Determination of compositeness with generalized weak-binding relation

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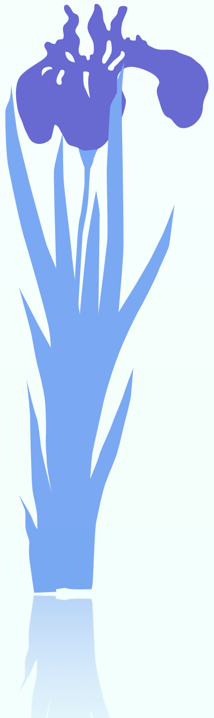
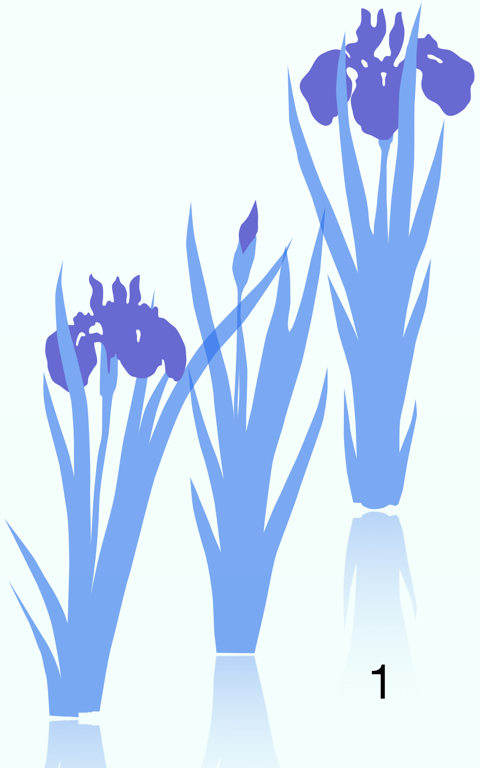
@Strangeness and charm
in hadrons and dense matter





Contents

- Introduction ~weak-binding relation for bound state~
- Extension to the quasibound state
- Extended relation with the CDD pole contribution
- Applications to exotic hadrons ~ $\Lambda(1405)$ ~

Introduction ~exotic hadrons~

- Exotic hadrons

Hadrons which do not coincide with the predictions of the quark model.

More complicated internal structure can be expected.

- tetra quark, penta quark
- hadron molecule ···

It is important to reveal the internal structure of exotics.

e.g. ; Λ(1405)

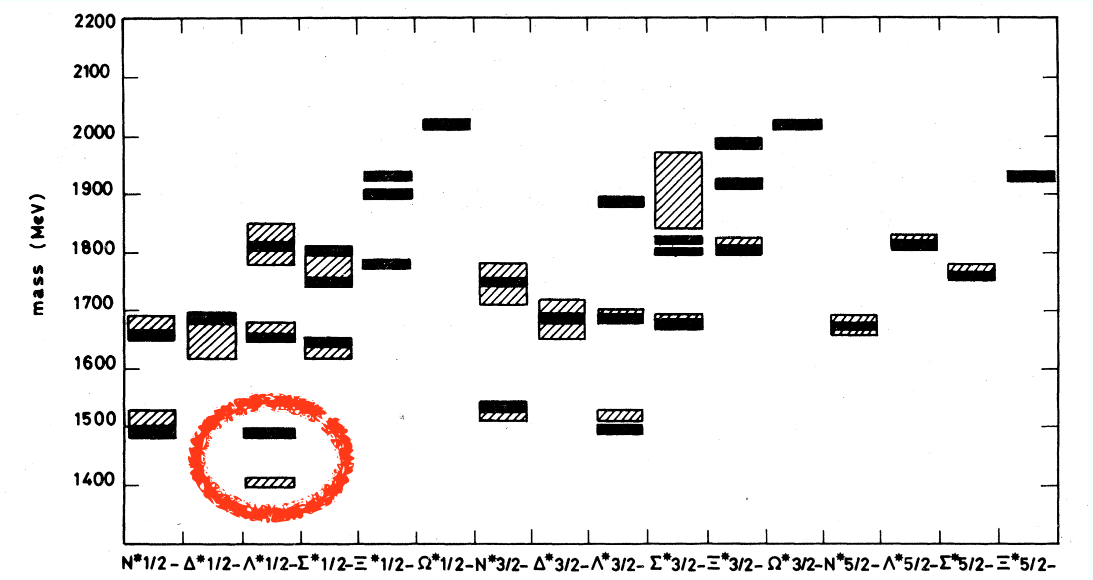
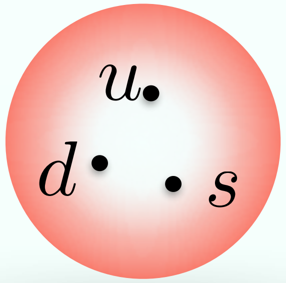


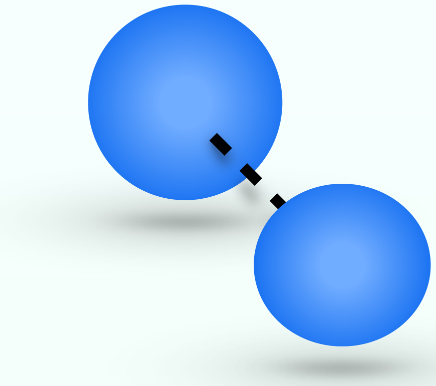
FIG. 1. Comparison of the predicted and observed spectrum of negative-parity baryons. The shaded regions corre-

N. Isgur, and G. Karl, Phys. Rev. D18, 4187 (1978)

exited Λ state(uds)



