma_+,$$

which renders effectively a 1D spin-orbit coupled system. The spin-orbit coupling firstly realized in cold atom experiments is of the above form, namely, the 1D spin-orbit coupling (with equal Rashba and Dresselhaus amplitudes).

1.3 Why study spin-orbit coupling?

1. Spintronics

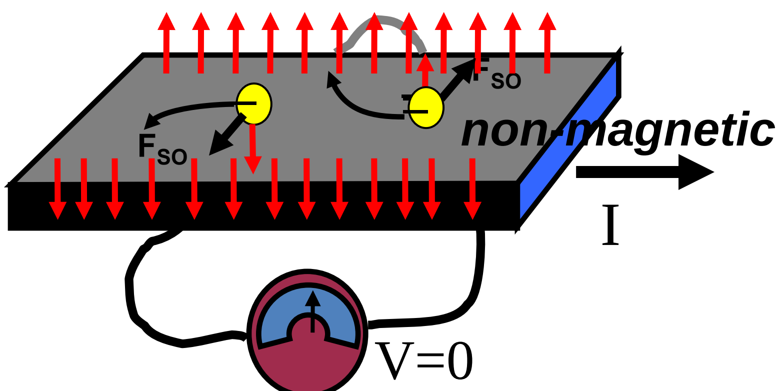
Effective Hamiltonian with a Rashba spin-orbit coupling term can be written as

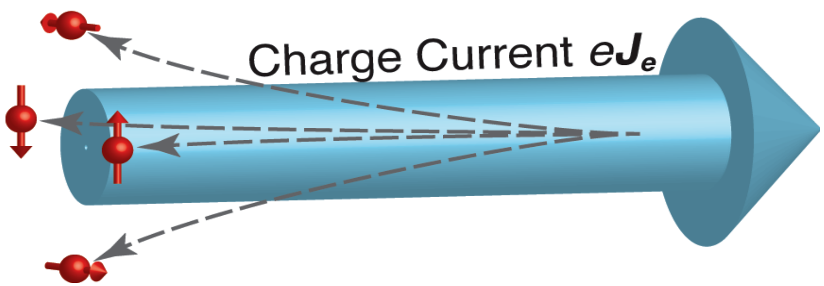
$$H = \frac{\hbar^2 k^2}{2m_{\text{eff}}} + \alpha_{so}\left(k_x\sigma_y - k_y\sigma_x\right) + V(\boldsymbol{r})$$
$$= \frac{1}{2m_{\text{eff}}}\left(\hbar k - \vec{A}_\sigma\right) + V(\boldsymbol{r}) + const.$$

The SU(2) non-Abelian spin-dependent gauge potential: $\vec{A}_\sigma = \frac{m_{\text{eff}}\alpha_{so}}{\hbar}\left(-\sigma_y\vec{e}_x + \sigma_x\vec{e}_y\right)$, which is associated a spin-dependent magnetic field, obtained through $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{ie}{\hbar c}[A_\mu, A_\nu]$:

$$\vec{B} = \frac{m_{\text{eff}}^2\alpha_{so}^2}{\hbar^3}\frac{e}{c}\sigma_z\vec{e}_z$$

When electrons are accelerated by electric field, their spins will be tilted to out-of-plane ($\vec{e}_z$) direction. The above formula implies that electron having nonzero spin polarization experiences an effective magnetic field along +z or –z direction depending on its polarization direction. Thus spin-up and spin-down electrons are deflected to opposite sides. This leads to a spin Hall effect.





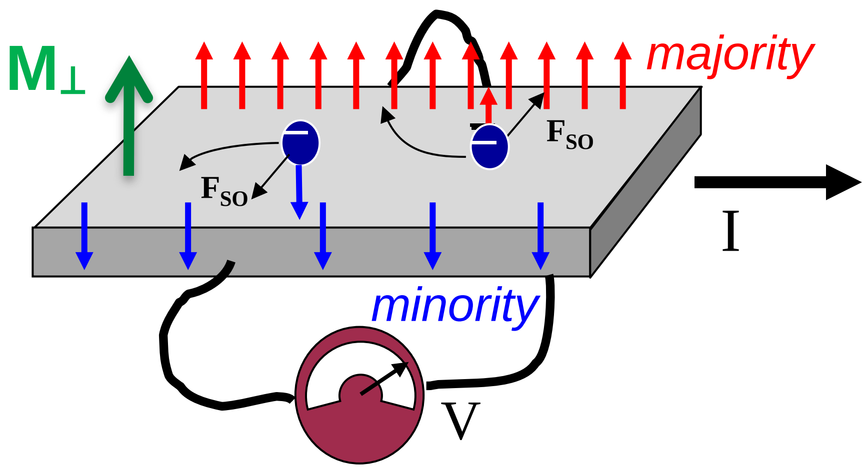
Murakami et al Science 2003; J. Sinova et al, PRL 2004.

When there is a Zeeman term, the Hamiltonian gives:

$$H = \frac{\hbar^2 k^2}{2m_{\text{eff}}} + \alpha_{so}\left(k_x\sigma_y - k_y\sigma_x\right) + \mathrm{M}_{\perp}\sigma_z + V(\boldsymbol{r})$$

In this case the electron having opposite spin polarization along z axis are still deflected to the opposite sides. However, the population of spin-up and spin-down electrons are different due to the Zeeman term. Thus the electron accumulation on one side (spin-up in the following figure) is more than that in the other side (spin-down), which gives rise to a transverse electric field. This leads to the anomalous Hall effect.

Hall resistance (Nagaosa, Sinova, etal. Rev. Mod. Phys. 2010)



$$\rho_H = R_0 B_{\perp} + 4\pi R_s M_{\perp}$$

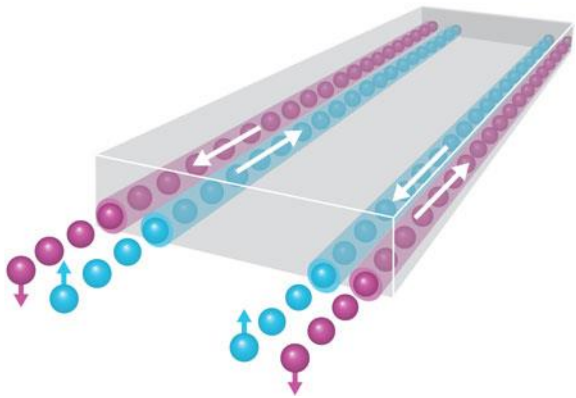
$$R_0 << R_s$$

Applications: spintronic devices; spin-current injection and spin-accumulation modulation