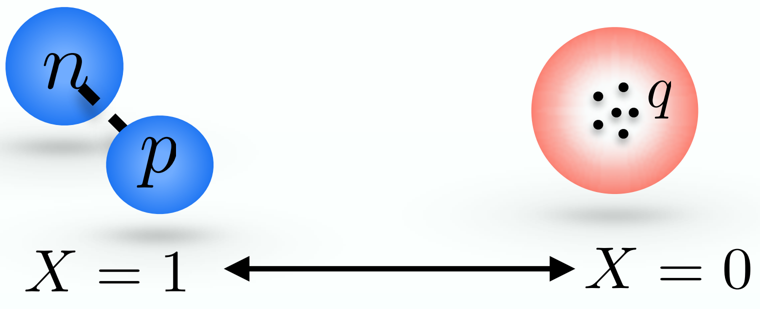
$\bar{K}N$ bound state



Compositeness

Weak-binding relation

S. Weinberg, Phys. Rev. 137, B672 (1965)



We consider the stable and s-wave bound state (deuteron) $|d\rangle$ in the n-p scattering.

$$Z \equiv |\langle B_0|d\rangle|^2$$

$|B_0\rangle$: bare state

$$1-Z=\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}|\langle\boldsymbol{p}|d\rangle|^2 \quad (=X)$$

$|\boldsymbol{p}\rangle$: (n-p) scattering state

eigenstates of $\mathcal{H}_{\text{free}}$

Without assuming the specific nuclear force, the following weak-binding relation is derived for the scattering length a_0, binding energy B, and compositeness X.

$$a_0 = R\left[\frac{2X}{1+X}+\mathcal{O}\left(\frac{1}{Rm_\pi}\right)\right] \qquad R=\frac{1}{\sqrt{2\mu B}}$$

μ ; reduced mass

When the binding energy is so small that $1/(Rm_\pi)$ can be neglected, the compositeness X can be determined only from experimental observables (a_0, B).

We can study the internal structure model-independently.

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Extension to the quasibound state

Effective field theory

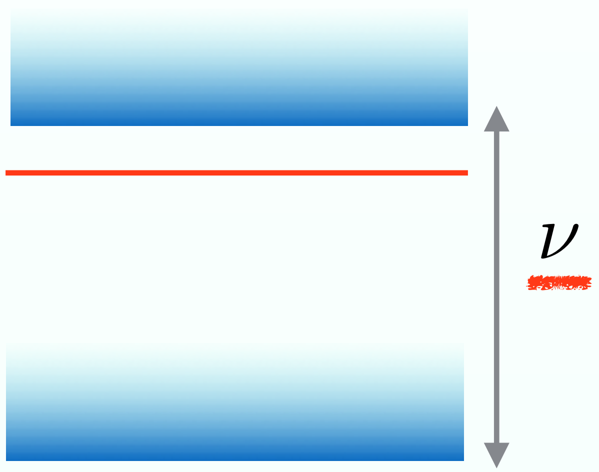
To discuss the near-threshold physics,
we use following non-relativistic EFT.

Eigenstate of H_{free}

$|\boldsymbol{p}\rangle$ scattering channel

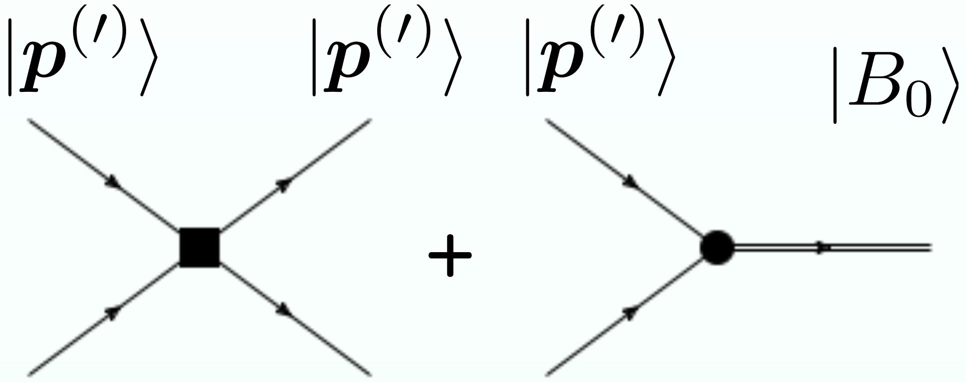
$|B_0\rangle$ discrete channel

$|\boldsymbol{p}'\rangle$ decay channel



Interaction H_{int}

$\mathcal{H}_{\text{int}} =$



point interaction

The interaction has a typical length scale R_{typ}.

Eigenstate of full H $\quad H = H_{\text{free}} + H_{\text{int}}$

Unstable quasibound state $|QB\rangle$ exists
near $|\boldsymbol{p}\rangle$ threshold.

$$H|QB\rangle = E_{QB}|QB\rangle$$

$$E_{QB} = -B - i\Gamma/2$$; complex

We consider the compositeness of $|\boldsymbol{p}\rangle$channel ; X.

Extension to the quasibound state

- Scattering amplitude for channel |p>

The T matrix T(E) in this theory is obtained by solving Lippmann-Schwinger Eq. for channel |p> :

$$T = v^{\rm eff} + v^{\rm eff} G T = [1/v^{\rm eff} - G(E)]^{-1}$$

$v^{\rm eff}(E)$: effective interaction for channel |p>
including the contribution of |p’> and |B0>

$G(E)$: loop function regularized with sharp momentum cutoff Λ

$$\mathcal{F}(E) = -\frac{\mu}{2\pi} T(E) = -\frac{\mu}{2\pi} \frac{1}{1/v^{\rm eff} - G(E)}$$

Extension to the quasibound state

Definition of compositeness

Bound state

Bound state $|B\rangle$ is normalized with $\langle B|B\rangle = 1$

$$X \equiv \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \langle B|\boldsymbol{p}\rangle\langle\boldsymbol{p}|B\rangle$$
$$= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} |\langle\boldsymbol{p}|B\rangle|^2$$

$$Z \equiv |\langle B_0|B\rangle|^2$$

- $X + Z = 1$
- $0 < X, Z < 1$

The probabilistic interpretation is guaranteed for X and Z.