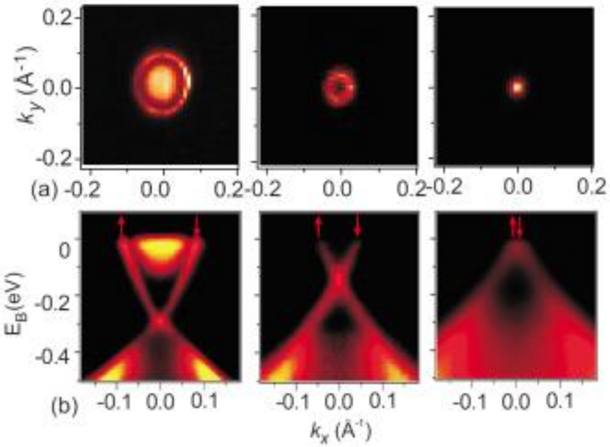
2. Topological insulator

Hasan, Kane, Rev. Mod. Phys. 2010; Qi, Zhang, Rev. Mod. Phys. 2011

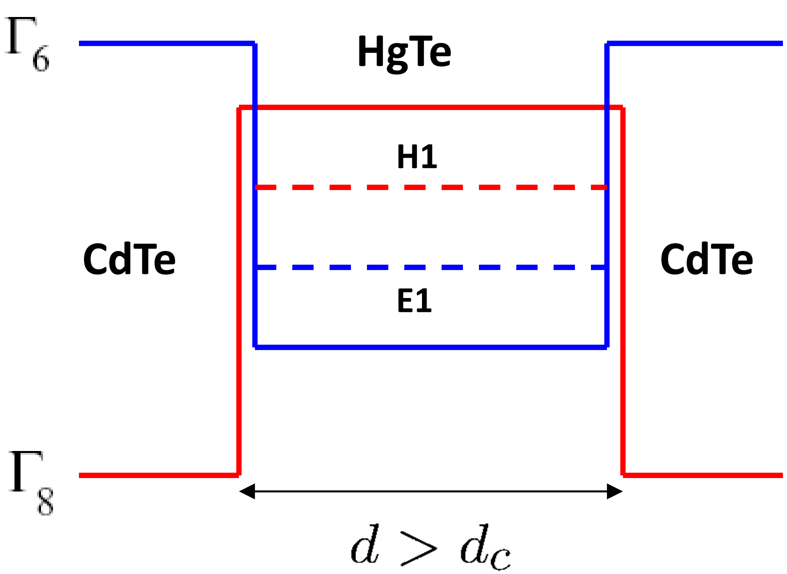
A. 2D Topological insulator



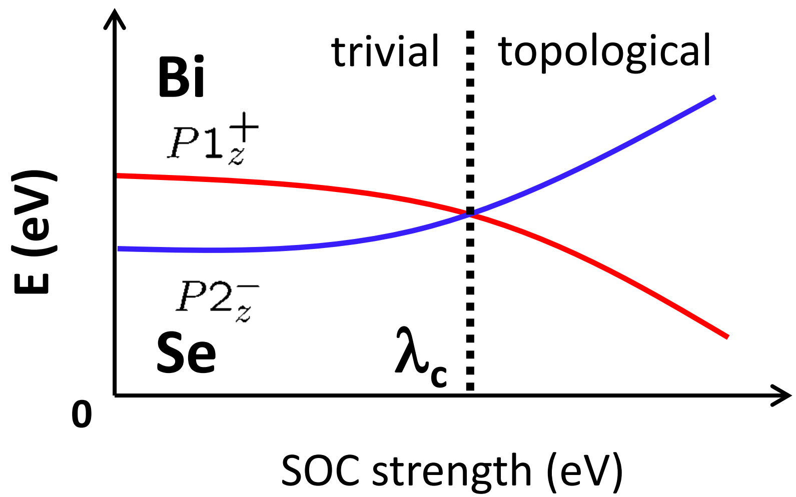
B. 3D Topological insulator



SO effect: band inversion and topological phase transition



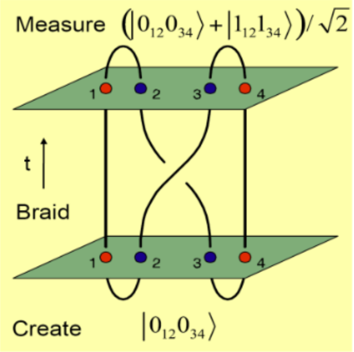
2D **HgTe:** E1 and H1 bands inverted



3D: **Band inversion in Bi_2Se_3**

3. Topological superconductor/superfluid

Non-Abelian Majorana quasiparticles

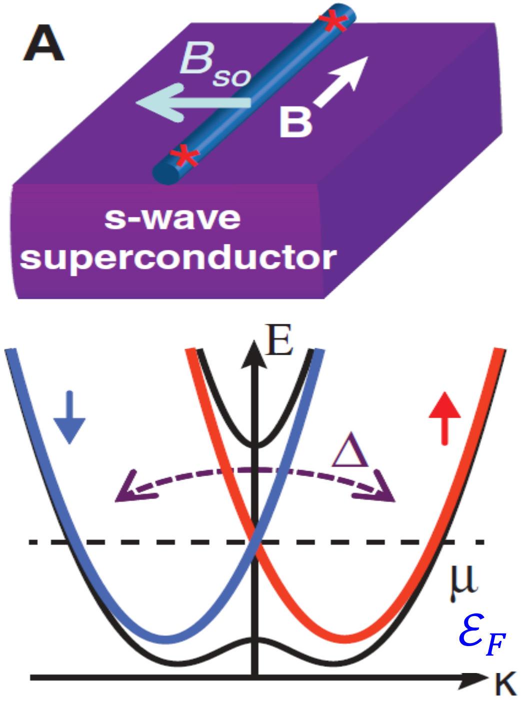


A. Intrinsic topological SC

B. Heterostructures: s-wave SC + spin-orbit coupling (+Zeeman term)

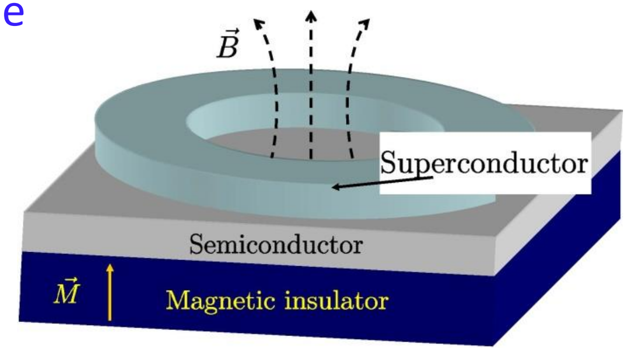
SO effect: drive s-wave pairing into p-wave order

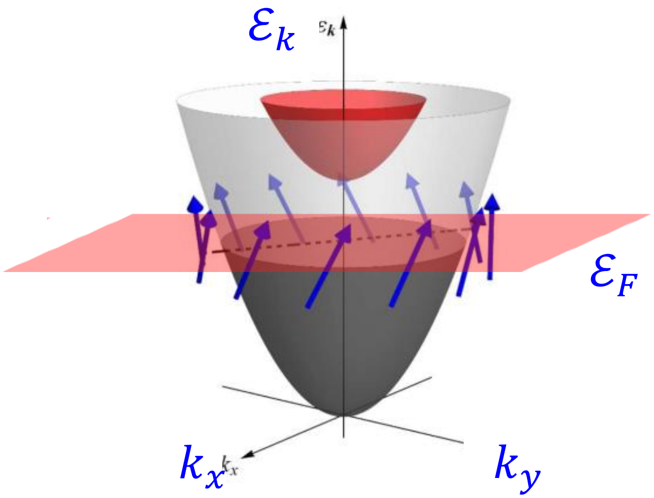
1D case



$$\gamma = \gamma^\dagger$$

2D case





J. Alicea, *Rep. Prog. Phys.* **75** 076501 (2012).

Why study spin-orbit coupling for cold atoms?

Motivation: Challenges in solids. New opportunities in cold atoms.

Compare cold atoms with solid state systems:

- **Advantages**

1. Simple and cleaning: *Exact models, no disorder, …*
2. Fully controllable: $N, U, \mu, L, t, \omega, \Omega$, ……
3. Dimensionality

Everything can be controlled!

4. Lattice configuratons: *square, honeycomb, kagome, …*
3. May go beyond condensed matter physics: *Large spin, high-orbital band, …*

……

- **Disadvantages**

1. Have to be in ex
2. Atom loss, short

Everything needs to be controlled!

3. Neutral particles: sometime difficult in manipulation and detection

Early references for artificial gauge potentials and SOC

Creation of synthetic magnetic fields:

1. D. Jaksch and P. Zoller, New J. Phys. 5, 56 (2003).
2. G. Juzeliunas and P. Ohberg, Phys. Rev. Lett., 93, 033602 (2004).

Creation of *spin-dependent* gauge potentials and spin-orbit coupling:

1. *X.-J. Liu*, H. Jing, X. Liu, and M.-L. Ge, Eur. Phys. J. D, 37, 261 (2005 online); arXiv:quant-ph/0410096 (2014).

(quantum) spin Hall effect for cold atoms:

1. *X.-J. Liu*, X. Liu, L. C. Kwek, and C. H. Oh, Phys. Rev. Lett. 98, 026602 (2007), arXiv:cond-mat/0603083 (2016).
2. S.-L. Zhu, H. Fu, C.-J. Wu, S.-C. Zhang, and L.-M. Duan, Phys. Rev. Lett. 97, 240401 (2006).

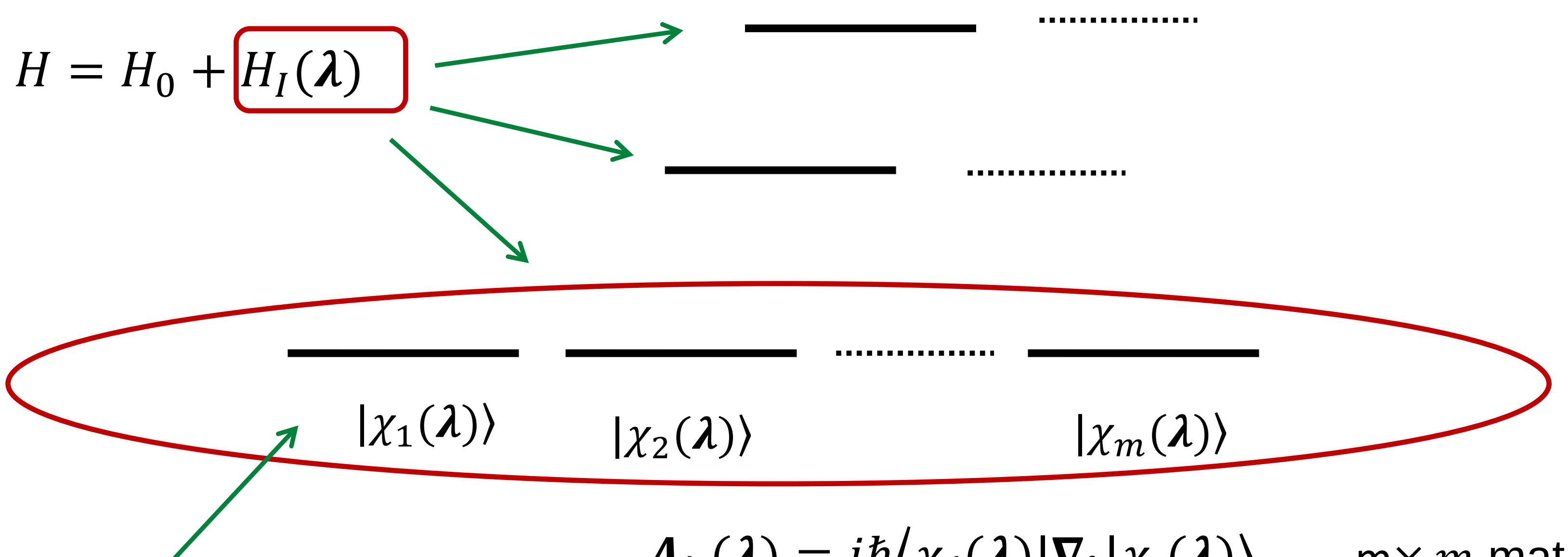
Non-Abelian gauge potentials/spin-orbit couplings:

1. K. Osterloh, M. Baig, L. Santos, P. Zoller, M. Lewenstein, Phys. Rev. Lett. 95, 010403 (2005).
2. J. Ruseckas, G. Juzeliunas, P. Ohberg, and M. Fleischhauer, Phys. Rev. Lett. 95, 010404 (2005).

2. Non-Abelian adiabatic gauge potential

2.1 Generic idea: spin-orbit coupling for cold atoms can be generated by realizing synthetic spin-dependent/non-Abelian gauge potentials.

Consider a many-state quantum system which couples to external field, with the interacting Hamiltonian $H_{int}(\boldsymbol{\lambda})$ depending on slowly varying parameter $\boldsymbol{\lambda}$. The eigenstates of $H_{int}(\boldsymbol{\lambda})$ are functions of $\boldsymbol{\lambda}$. In the case that the ground states of $H_{int}(\boldsymbol{\lambda})$ have m-fold degeneracy, one can obtain a non-Abelian adiabatic gauge potential (Berry's connection) which is generically a $m \times m$ matrix.



In the ground state manifold:

$$\boldsymbol{A}_{jl}(\boldsymbol{\lambda}) = i\hbar\langle\chi_j(\boldsymbol{\lambda})|\boldsymbol{\nabla}_{\lambda}|\chi_l(\boldsymbol{\lambda})\rangle,$$ m× m matrix

For spin-orbit coupling: $\boldsymbol{\lambda} = \boldsymbol{r}$, $\quad i\hbar\boldsymbol{\nabla}_{\boldsymbol{r}} \to i\hbar\boldsymbol{\nabla}_{\boldsymbol{r}} + \boldsymbol{A}(\boldsymbol{r})$ *Wilczek and Zee, (1984)*

Unitary transformation For a review: X.-J. Liu et al., Front. Phys. China, **3**, 113 (2008).

For a N-state quantum system, the wave function can be generically described as

$$\tilde{\Psi}(\boldsymbol{r},t) = \sum_{k=1}^{N} \Psi_k(\boldsymbol{r}) \exp(-\mathrm{i}\mathcal{E}_k t/\hbar)|\chi_k\rangle$$

We can define the N-component vector by

$$\Psi = (\Psi_1, \Psi_2, \cdots, \Psi_N)^{\mathrm{T}}$$

which is governed by the Schrodinger equation under rotating wave approximation:

$$\mathrm{i}\hbar\frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r})\right]\Psi + H_I(\boldsymbol{r})\Psi$$

where V(**r**) is trapping potential. Assume that the interacting Hamiltonian is slowly dependent on the position ***r***. we diagonalize the interacting Hamiltonian by a position dependent unitary transformation:

$$\Psi = U(\boldsymbol{r})\Phi, \quad H_I^d = U^\dagger H_I U$$

This gives the N eigenbases:

$$\Phi = (\Phi_1, \Phi_2, \cdots, \Phi_N)$$

With the eigenenergies

$$(H_I^d)_{kk} = E_k \qquad k = 1,2,\ldots,N$$

Note that after diagonalizing the interacting Hamiltonian, the kinetic part generically becomes off-diagonal. The new wave function Φ satisfies

$$i\hbar\frac{\partial \Phi}{\partial t} = -\frac{\hbar^2}{2m}U^\dagger\nabla^2 U\,\Phi + U^\dagger V(\boldsymbol{r})U\,\Phi + H_I^d(\boldsymbol{r})\,\Phi$$

The above equations can be simplified by introducing a non-Abelian gauge potential

$$\boldsymbol{A}(\boldsymbol{r}) = i\frac{\hbar c}{e}U^\dagger(\boldsymbol{r})\nabla U(\boldsymbol{r})$$

which is an $N \times N$ matrix. We obtain

$$i\hbar\frac{\partial \Phi}{\partial t} = \frac{1}{2m}\left[i\hbar\nabla + \frac{e}{c}\boldsymbol{A}(\boldsymbol{r})\right]^2 \Phi + V'(\boldsymbol{r})\,\Phi + H_I^d(\boldsymbol{r})\,\Phi$$

$$V' = U^\dagger V(\boldsymbol{r})U$$

$\boldsymbol{A}(\boldsymbol{r})$ is called *SU(N)* Berry's connection, and the associated Berry's curvature reads

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\frac{e}{c}[A_\mu, A_\nu]$$

The corresponding artificial magnetic field magnetic field is given by

$$B_j = 1/2\epsilon_{jkl}F_{kl}$$

Adiabatic condition

Note that we have not considered any approximation, and in this stage the non-Abelian gauge potential is a pure gauge. We can verify that

$$F_{\mu\nu} = 0$$

which can be become nonzero after introducing the adiabatic condition. We consider below two situations.

1) Abelian gauge potential

When all the N eigenstates of the interacting Hamiltonian are non-degenerate, and under the following condition:

$$\left|\frac{\boldsymbol{P}\cdot\boldsymbol{A}_{jk}}{m(E_j - E_k)}\right| \ll 1$$

The transitions between two different eigenstates are negligible, namely, we can apply the adiabatic condition. Then we ignore the off-diagonal terms of the gauge potential and obtain

$$\begin{aligned}\boldsymbol{A}_j(\boldsymbol{r}) &= \mathrm{i}\frac{\hbar c}{e}(U^\dagger(\boldsymbol{r})\nabla U(\boldsymbol{r}))_{jj} \\ &= \mathrm{i}\hbar\langle\Phi_j|\nabla|\Phi_j\rangle, \quad j = 1, 2, \cdots, N\end{aligned}$$

This implies that under the adiabatic condition we reduce the symmetry of the gauge by

$$SU(N) \rightarrow U(1) \times U(1) \ldots \times U(1)$$

The Schrodinger equation under the adiabatic condition reads

$$i\hbar \frac{\partial \Phi_j}{\partial t} = \frac{1}{2m}\left[i\hbar\nabla + \frac{e}{c}\boldsymbol{A}_j(\boldsymbol{r})\right]^2 \Phi_j + \bar{V}_j'(\boldsymbol{r})\Phi_j$$

where the scalar potential is obtained by

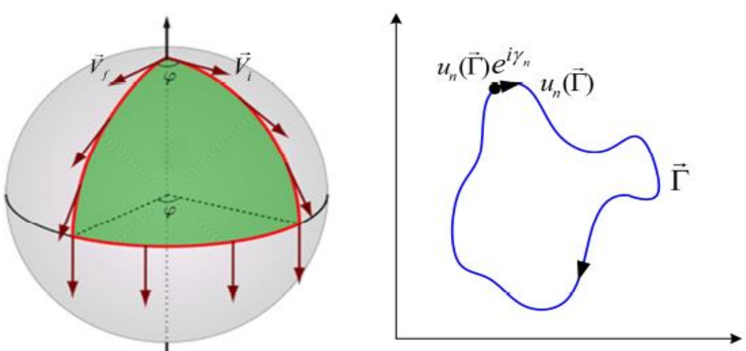
$$V_j'(\boldsymbol{r}) = V'(\boldsymbol{r}) + \phi_j + E_j$$

$$\phi_j(\boldsymbol{r}) = \frac{1}{2m}\sum_{k\neq j}\boldsymbol{A}_{jk}\cdot\boldsymbol{A}_{kj}, \quad j = 1, 2, \cdots, N$$

Now the U(1) Berry's curvature is given by

$$F_{\mu\nu}^{(j)} = \partial_\mu A_\nu^{(j)} - \partial_\nu A_\mu^{(j)}$$

which is generically nonzero. The nonzero Berry's curvature can lead to nontrivial Berry's phase.



γ_n : Berry Phase

(a) (b)

If the parameter is position:

$$\gamma = \frac{e}{\hbar c}\int_c \vec{A}\cdot d\vec{r}$$

Similar as the AB phase.

1) non-Abelian gauge potential

If the ground state subspace has a *m*-fold degeneracy. Then the adiabatic condition is expressed as

$$\left|\frac{\boldsymbol{P}\cdot\boldsymbol{A}_{ij}}{m(E_g-E_j)}\right| \ll 1$$

where $i = 1,2,\ldots,m$ and $j = m+1, m+2, \ldots, N$.

Similarly, for the wave function of the ground state subspace: $\Phi' = (\Phi_1, \Phi_2, \cdots, \Phi_m)^{\mathrm{T}}$, we have

$$\mathrm{i}\hbar\frac{\partial\Phi'}{\partial t} = \frac{1}{2m}\left[\mathrm{i}\hbar\nabla + \frac{e}{c}\boldsymbol{A}'(\boldsymbol{r})\right]^2\Phi' + [V'(\boldsymbol{r}) + \phi' + E_g]\Phi'$$

$$\boldsymbol{A}'_{jk}(\boldsymbol{r}) = \mathrm{i}\frac{\hbar c}{e}(U^\dagger(\boldsymbol{r})\nabla U(\boldsymbol{r}))_{jk}$$

$$\phi'_{jk} = \frac{1}{2m}\sum_{l>m}\boldsymbol{A}_{jl}\cdot\boldsymbol{A}_{lk}$$

In this case we reduce the symmetry of the gauge by $SU(N) \rightarrow U(m) \times \ldots$

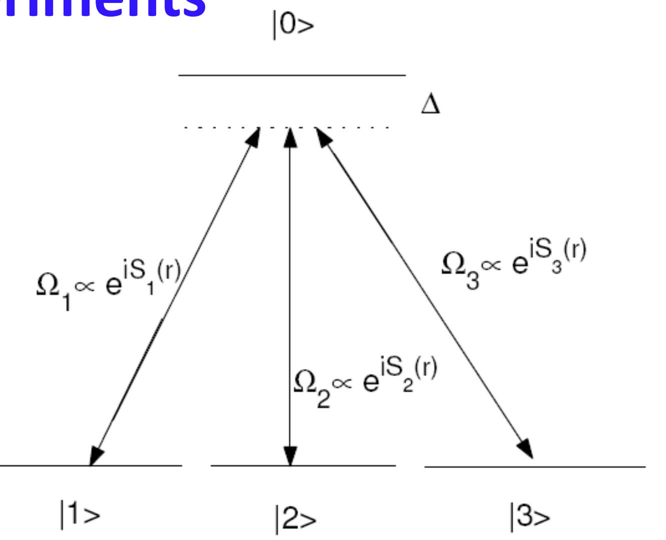
If m=2, $|\phi_1\rangle$ and $|\phi_2\rangle$ consist of a pseudo-spin ½ system. The U(2) adiabatic gauge takes the form

$$\vec{A} = \vec{A}_1\sigma_x + \vec{A}_2\sigma_y + \vec{A}_3\sigma_z \quad \Longrightarrow \quad \boxed{\text{Spin-orbit coupling}}$$

3. Theoretical models and Experiments

3. 1. Four-level tripod system

Ruseckas et al., PRL (2005); Stanescu, Zhang, and Galitski, PRL (2007).



The atoms coupled to three laser fields, with interacting Hamiltonian:

$$H_I = \hbar\Delta|0\rangle\langle 0| - (\hbar\Omega_1|0\rangle\langle 1| + \hbar\Omega_2|0\rangle\langle 2| + \hbar\Omega_3|0\rangle\langle 3| + h.c.)$$

which has degenerate two dark states:

$$|\chi_1(\boldsymbol{\lambda})\rangle = \cos\theta e^{-iS_1(r)}|1\rangle - \sin\theta\, e^{-iS_2(r)}|2\rangle$$

$$|\chi_2(\boldsymbol{\lambda})\rangle = \cos\phi[\sin\theta\, e^{-iS_1(r)}|1\rangle + \cos\theta\, e^{-iS_2(r)}|2\rangle] - \sin\phi e^{-iS_3(r)}|3\rangle$$

The parameters are defined through

$$\Omega_1 = \Omega\sin\theta\cos\phi e^{iS_1} \qquad \Omega_2 = \Omega\sin\theta\sin\phi e^{iS_2}$$

$$\Omega_3 = \Omega\cos\theta e^{iS_3} \qquad \Omega = \sqrt{|\Omega_1|^2 + |\Omega_2|^2 + |\Omega_3|^2}$$

The two darks states consist of a pseudospin-1/2 system, for which a U(2) non-Abelian gauge potential can be obtained.