Quasibound state

To normalize unstable state, we introduce Gamow state $|\overline{QB}\rangle$.

Normalization condition becomes

$$\langle\overline{QB}|QB\rangle = \langle QB^*|QB\rangle = 1.$$

T. Berggren, Nucl. Phys. A 109 (1968)

The expectation value of any operator becomes complex number.

- $X + Z = 1$
- $\cancel{0 < X, Z < 1}$ $X, Z \in C$

$$X \equiv \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \langle\overline{QB}|\boldsymbol{p}\rangle\langle\boldsymbol{p}|QB\rangle$$

The probabilistic interpretation is not guaranteed!

Extension to the quasibound state

- Definition of compositeness

$$X \equiv \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \langle \overline{QB}|\boldsymbol{p}\rangle\langle \boldsymbol{p}|QB\rangle$$

- Schrödinger Eq. for eigenstate

$$H|QB >= E_h|QB >$$

Compositeness X can be expressed with the terms of scattering:

$$X = \frac{G'(E_{QB})}{G'(E_{QB}) - [1/v^{\mathrm{eff}}(E_{QB})]'}$$

Assuming $|E_{QB}|$ is small,

we expand $1/a_0$ with respect to E_{QB}.

$$\frac{1}{a_0} = -\frac{2\pi}{\mu}\left[1/v^{\mathrm{eff}}(E_{QB}) - G(E_{QB}) - \underline{\left([1/v^{\mathrm{eff}}(E_{QB})]' - G'(E_{QB})\right)} E_{QB} \underline{+ \cdots}\right]$$

higher order terms

Extension to the quasibound state

- Extended Weak binding relation

Y. Kamiya and T. Hyodo, PTEP 023D02 (2017).

$$a_0 = R\left[\underbrace{\frac{2X}{1+X} + \mathcal{O}\left(\left|\frac{R_{\rm typ}}{R}\right|\right)}_{\text{original}} + \underbrace{\mathcal{O}\left(\left|\frac{l}{R}\right|^3\right)}_{\text{new}}\right]$$

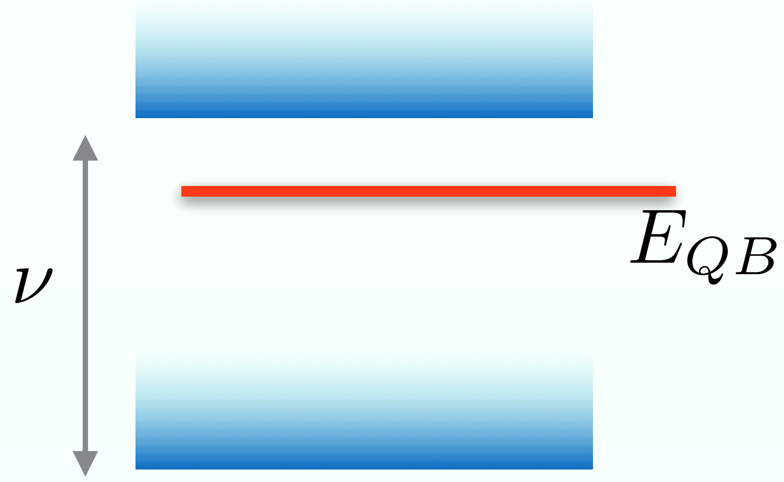
$$R = \frac{1}{\sqrt{-2\mu E_{QB}}}$$

$$l = \frac{1}{\sqrt{2\mu\nu}}$$

If $|R_{\rm typ}/R|$ and $|l/R|^3$ are

sufficiently smaller than 1,

we can determine X from a_0 and E_{QB}.



* Note
  - a_0 , E_{QB}, X are all complex numbers,
    then above relation is established among them.
  - The same argument is valid for the case with $\mathrm{Re}\ E_h > 0$.

Interpretation of X

Our proposal

c.f. T. Berggren, Phys. Lett. B 33 (1979) 8

For probabilistic interpretation we define the following real quantities.

$\tilde{X}$; probability of finding the scattering state in physical state

$\tilde{Z}$; probability of finding the other states

U ; degree of uncertainty of the interpretation

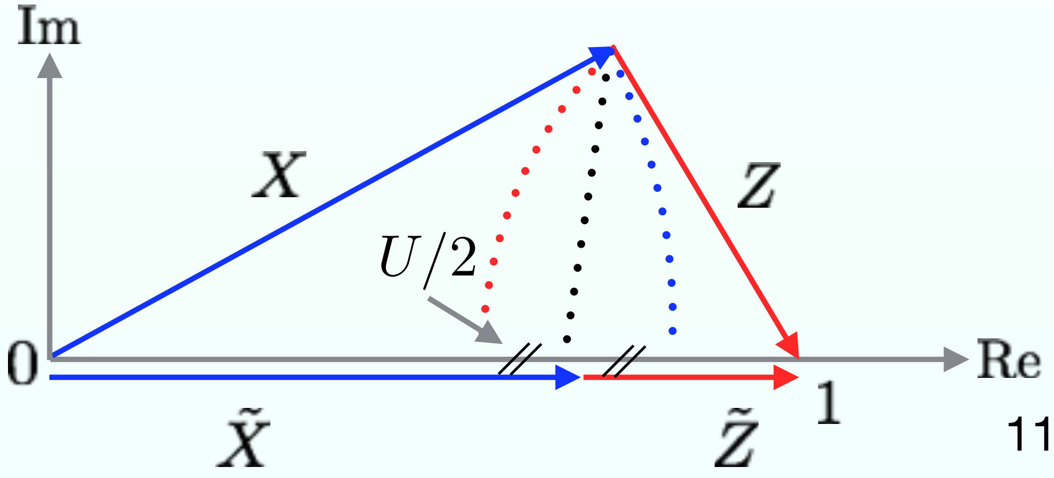
$$\tilde{X} \equiv \frac{1-|Z|+|X|}{2}$$

$$\tilde{Z} \equiv \frac{1-|X|+|Z|}{2}$$

$$U \equiv |Z|+|X|-1$$

conditions :