The tripod system is experimentally challenging!

3. 2. A minimal scheme: Λ system

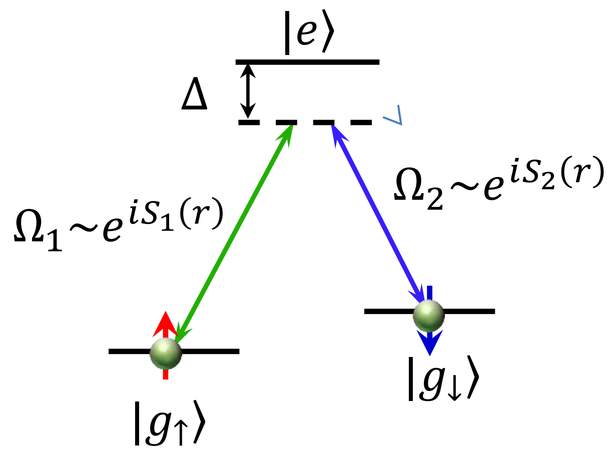
X.-J. Liu, M. F. Borunda, X. Liu, and J. Sinova, PRL, 102, 046402 (2009); arXiv: 0808.4137.

The atoms are coupled to two laser fields with a detuning Δ. The interacting Hamiltonian reads

$$H_I = \hbar\Delta - (\hbar\Omega_1|g_\uparrow\rangle\langle g_\downarrow| + \hbar\Omega_2|g_\uparrow\rangle\langle|g_\downarrow| + h.c.)$$

The optical dipole transition Rabi-frequencies:

$$\Omega_1 = \langle g_\uparrow|\boldsymbol{d}_{1F}\cdot\boldsymbol{E}_1|e\rangle, \quad \Omega_2 = \langle g_\downarrow|\boldsymbol{d}_{1F}\cdot\boldsymbol{E}_2|e\rangle$$



1D spin-orbit coupling via Λ system.

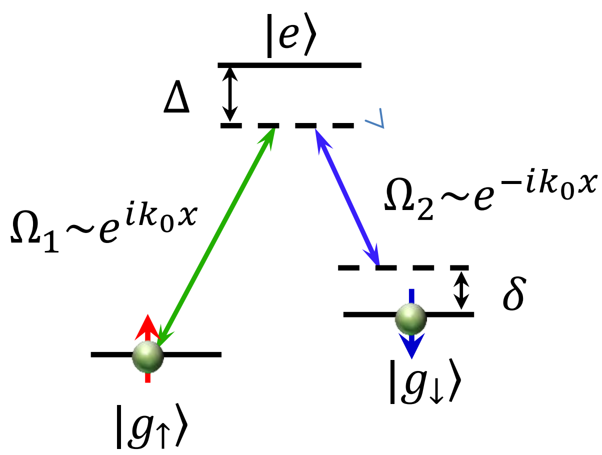
When Δ is much larger than the optical dipole transition strength: $|\Delta|^2 \gg |\Omega_1|^2 + |\Omega_2|^2$, the above Hamiltonian has two nearly degenerate ground states:

$$|\chi_1(\boldsymbol{\lambda})\rangle = e^{iS_2(r)}\cos\theta|g_\uparrow\rangle - \sin\theta\, e^{iS_1(r)}|g_\downarrow\rangle$$

$$|\chi_2(\boldsymbol{\lambda})\rangle \approx e^{iS_2(r)}\sin\theta|g_\uparrow\rangle + \cos\theta\, e^{iS_1(r)}|g_\downarrow\rangle$$

which also consist of a pseudospin-1/2 system. Actually, the physical picture of this result is simple. Under the condition: $|\Delta|^2 \gg |\Omega_1|^2 + |\Omega_2|^2$, the single-photon transitions from ground states to the excited state is greatly suppressed, while the two-photon transition from one ground state (e.g. $|g_\uparrow\rangle$) to another ground state (e.g. $|g_\downarrow\rangle$) can happen. This effectively leads to a two-state ($|g_\uparrow\rangle$ and $|g_\downarrow\rangle$) system with Raman coupling induced by two-photon process. As a result, we can obtain adiabatic gauge potential in this pseudospin-1/2 system.

For the simplest situation, we can consider the following configuration for the laser fields:



$$\Omega_1 = |\Omega_0| e^{ik_0 x}, \qquad \Omega_2 = |\Omega_0| e^{-ik_0 x}$$

As described previous, when $|\Delta|^2 \gg |\Omega_1|^2 + |\Omega_2|^2$ we can adiabatically remove the excited state and obtain the effective Hamiltonian in two ground states

$$H_{\text{eff}} = \begin{bmatrix} \frac{k_x^2}{2m} + \frac{\delta}{2}, & \Omega_R e^{i2k_0 x} \\ \Omega_R e^{-i2k_0 x}, & \frac{k_x^2}{2m} - \frac{\delta}{2} \end{bmatrix}$$

The Raman coupling strength:

$$\Omega_R = \frac{|\Omega_1 \Omega_2|}{\Delta}$$

δ is a small two-photon detuning. Using the following transformation

$$U = \begin{pmatrix} e^{-ik_0 x} & 0 \\ 0 & e^{ik_0 x} \end{pmatrix}$$

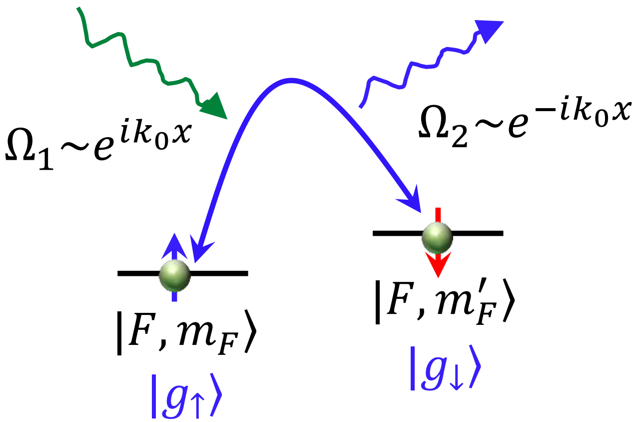
we obtain the 1D (equal Rashba-Dresselhaus) spin-orbit coupling

$$H_0 = U H_{\text{eff}} U^\dagger = \frac{(k_x + k_0 \sigma_z)^2}{2m} + \frac{\delta}{2}\sigma_z + \Omega_R \sigma_x$$

Picture of the snthetic spin-orbit coupling generated by two-photon Raman process

Effects of the Raman coupling:

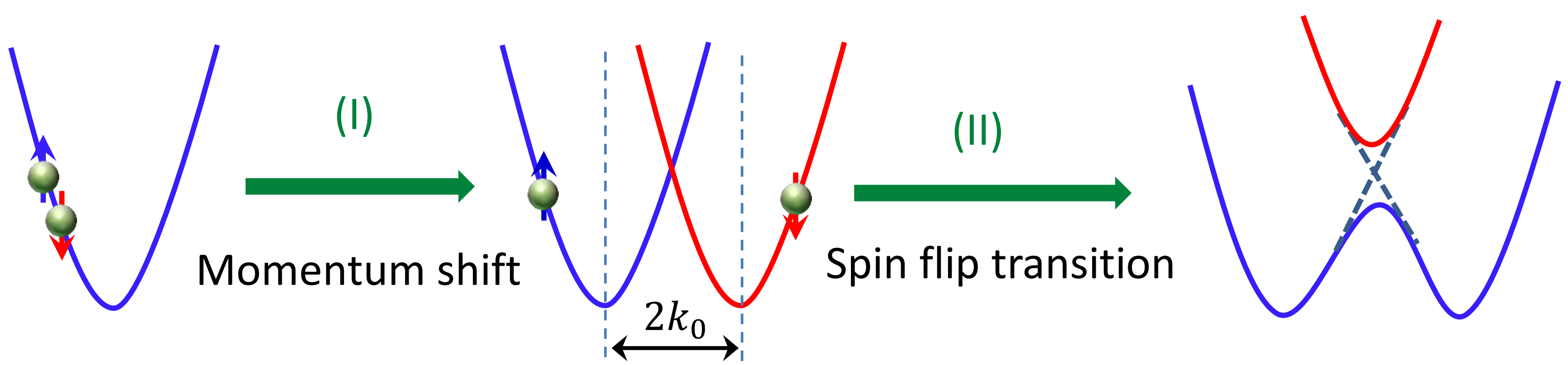
(I) $2k_0$ momentum transfer; (II) spin-flip transition.



1D spin-orbit coupling plus Zeeman coupling

$$H_0 = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{k_0}{m} p_x \sigma_z + \frac{\Omega_R}{2} \sigma_x$$

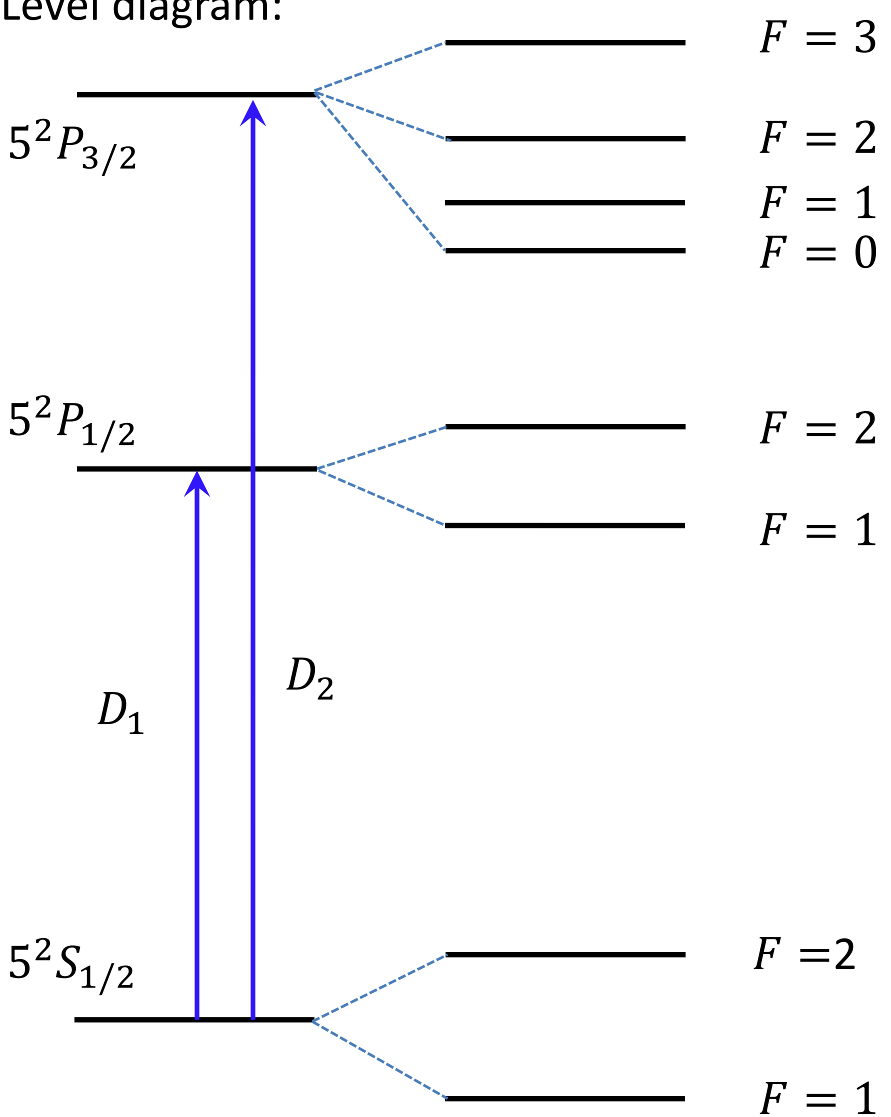
Illustration of 1D SO coupling:



3.3 Realize a Λ-type system with cold atoms

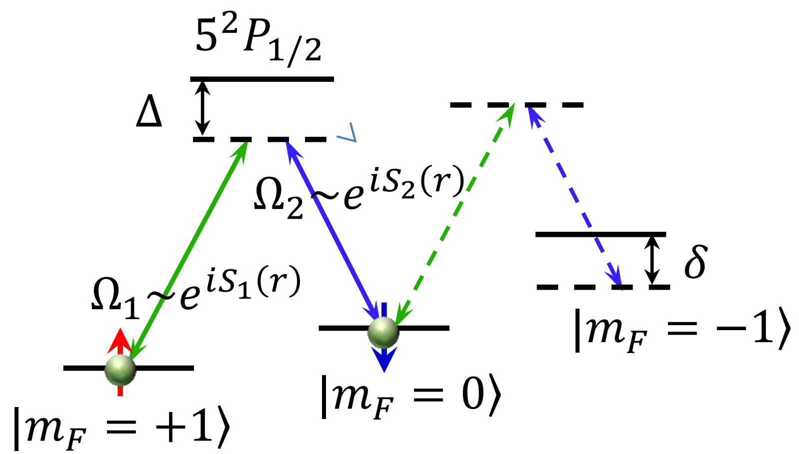
Candidate: ^{87}Rb

Level diagram:



Note that the ground state corresponds to $F = 1$, which has 3 degenerate states. To separate a Λ-type system from other states, one can apply a magnetic field to induce a energy shift in the ground states. The energy shift is nonlinear, hence when two of the three ground states are coupled in two-photon resonance, they are detuned from the third state. Thus a Λ-type system is resulted.

$$\Delta E_{|F\ m_F\rangle} = \mu_B\, g_F\, m_F\, B_z + \frac{(g_J - g_I)^2 \mu_B^2}{2\hbar \Delta E_{\rm hfs}} B^2$$



When $\delta \gg |\Omega_{1,2}|$, the coupling to the state $|m_F = -1\rangle$ can be neglected, and then a pseudo spin-1/2 system is achieved.

Cancellation of Raman couplings through D1 and D2 lines.

Note that the net Raman coupling is obtained by taking into account the contributions through both the D1 and D2 lines. The total Raman coupling between two ground states $|m_F = +1\rangle$ and $|m_F = 0\rangle$ is given by

$$\Omega_R = \sum_F \frac{\Omega^*_{2F,D1}\Omega_{1F,D1}}{\Delta} + \sum_F \frac{\Omega^*_{2F,D2}\Omega_{1F,D2}}{\Delta + E_s}.$$

in the above formula E_s represents the frequency difference between D1 and D2 lines (i.e. the fine structure splitting of excited states), $\Omega_{1F,D1}$ is the Rabi-frequency of the laser induced transition from the ground state $|m_F = +1\rangle$ to an excited state of quantum number F in the D1 line, and similar for other related notations.

We can show that (denote $|e_{F,Dj}\rangle$ as an excited state corresponding to Dj (j=1,2) line):

$$\sum_F \Omega^*_{2F,D1}\Omega_{1F,D1} + \sum_F \Omega^*_{2F,D2}\Omega_{1F,D2} = \sum_{F,j=1,2} \langle\downarrow|\boldsymbol{d}_{2F}\cdot\boldsymbol{E}_2|e_{F,Dj}\rangle\langle e_{F,Dj}|\boldsymbol{d}_{1F}\cdot\boldsymbol{E}_1|\uparrow\rangle$$

$$= \langle\downarrow|(\boldsymbol{d}_{2F}\cdot\boldsymbol{E}_2)(\boldsymbol{d}_{1F}\cdot\boldsymbol{E}_1)|\uparrow\rangle$$

$$= 0$$

In the second line we have used the identity

$$\sum_{F;j=1,2} |e_{F,Dj}\rangle\langle e_{F,Dj}| \quad = 1.$$

Thus the couplings through the D1 and D2 lines contribute oppositely to the Raman transition. We obtain that

$$\Omega_R = \sum_F \frac{E_s}{\Delta(\Delta + E_s)} \Omega^*_{2F,D1}\Omega_{1F,D1}$$

When $|\Delta| < E_s$, we have

$$\Omega_R \propto \frac{|\Omega_1\Omega_2|}{\Delta} \propto \frac{1}{\tau_{\text{life}}}\frac{\Delta}{\Gamma},$$

where the life time is given via $\frac{1}{\tau_{\text{life}}} \sim \frac{|\Omega_j|^2}{\Delta^2}\Gamma$, and Γ is the decaying rate. In this regime, both Ω_R and τ_{life} can be enhanced by properly increasing Δ and Rabi-frequencies $|\Omega|$ accordingly.

When $|\Delta| \gg E_s$, we have

$$\Omega_R \propto \frac{|\Omega_1\Omega_2|}{\Delta^2} \propto \frac{1}{\tau_{\text{life}}}\frac{E_s}{\Gamma}$$

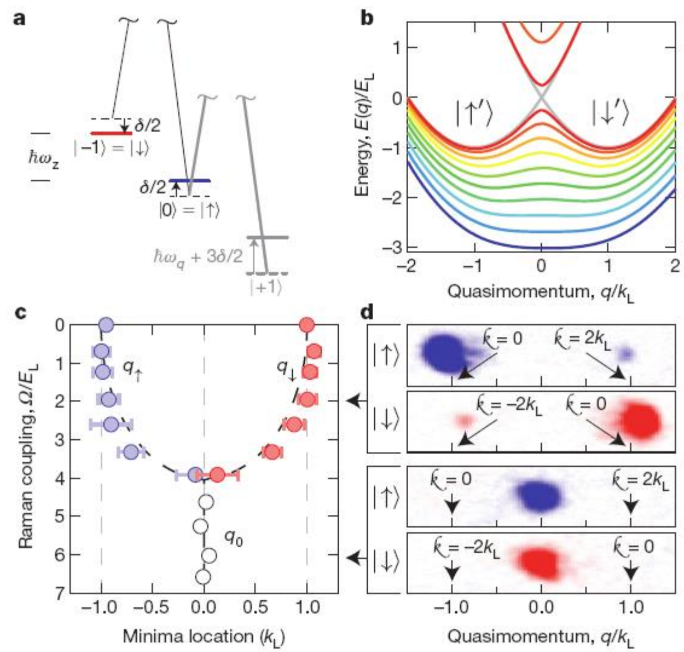
In this regime, the Raman coupling strength Ω_R and life time τ_{life} cannot be enhanced at the same time by increasing Δ and Rabi-frequencies $|\Omega|$. Thus we have the following conclusions:

1) To induce an appreciable Raman coupling strength V_R, the detuning Δ cannot be much larger than fine structure splitting E_s of the excited states.
2) A large enough life time τ_{life}, however, requires that Δ should be much larger than $|\Omega|$.
3) For Alkali atoms, The proper parameter regime is that $|\Delta| \lesssim E_s$.
4) The atomic candidates with large fine structure splitting E_s are preferred for the generation of spin-orbit coupling.