- $\tilde{X}+\tilde{Z}=1$
- $0 \leq \tilde{X}, \tilde{Z} \leq 1$
- When there is no cancellation in X + Z,
  $\tilde{X}=X, \tilde{Z}=Z, U=0$.
- U becomes large
  when the cancellation becomes large.



Solid interpretation is possible only when U is small.

Error estimation of compositeness

For the actual application to Hadrons,
the Higher-order terms are finite and give the correction.

$$a_0 = R\left[\frac{2X}{1+X} + \underline{\mathcal{O}\left(\left|\frac{R_{\rm typ}}{R}\right|\right) + \mathcal{O}\left(\left|\frac{l}{R}\right|^3\right)}\right]$$

The magnitude of the higher-order terms cannot be determined from the observables.

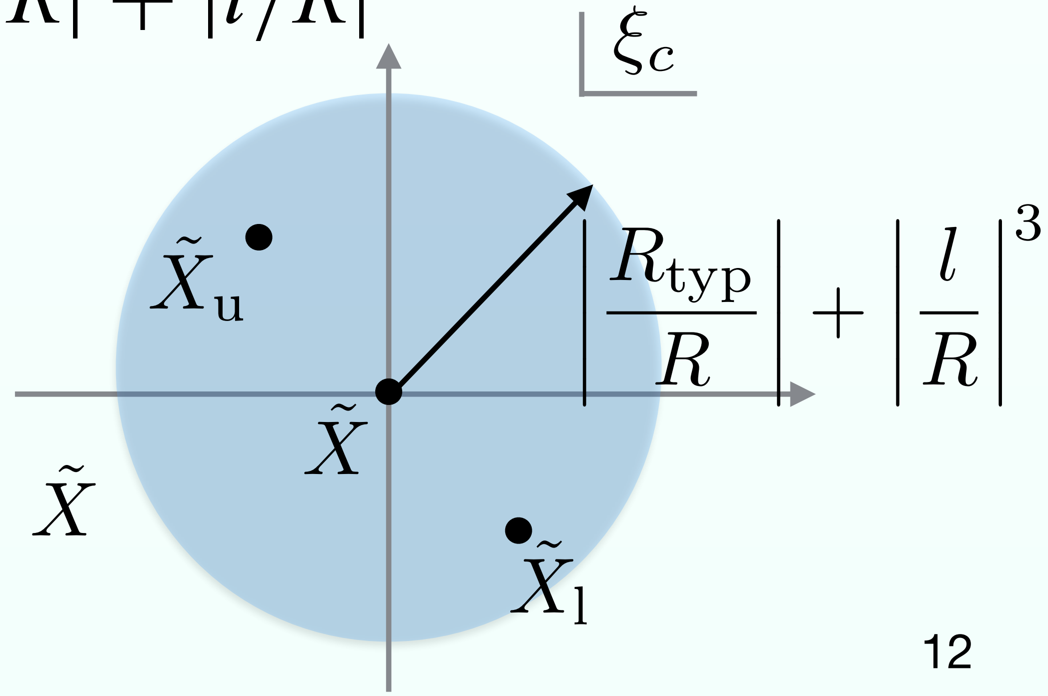
—> We give the uncertainty band $\tilde{X}_{\rm l} < \tilde{X} < \tilde{X}_{\rm u}$ as follows.

(1) We vary ξ_c in the region : $|\xi_c| \leq |R_{\rm typ}/R| + |l/R|^3$

(2) calculate $\tilde{X}$ at each ξ_c with

$$X = \frac{a_0/R + \xi_c}{2 - a_0/R - \xi_c} \quad \tilde{X} = \frac{1+|X|-|1-X|}{2}$$

(3) assign the maximum (minimum) value of $\tilde{X}$ as $\tilde{X}_{\rm u}(\tilde{X}_{\rm l})$.



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CDD pole and weak-binding relation

CDD (Castillejo Dalitz Dyson) pole(E_c) and internal structure

CDD pole : $f(E_c) = 0$

L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. 101, 453 (1956).

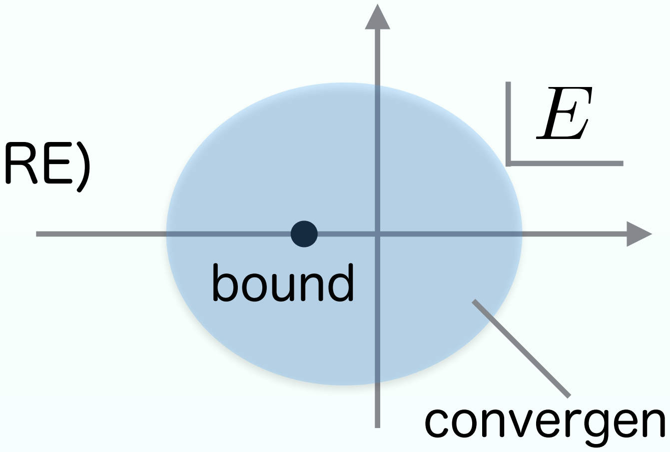
G. F. Chew and S. C. Frautschi, Phys. Rev. 124, 264 (1961).