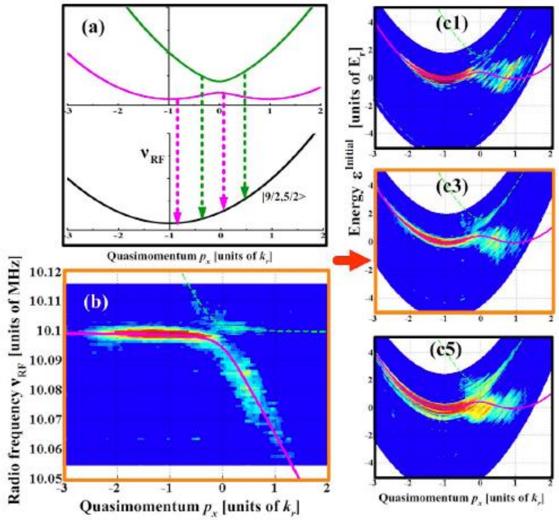
^{87}Rb: good	^{40}K: marginal	^{6}Li: strong heating	Alkali-earth atoms (Yb)
$E_s \sim 7.1$THz	$E_s \sim 1.8$THz	$E_s \sim 10$GHz	$E_s > 100$THz

3.4 Experimental realization of 1D SOC

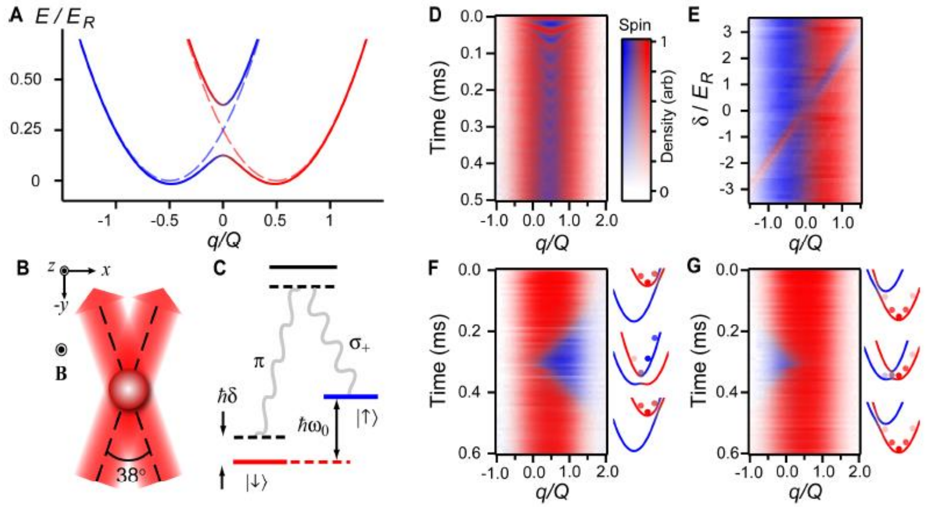
^{87}Rb boson: I. Spielman group, 2011
Shuai Chen, Jianwei Pan group, 2012
P. Engels' group, Washington State U.
Y. P. Chen, Puedue U
^{40}K fermion: J. Zhang group, 2012
^{6}Li fermion: M. Zwierlein group, 2012.
^{161}Dy fermion: Lev, 2016;
^{173}Yb fermion: G.-B. Jo & XJL etal, 2016; ….



^{40}K fermions:
J. Zhang group, PRL. **109**, 095301 (2012).



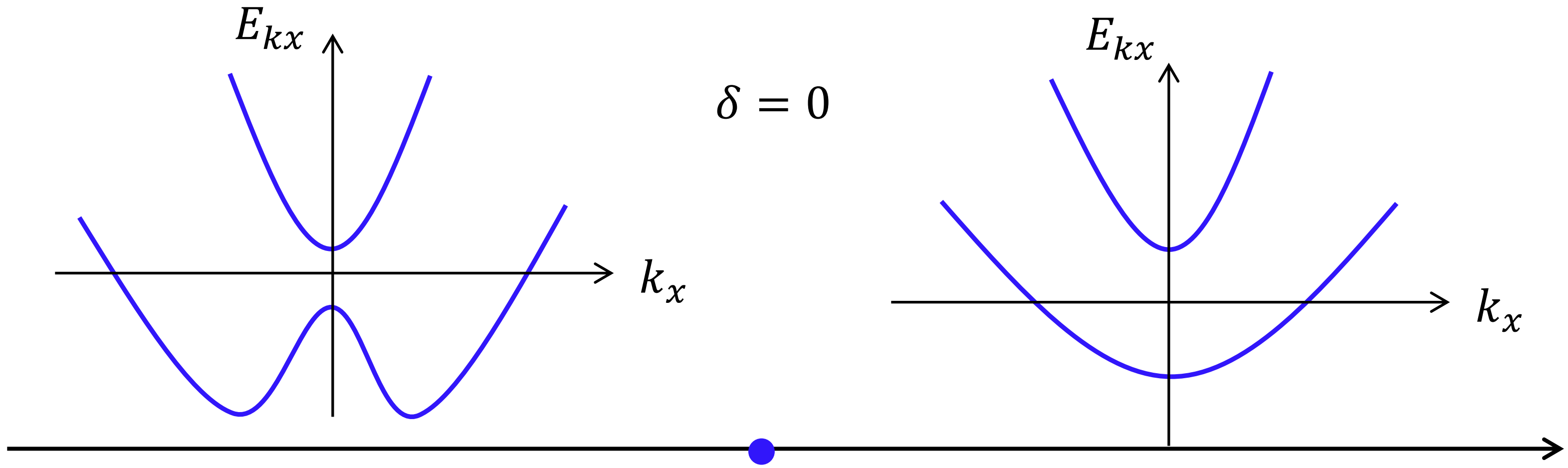
^{6}Li fermions: M. Zwierlein group, PRL 1**09**, 095301 (2012).



4. Effect of the 1D spin-orbit coupling

The spin-orbit coupled Hamiltonian: $$H_0 = \frac{(k_x + k_0\sigma_z)^2}{2m} + \frac{\delta}{2}\sigma_z + \frac{\Omega_R}{2}\sigma_x$$

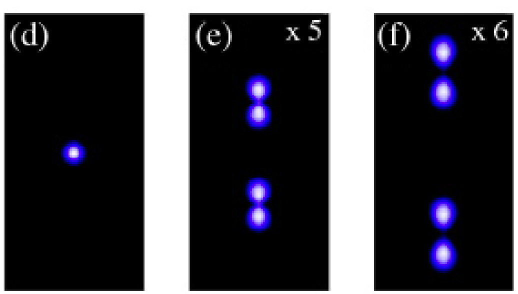
The single-particle spectra: single-well vs double-well dispersion relation (X. -J. Liu, M. F. Borunda, X. Liu, and J. Sinova, PRL 102, 046402 (2009)).

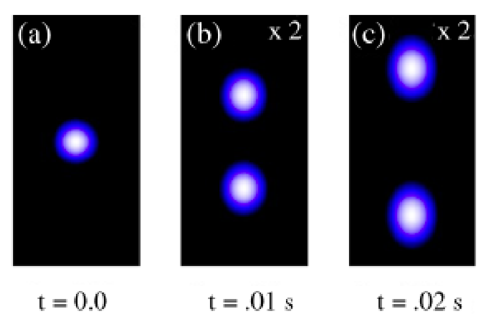


$\Omega_R < \Omega_c$ $\qquad \Omega_c = 2E_R = k_0^2/m$ $\qquad \Omega_R > \Omega_c$

4.1 Fermions with the Fermi energy $\mu_F = 0$. Time-of-flight expansion by switching off trap:

$\Omega_R < \Omega_c$





$\Omega_R > \Omega_c$

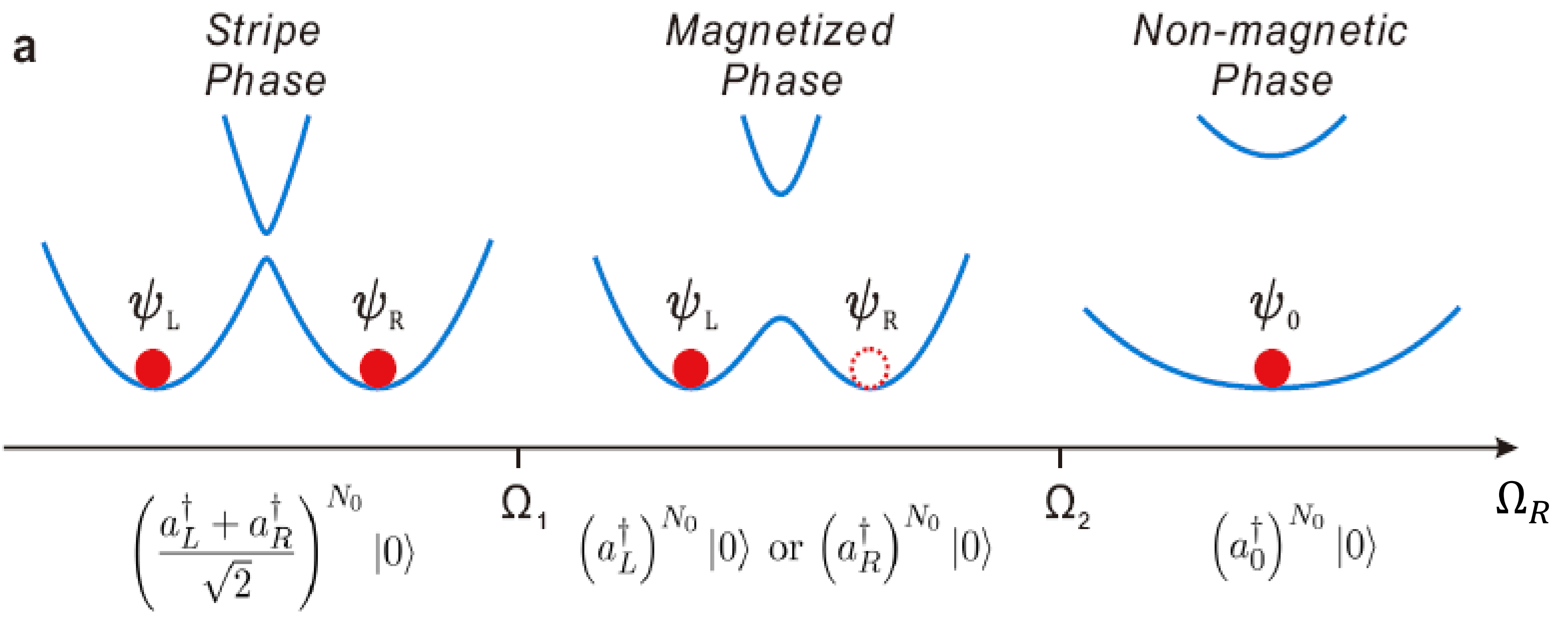
X. -J. Liu, etal, PRL 102, 046402 (2009)

4.2 Bosons: magnetized phase vs strip phase (supersolid)

Refs: Wang, Gao, Jian & Zhai, PRL 105, 160403 (2010); C. -J. Wu, Mondragon-Shem, & Zhou, Chin. Phys. Lett. 28, 097102 (2011); Ho & Zhang, PRL, 107, 150403 (2011); Y. Li, Pitaevskii, & Stringari, PRL 108, 225301 (2012).

The bosonic system is very different from the fermions with spin-orbit coupling. At low temperature the bosons are condensed in the state with total energy being minimized. Due to the nontrivial band structure induced by spin-orbit coupling, the ground states of the interacting bosons can be nontrivial.