- represents the contribution from outside the model space

V. Baru et al, Eur. Phys. J. A44, 93 (2010), 1001.0369.
T. Hyodo, Phys. Rev. Lett. 111 (2013) 132002.
Z.-H. Guo and J. A. Oller, Phys. Rev. D93, 054014 (2016), 1601.00862.

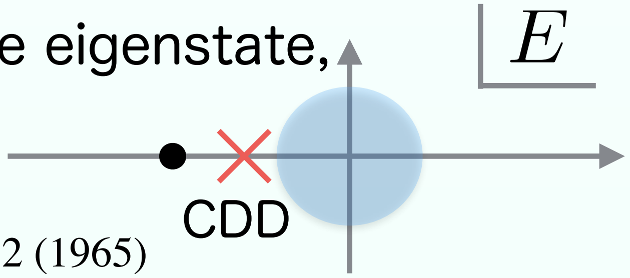
Condition of the weak-binding relation

In the derivation of the relation
we assume that the effective range expansion (ERE)
works well at the pole of the eigenstate.



$$f(E) = \left[-\frac{1}{a_0} + \frac{r_e}{2}p^2 - ip\right]^{-1}$$ (s-wave)

When the CDD pole lies near the threshold and ERE fails to describe the eigenstate,
the weak-binding relation is not applicable.



S. Weinberg, Phys. Rev. 137, B672 (1965)

To include the CDD pole contribution to the estimation of X,
the extension of the weak-binding relation is needed.

Derivation without convergence of ERE

For simplicity, we consider the stable bound state case.

$$X \equiv \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} |\langle \boldsymbol{p}|B\rangle|^2$$

$$H|B\rangle = E_B|B\rangle (E_B < 0)$$

$|B\rangle$: bound state

Another derivation of relation

The expression of compositeness with the loop fcn.

and the coupling constant is given as

$$X = -g^2 G'(E_B)$$

$G(E)$: loop function

g^2 : coupling constant between $|\boldsymbol{p}\rangle$ and $|B\rangle$.

(or residue of bound state pole)

T. Sekihara, T. Hyodo, and D. Jido, PTEP 2015, 063D04 (2015)

- Two factors are expressed with observables as follows
  - G(E)

$$G'(E_B) = \frac{\mu}{4\pi E_B R}\left\{1 + \mathcal{O}\left(\frac{R_{\mathrm{typ}}}{R}\right)\right\}$$

$R \equiv 1/\sqrt{-2\mu E_B}$

R_{typ} : typical length scale of int. $(\sim 1/\Lambda)$

  - coupling constant g^2

$$g^2 = -\lim_{E \to E_B} \frac{2\pi}{\mu}(E - E_B) f(E)$$

<— if the approximation of f(E) with physical observables is given, g^2 can be expressed

Derivation without convergence of ERE

- Compositeness

T. Sekihara, T. Hyodo, and D. Jido, PTEP 2015, 063D04 (2015), 1411.2308.

$$X \equiv \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} |\langle \boldsymbol{p}|B\rangle|^2$$

$$H|B\rangle = E_B|B\rangle (E_B < 0)$$

$|B\rangle$: bound state

$$\boxed{X = \underline{-g^2} G'(E_B)}$$

If we approximate g^2 with ERE

$$g^2 = -\lim_{E \to E_B} \frac{2\pi}{\mu}(E - E_B) f(E)$$

$$f(E) = [\underline{p\cot\delta - ip}]^{-1}$$

$$-\frac{1}{a_0} + \frac{r_e}{2}p^2 + \mathcal{O}(R_{\text{eff}}^3 p^4)$$

R_{eff}: range scale characterizing ERE

$$g^2 = \frac{2\pi}{\mu^2} \frac{1}{R - r_e + R\mathcal{O}((R_{\text{eff}}/R)^3)}$$

$$\boxed{X = \frac{1}{1 - \frac{r_e}{R} + \mathcal{O}\left(\left(\frac{R_{\text{eff}}}{R}\right)^3\right)} \left[1 + \mathcal{O}\left(\frac{R_{\text{typ}}}{R}\right)\right]}$$

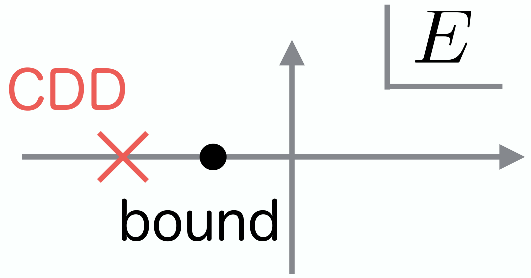
equivalent to the original Weinberg's relation.

In this approximation, the CDD pole contribution drops out from the weak-binding relation.

To include the CDD pole contribution,
a better approximation for g^2 is needed .

Extended relation with the CDD pole contribution

- To take account of the contribution of CDD pole



$$\boxed{X = -\underline{g}^2 G'(E_B)}$$

$$f(E) = [\underline{p\cot\delta} - ip]^{-1}$$

$$\frac{b_0 + b_1 p^2}{1 + c_1 p^2} + \mathcal{O}(R_{\text{Padè}}^5 p^6)$$ Pade approximant

Y. Kamiya and T. Hyodo, PTEP 023D02 (2017).

- CDD pole position : $p_{\mathrm{CDD}} = i/\sqrt{c_1}$

  (In the limit of $p_{\mathrm{CDD}} \to \infty$, this reduces to ERE.)

- Relation to the threshold parameters : $a_0 = -\dfrac{1}{b_0}$ $r_{\mathrm{e}} = 2(b_1 - b_0 c_1)$

$$\boxed{X = \left[1 - \frac{4R(a_0 - R)^2}{a_0^2 r_e} + \mathcal{O}\left(\left(\frac{R_{\text{Padè}}}{R}\right)^5\right)\right]^{-1} \left(1 + \mathcal{O}\left(\frac{R_{\text{typ}}}{R}\right)\right)}$$

Even when a CDD pole lies near the threshold,
we can estimate the compositeness using experimental observables.

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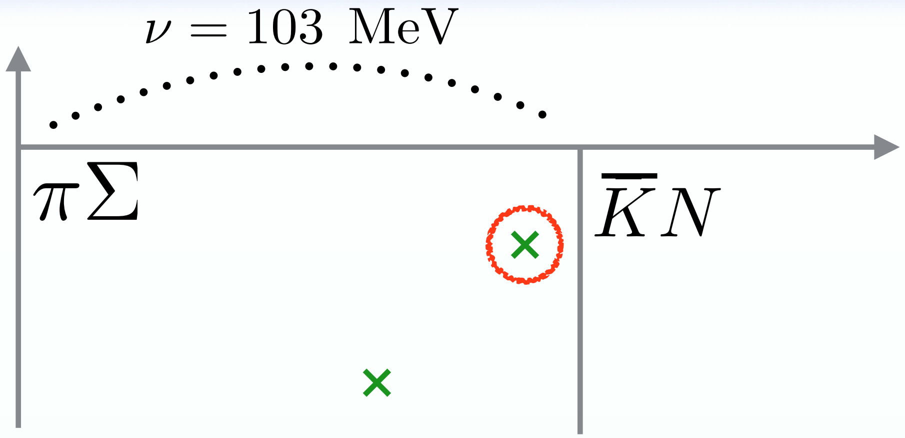
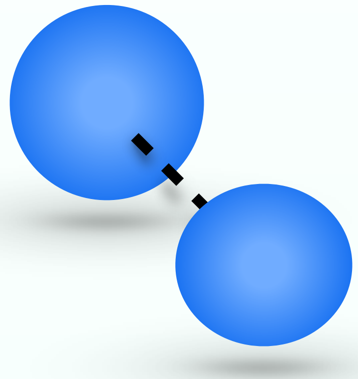
Applications to hadrons

- $\Lambda(1405)$ ($I=0$ $\bar{K}N$ scattering)

$J^P = \frac{1}{2}^-$



$\bar{K}N$ molecule?



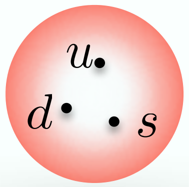
$\tilde{X} = 1$

or

other components?

$\tilde{X} = 0$

e.g.



- Λ excited state (uds)
- penta-quark state

…

R

R_{typ} is estimated from rho meson exchange int. $(R_{\mathrm{typ}} \sim 0.25\ \mathrm{fm})$ → $\left|\frac{R_{\mathrm{typ}}}{R}\right| \lesssim 0.17$

$R = \frac{1}{\sqrt{-2\mu E_{QB}}}$

$l = \frac{1}{\sqrt{2\mu\nu}}$

l is estimated from difference of the threshold energy → $\left|\frac{l}{R}\right|^3 \lesssim 0.14$

$$a_0 = R\left[\frac{2X}{1+X} + \mathcal{O}\left(\left|\frac{R_{\mathrm{typ}}}{R}\right|\right) + \mathcal{O}\left(\left|\frac{l}{R}\right|^3\right)\right] \Rightarrow X = \frac{a_0}{2R - a_0} \Rightarrow \tilde{X}, U$$

Applications to hadrons

- $\Lambda(1405)$ in $I=0$ $\bar{K}N$ scattering

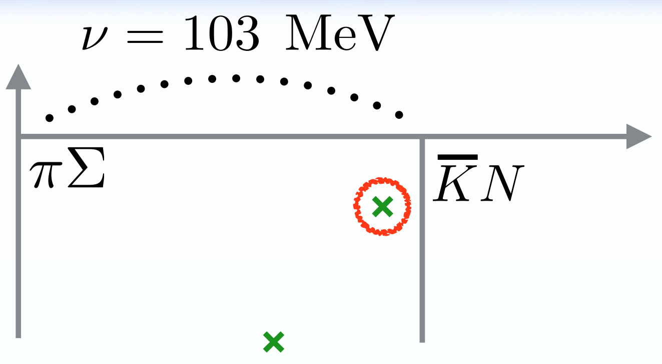
We use E_{QB} and a_0 in the following papers.

Set 1 : Y. Ikeda, T. Hyodo and W. Weise, Nucl. Phys. A 881 98 (2012)

Set 2 : M. Mai and U. G. Meissner, Nucl. Phys. A 900, 51 (2013)

Set 3 : Z. H. Guo and J. A. Oller, Phys. Rev. C 87, 035202 (2013)

Set 4 and 5 : M. Mai and U.-G. Meißner, Eur. Phys. J. A 51, 30 (2015).



Ref.	E_{QB} (MeV)	a_0 (fm)	X	$\tilde{X}$	$U/2$
Set 1	-10-i26	1.39-i0.85	1.3+i0.1	1.0	0.3
Set 2	-4-i8	1.81-i0.92	0.6+i0.1	0.6	0.0
Set 3	-13-i20	1.30-i0.85	0.9-i0.2	0.9	0.1
Set 4	2-i10	1.21-i1.47	0.6+i0.0	0.6	0.0
Set 5	- 3-i12	1.52-i1.85	1.0+i0.5	0.8	0.3

- U is small enough. —> $\tilde{X}$ can be considered as the probability.
- $\tilde{X}$ is close to 1.