The results are illustrated in the following figure:



The total Hamiltonian:

$$H = H_0 + G = \int d^3r \psi^\dagger h_0 \psi + \frac{g_s}{2}\left(n_\uparrow^2 + n_\downarrow^2\right) + g_a n_\uparrow n_\downarrow$$

$$h_0 = \frac{p_y^2+p_z^2}{2m} + \frac{(k_x+k_0\sigma_z)^2}{2m} + \frac{\Omega_R}{2}\sigma_x, \qquad g_a < g_s = (g_{\uparrow\uparrow} + g_{\downarrow\downarrow})/2$$

The ground state of the present bosonic system can be determined by variation method. The wave function of the BEC is taken as:

$$\psi = \sqrt{\frac{N}{V}}\left[C_1\begin{pmatrix}\cos\theta \\ -\sin\theta\end{pmatrix}e^{ikx} + C_2\begin{pmatrix}\sin\theta \\ -\cos\theta\end{pmatrix}e^{-ikx}\right]$$

Where $C_{1,2}$ and k are variation parameters, and θ is defined through

$$\cos^2\theta = \frac{1}{2}\left(1 + \frac{k}{\sqrt{k^2 + \frac{\Omega^2}{4}}}\right)$$

The atomic densities of the spin-up and spin-down components:

$$n_\uparrow = \frac{N}{V}\left(|C_1|^2\cos^2\theta + |C_2|^2\sin^2\theta + 2|C_1C_2|\cos\theta\sin\theta\cos(2kx + 2\phi)\right)$$

$$n_\downarrow = \frac{N}{V}\left(|C_1|^2\sin^2\theta + |C_2|^2\cos^2\theta + 2|C_1C_2|\cos\theta\sin\theta\cos(2kx + 2\phi)\right)$$

How to minimize the energy (Jeffrey C. F. Poon and XJL, PRA **93**, 063420 (2016)**)?**

1. The kinetic energy is minimized at the single-particle band bottom.
2. The interaction term favors an equal distribution of spin-up and spin-down atoms.
3. If atoms are distributed in both left and right minima. A density modulation in the position space is resulted due to interference. This costs extra energy.
4. Increasing the Zeeman term enhances the interference between two states at the single-particle band bottom.

$$\epsilon(k,\gamma) = \frac{k^2}{2} - \sqrt{k^2 + \frac{\Omega^2}{4}} + G_1\left(1 + 2\gamma\frac{\frac{\Omega^2}{4}}{k^2+\frac{\Omega^2}{4}}\right) + G_2\left(1-4\gamma\right)\frac{k^2}{k^2+\frac{\Omega^2}{4}}$$

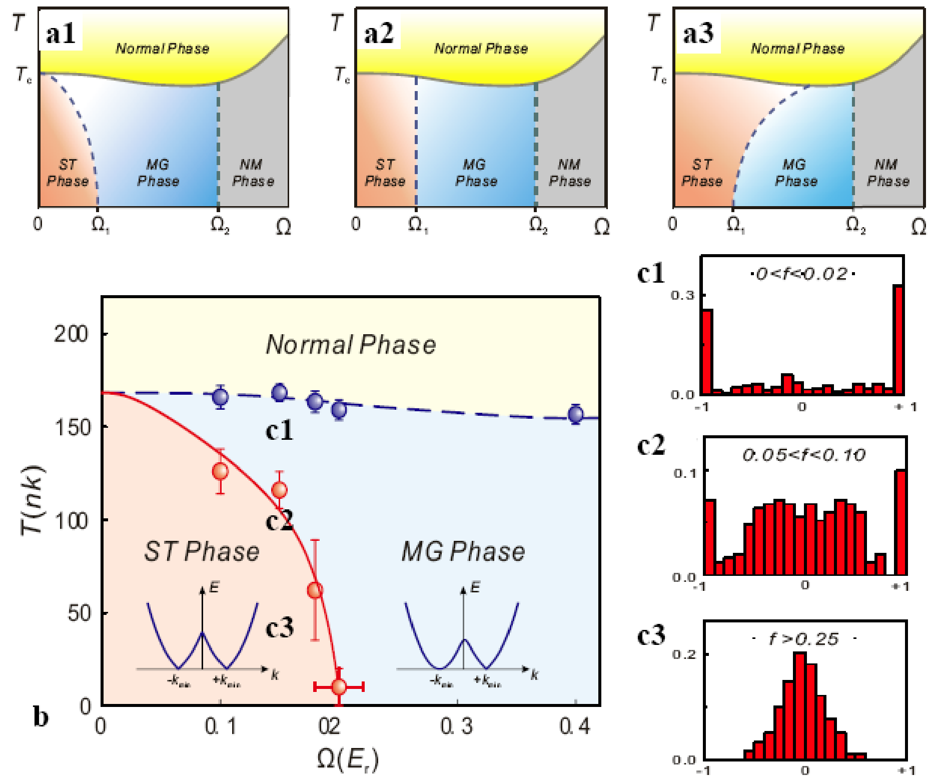
with $\gamma = |C_1|^2\,|C_2|^2$

$$\Theta = \cos^2\theta \sin^2\theta$$

$$G_1 = \frac{n(g_s+g_a)}{4},\; G_2 = \frac{n(g_s-g_a)}{4}$$

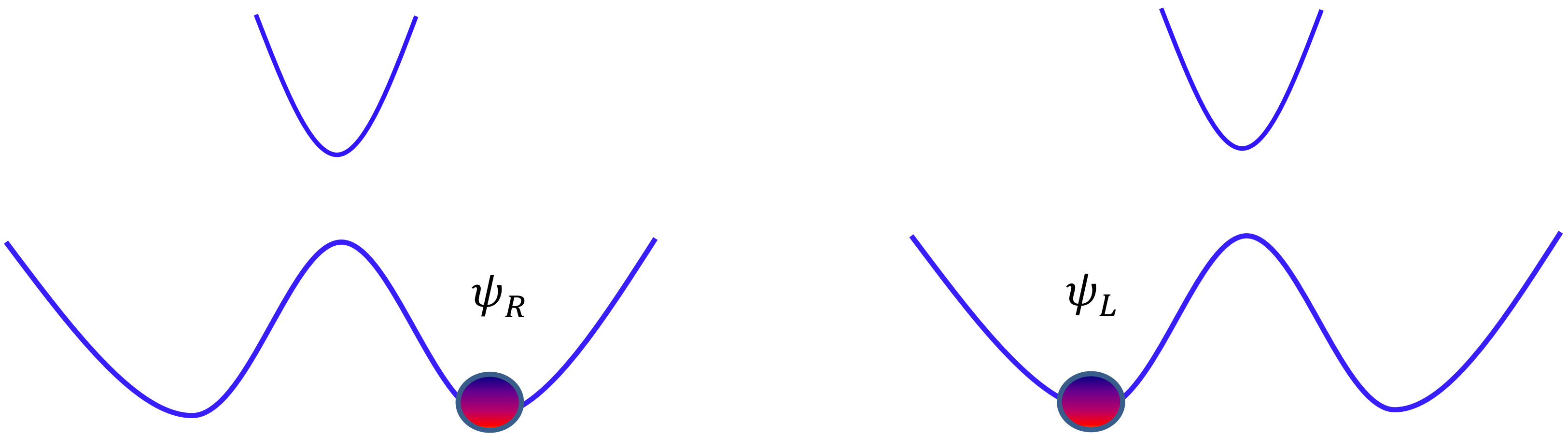
Experiment observation

Shuai Chen and Jian-Wei Pan group, 87Rb atoms, Nature Phys. 10, 31420 (2014)



A Question

The magnetized phase has a double degeneracy, consisting of a many-particle two-state system. Can nontrivial dynamics be induced between such degenerate ground states by external perturbation?



Magnetized phase A

Magnetized phase B

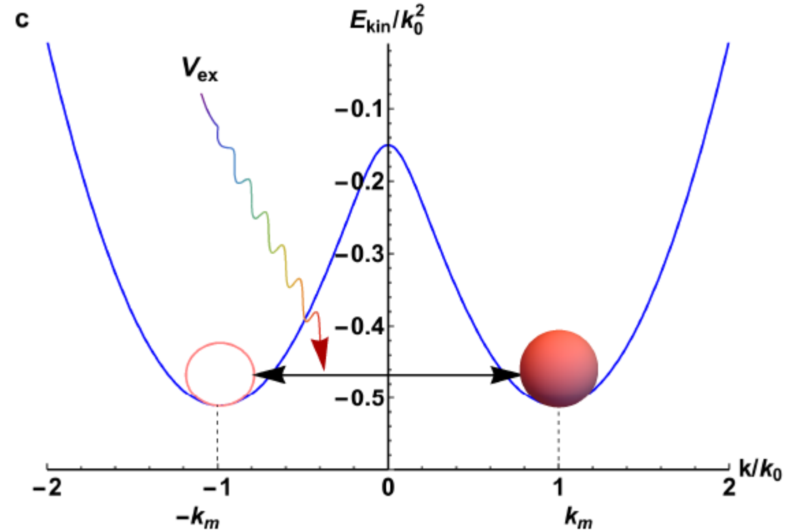
Induced quantum dynamics for magnetized phases

Start with a magnetized phase, and apply a perturbation to couple the two magnetized phases

$$H_{\rm BE} = H_0 + H_{\rm int},$$

$$H_0 = \sum_{ss'=\uparrow,\downarrow} \int d^3\mathbf{r}\,\psi_s^\dagger\Big(-\frac{\nabla_\mathbf{r}^2}{2} + ik_0\partial_x\sigma_z + \frac{\Omega}{2}\sigma_x\Big)_{ss'}\psi_{s'},$$

$$H_{\rm int} = \int d^3\mathbf{r}\frac{g_s}{2}\left(\psi_\downarrow^\dagger\psi_\downarrow^\dagger\psi_\downarrow\psi_\downarrow + \psi_\uparrow^\dagger\psi_\uparrow^\dagger\psi_\uparrow\psi_\uparrow\right) + \int d^3\mathbf{r}\,g_a\psi_\downarrow^\dagger\psi_\uparrow^\dagger\psi_\uparrow\psi_\downarrow.$$



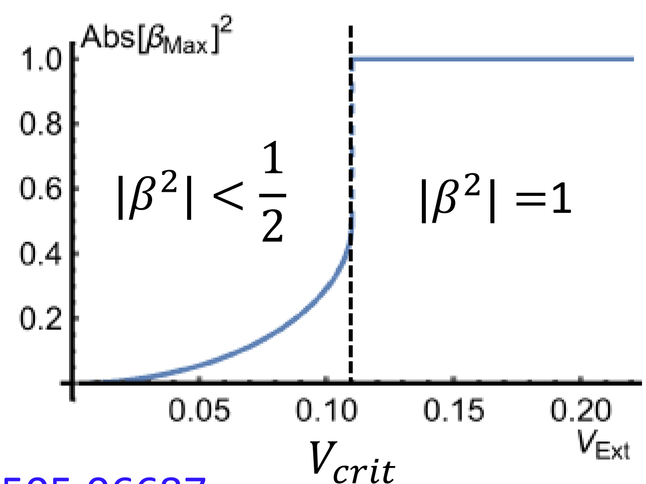
Time-dependent perturbation

$$V_{\rm ex}(\mathbf{r},t) = \begin{cases} 0, & t<0 \\ V_0\int d^3\mathbf{r}\cos^2 k_m x\left(\psi_\uparrow^\dagger\psi_\uparrow + \psi_\downarrow^\dagger\psi_\downarrow\right), & t>0 \end{cases}$$

Evolution of the BEC with N particles

$$\Psi(t) = [\alpha(t)\hat{\psi}_R^\dagger + \beta(t)\hat{\psi}_L^\dagger]^N|vac\rangle$$

$$\alpha(0) = 1, \beta(0) = 0.$$



Jeffrey C. F. Poon and XJL, PRA **93**, 063420 (2016); arXiv: 1505.06687.