$\Lambda(1405)$: $\bar{K}N$ composite dominance

Applications to hadrons

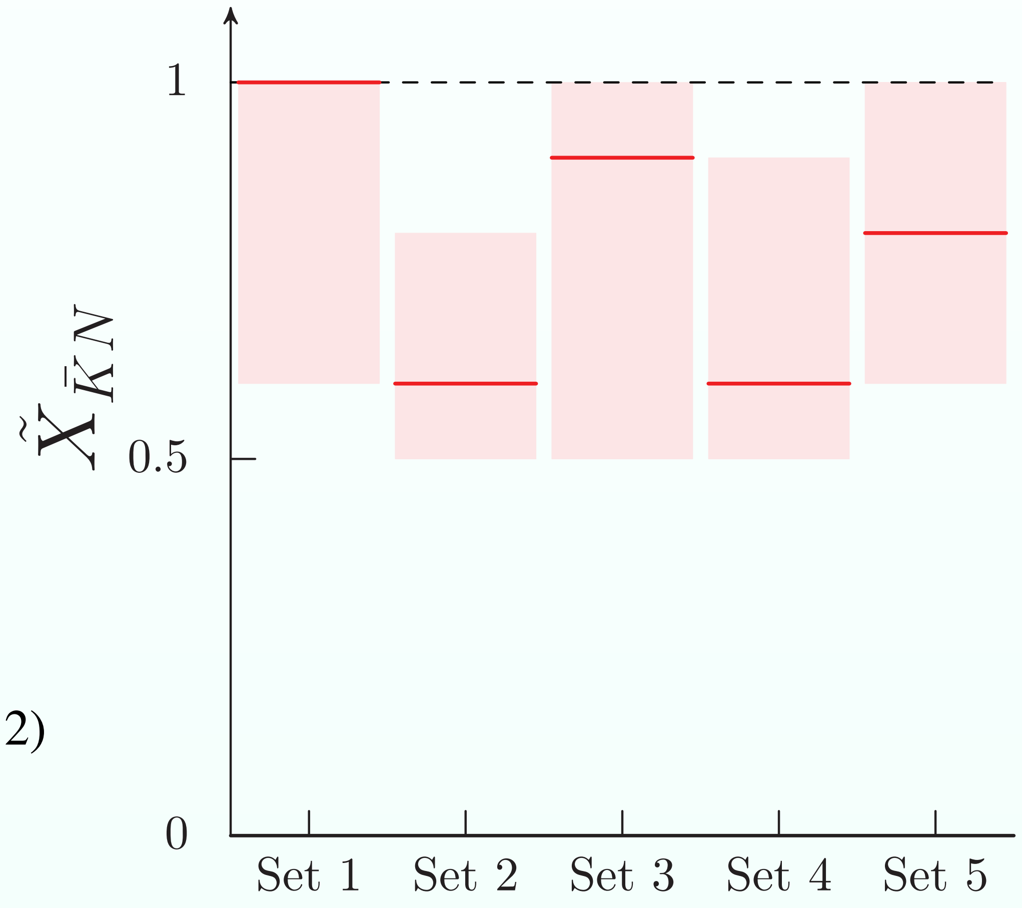
- $\Lambda(1405)$ in $I=0$ $\bar{K}N$ scattering

Uncertainty from the higher order terms is estimated.

$$X=\frac{a_0/R+\xi_c}{2-a_0/R-\xi_c} \qquad \tilde{X}=\frac{1+|X|-|1-X|}{2}$$

$$|\xi_c|\leq|R_{\rm typ}/R|+|l/R|^3$$

	$\lvert R_{\rm typ}/R\rvert$	$\lvert l/R\rvert^3$	$\tilde{X}_{\bar{K}N}$
Set 1	0.17	0.14	$1.0^{+0.0}_{-0.4}$
Set 2	0.10	0.03	$0.6^{+0.2}_{-0.1}$
Set 3	0.16	0.11	$0.9^{+0.1}_{-0.4}$
Set 4	0.10	0.03	$0.6^{+0.3}_{-0.1}$
Set 5	0.12	0.04	$0.8^{+0.2}_{-0.2}$



Set 1 : Y. Ikeda, T. Hyodo and W. Weise, Nucl. Phys. A 881 98 (2012)

Set 2 : M. Mai and U. G. Meissner, Nucl. Phys. A 900, 51 (2013)

Set 3 : Z. H. Guo and J. A. Oller, Phys. Rev. C 87, 035202 (2013)

Set 4 and 5 : M. Mai and U.-G. Meißner, Eur. Phys. J. A 51, 30 (2015).

Conclusion of the composite dominance still holds.

CDD pole contribution

- The CDD pole contribution to $\Lambda(1405)$

We calculate the compositeness using extended relations with a_0, r_e, E_h.

Y. Ikeda, T. Hyodo and W. Weise, Nucl. Phys. A 881 98 (2012)

original relation

$$X_{R,a_0} = \frac{a_0}{2R - a_0}$$

extended relation with CDD

$$X_{\text{Padè}} = \left[1 - \frac{4R(a_0 - R)^2}{a_0^2 r_e}\right]^{-1}$$

	X_{R,a_0}	$X_{\text{Padé}}$
estimated value of X	1.2+i0.2	1.4+i0.2
$\tilde{X}$	1.0	1.0

This small deviation means that the ERE converges well

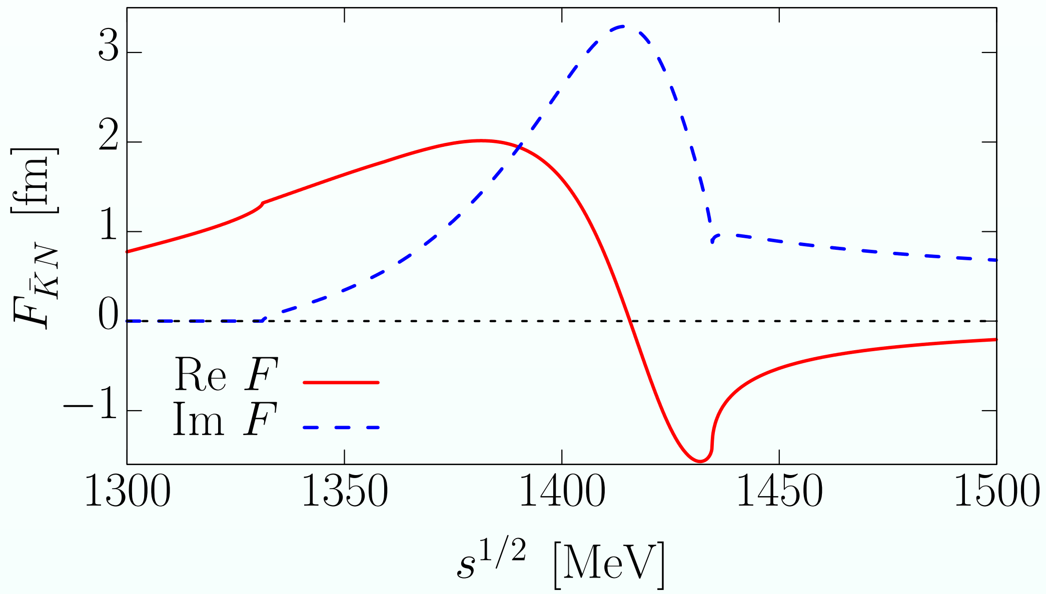
and the CDD pole contribution in the $\bar{K}N$ channel can be neglected.

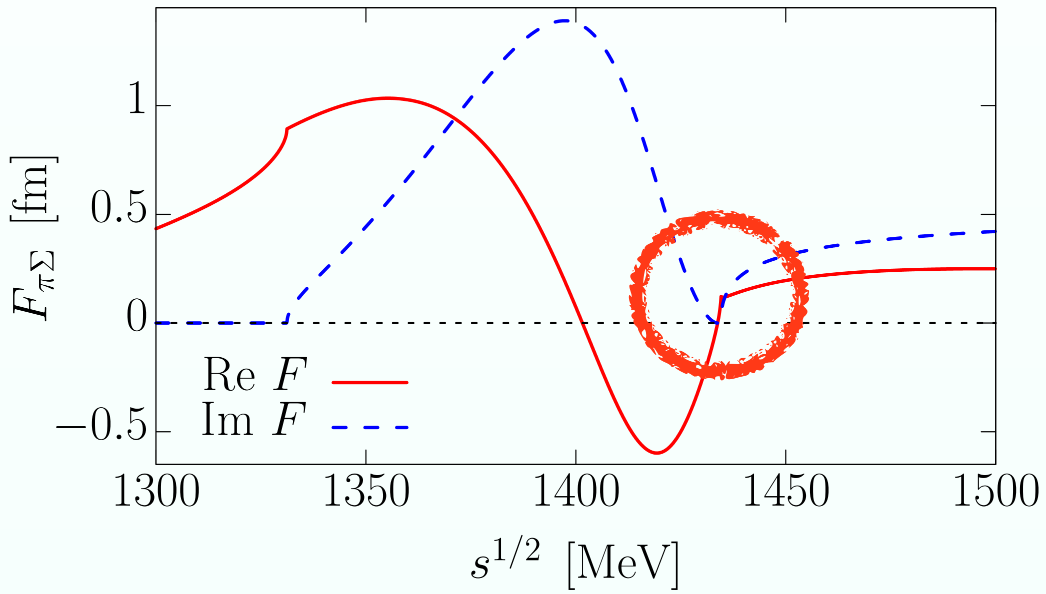
CDD pole contribution

- The CDD pole contribution to $\Lambda(1405)$

In the I = 0 scattering amplitude in the diagonal $\bar{K}N$ channel $F_{\bar{K}N}$, the CDD pole does not appear in the $\bar{K}N$ threshold energy region.

Y. Ikeda, T. Hyodo and W. Weise, Nucl. Phys. A 881 98 (2012)





In the $\pi\Sigma$ amplitude, the CDD pole appears at E = 1434 MeV.
The ERE description of the $\pi\Sigma$ amplitude around its threshold will not reach the $\bar{K}N$ threshold.

c. f. Yuki Kamiya, Kenta Miyahara, Shota Ohnishi et al. Nucl. Phys. A954, 41 (2016)

Conclusions

Conclusions

Y. Kamiya and T. Hyodo, Phys. Rev. C. 93.035203

Y. Kamiya and T. Hyodo, PTEP 023D02 (2017).

- We extend the weak-binding relation to quasi-bound states and we propose an interpretation of complex X introducing real quantities $\tilde{X}$ and U.

$$a_0 = R\left\{\frac{2X}{1+X} + \mathcal{O}\left(|R_{\rm typ}/R|\right) + \mathcal{O}\left(|l/R|^3\right)\right\}$$

- Using the Pade approximant, we take into account the contribution of the near-threshold CDD pole and derive the extended weak-binding relation.

- We apply the method to hadrons and discuss the internal structures.

$\Lambda(1405):\bar{K}N$ composite dominance

- We show that the CDD pole contribution to the $\Lambda(1405)$ in the $\bar{K}N$ channel is small with the extended weak-binding relation.