4.3 BCS-BEC crossover

Consider a model with a 3D Fermi system with 2D isotropic spin-orbit coupling:

$$\hat{H}_0 = \mathbf{p}^2/(2m) + \lambda \mathbf{p}_\perp \cdot \boldsymbol{\sigma}_\perp / m, \qquad \mathbf{p}_\perp = (p_x, p_y)$$

In the second quantization picture:

$$\hat{\mathcal{H}}_0 = \sum_{\mathbf{p}} [\epsilon_{\mathbf{p}} (c^\dagger_{\mathbf{p}\uparrow} c_{\mathbf{p}\uparrow} + c^\dagger_{\mathbf{p}\downarrow} c_{\mathbf{p}\downarrow}) + \lambda p_\perp (e^{-i\varphi_{\mathbf{p}}} c^\dagger_{\mathbf{p}\uparrow} c_{\mathbf{p}\downarrow} + e^{i\varphi_{\mathbf{p}}} c^\dagger_{\mathbf{p}\downarrow} c_{\mathbf{p}\uparrow})],$$

with $\epsilon_{\mathbf{p}} = p^2/(2m), \quad \varphi_{\mathbf{p}} = \arg(p_x + i p_y).$

The contact interaction:

$$\hat{\mathcal{H}}_{\text{int}} = (g/V) \sum_{\mathbf{p}\mathbf{p}'\mathbf{q}} c^\dagger_{\mathbf{q}/2+\mathbf{p}\uparrow} c^\dagger_{\mathbf{q}/2-\mathbf{p}\downarrow} c_{\mathbf{q}/2-\mathbf{p}'\downarrow} c_{\mathbf{q}/2+\mathbf{p}'\uparrow},$$

The bare interaction

$$1/g = m/(4\pi a_s) - \sum_{\mathbf{k}} 1/(2\epsilon_{\mathbf{k}})$$

J. P. Vyasanakere and V. B. Shenoy, PRB 83, 094515 (2011); M. Gong, S. Tewari, and C. Zhang, this issue, PRL 107, 195303 (2011); H. Hu, L. Jiang, X. -J. Liu, and H. Pu, PRL 107, 195304 (2011); Z-Q. Yu and H. Zhai, PRL 107, 195305 (2011).

1) Two-body problem: bound state ("Rashbon") due to SO coupling (J. P. Vyasanakere and V. B. Shenoy, PRB 83, 094515 (2011)), with the energy solved by $(\hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{\text{int}})|\Psi\rangle_{\mathbf{q}} = E_{\mathbf{q}}|\Psi\rangle_{\mathbf{q}}$

$$\frac{2}{a_s} = 2\sqrt{(-E_0)m} - \lambda \ln\frac{\sqrt{(-E_0)m} + \lambda}{\sqrt{(-E_0)m} - \lambda}.$$

This give a bound state as long as the SOC is nonzero. In the strong scattering regime, one has:

$$E_0 = -2.88\lambda^2/(2m) < -\lambda^2/m.$$

2) Many-body problem: mean-field calculation for the order parameter

$$\Delta = (g/V)\textstyle\sum_{\mathbf{p}}\langle c_{-\mathbf{p}\downarrow}c_{\mathbf{p}\uparrow}\rangle,$$

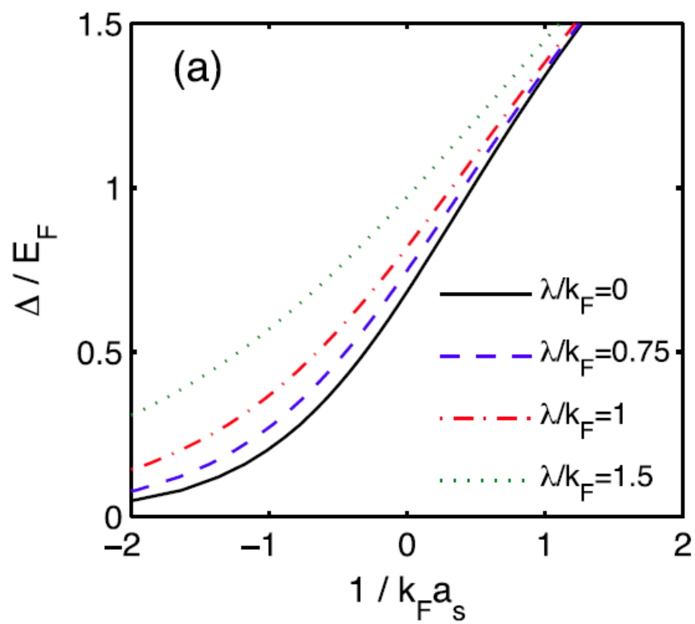
$$-\frac{m}{4\pi a_s} = \frac{1}{4V}\sum_{\mathbf{p}}\left[\frac{f_{\mathbf{p},+}}{\varepsilon_{\mathbf{p},+}} + \frac{f_{\mathbf{p},-}}{\varepsilon_{\mathbf{p},-}} - \frac{2}{\epsilon_{\mathbf{p}}}\right],$$

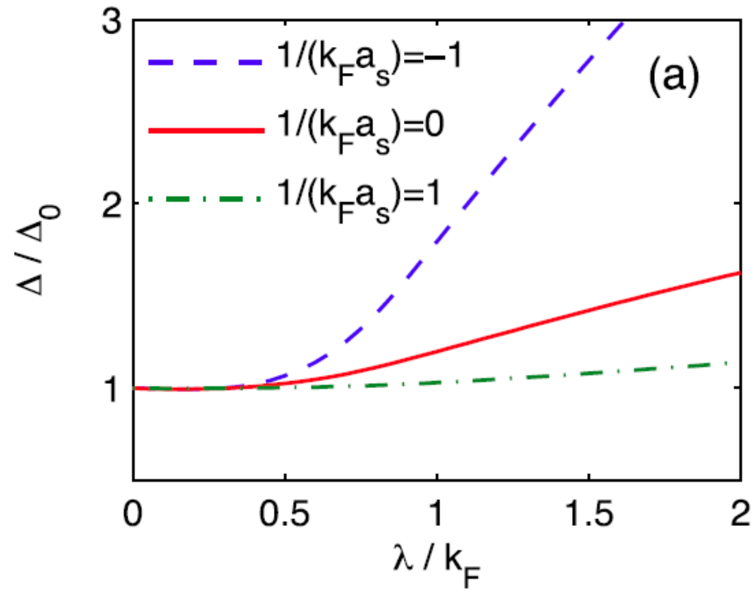
$$n = \frac{1}{V}\sum_{\mathbf{p}}\left[1 - \frac{\xi_{\mathbf{p},+}f_{\mathbf{p},+}}{2\varepsilon_{\mathbf{p},+}} - \frac{\xi_{\mathbf{p},-}f_{\mathbf{p},-}}{2\varepsilon_{\mathbf{p},-}}\right],$$

$$\varepsilon_{\mathbf{p},\pm} = \sqrt{(\xi_{\mathbf{p},\pm} - \mu)^2 + \Delta^2}$$

Enhancement of SF order by SO coupling

$$|\mathrm{BCS}\rangle \propto \exp\left[\sum_{\mathbf{k}}{}' \frac{v_{\mathbf{k},+}}{u_{\mathbf{k},+}} h^\dagger_{\mathbf{k},+} h^\dagger_{-\mathbf{k},+} + \frac{v_{\mathbf{k},-}}{u_{\mathbf{k},-}} h^\dagger_{\mathbf{k},-} h^\dagger_{-\mathbf{k},-}\right]|0\rangle,$$





The enhancement of the SF order is a consequence of the enhancement of DOS by the SO coupling.

M. Gong, S. Tewari, and C. Zhang, this issue, PRL 107, 195303 (2011); H. Hu, L. Jiang, X. -J. Liu, and H. Pu, PRL 107, 195304 (2011); Z-Q. Yu and H. Zhai, PRL 107, 195305 (2011).

Other related issues of spin-orbit coupling

- Synthetic spin-orbit coupling in optical lattices (next lecture).
- Spin-orbit coupling through shaken lattice; SOC for high orbital bands in optical lattices.
- Exotic magnetic phases with spin-orbit couplings.
- Dynamical non-Abelian gauge potentials/fields

Review articles: X.-J. Liu et al., Front. Phys. China, **3**, 113 (2008); J. Dalibard, F. Gerbier, G. Juzeliunas, P. Ohberg, Rev. Mod. Phys. 83, 1523 (2011); N Goldman, G Juzeliunas , P Ohberg and I B Spielman, Rep. Prog. Phys. 77 126401 (2014); X. Zhou, Y. Li, Z. Cai, and C. Wu, J. Phys. B: At. Mol. Opt. Phys. 46 , 134001 (2014); H. Zhai, Rep. Prog. Phys. 78, 026001 (2015).

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Thank you for your attention!