Microscopic derivation of (non-)relativistic second-order hydrodynamics from Boltzmann Equation

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based on work done with
S. Ei, K. Fujii, and K. Ohnishi,
K.Tsumura and **Y. Kikuchi**

Strangeness and charm in hadrons and dense matter
2017-05-15 — 2017-05-26

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Introduction

Separation of scales in the time evolution of a physical system

Liouville → **Kinetic (Boltzmann eq.)** → **Fluid dyn.**

Hamiltonian (i) (ii)

One-body dist. fn.

Hydrodynamic variables,u^{μ} T ,n and so on

(BBGKY hierarchy)

→

Slower dynamics with fewer variables

(i) From Liouville (BBGKY) to Boltzmann (Bogoliubov) The relaxation time of the s-body distribution function F_s (s>1) should be short and hence slaving variables of F_1. The reduced dynamics is described solely with the one-body distribution function F_1 as.the coordinate of the attractive manifold. N.N. Bogoliubov, in "Studies in Statistical Mechanics", (J. de Boer and G. E. Uhlenbeck, Eds.) vol2, (North-Holland, 1962)

(ii) Boltzmann to hydrodynamics (Hilbert, Chapman-Enskog,Bogoliubov)

After some time, the one-body distribution function is asymptotically well described by local temperature T(x), density n(x), and the flow velocity ***u* ,**
i.e., the hydrodynamic variables

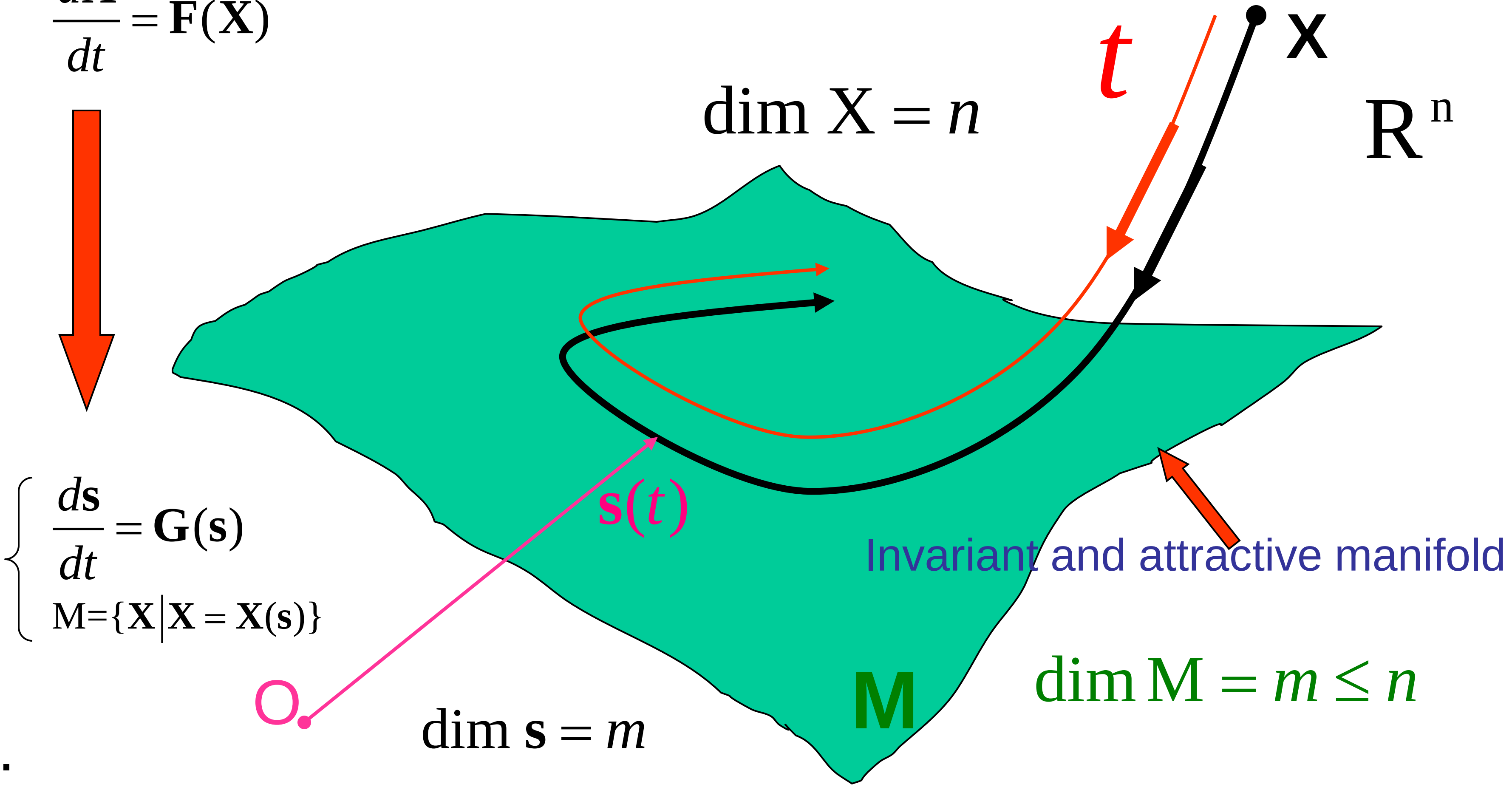
(iii) Langevin to Fokker-Planck equation,
(iv) Critical dynamics as described by TDGL etc…..

Geometrical image of reduction of dynamical systems

(including Hydrodynamic limit of Boltzmann equation)

n-dimensional dynamical system:

$$\frac{d\mathbf{X}}{dt} = \mathbf{F}(\mathbf{X})$$



$$\begin{cases} \dfrac{d\mathbf{s}}{dt} = \mathbf{G}(\mathbf{s}) \\ \mathrm{M}=\{\mathbf{X} \,|\, \mathbf{X} = \mathbf{X}(\mathbf{s})\} \end{cases}$$

Invariant and attractive manifold

$\dim \mathbf{s} = m$

$\dim \mathrm{M} = m \leq n$

eg.

$\mathbf{X} = f(\mathbf{r}, \mathbf{p})$; **distribution function in the phase space (infinite dimensions)**

$\mathbf{s} = \{u^{\mu}, T, n\}$; **the hydrodinamic quantities or conserved quantities for 1st-order eq.**

Introduction (continued)

The problems listed above maybe formulated as a construction of an asymptotic invariant/attractive manifold with possible space-time coarse-graining,

and

it may be interpreted

as a geometrical resolution to Hilbert's 6^{th} problem,

which is based on a similarity of geometry and physics.

c.f. Leo Corry's talk; Arch. Hist. exact. Sci. 51 (1997) 83.

We adopt the Renormalization Group method (Chen et al, 1995; T.K. (1995)) to construct the attractive/invariant manifolds and extend it so as to incorporate excited modes as well as the would-be zero modes as the slow/collective variables

and

thereby derive the second-order hydrodynamics as the mesoscopic dynamics.

Some references

The talk is based on the following work done with Tsumura, Kikuchi and K.Ohnishi;

K. Tsumura, K. Ohnishi and TK, PLB46 (2007), 134:

The original. 1st-order eq.

Tsumura, Kikuchi and TK, Physica D336 (2016),1;

The doublet scheme with application to derivation of second-order non-rel hydro in classical statistics

Tsumura, Kikuchi, TK, PRD92 (2015);

Quantum and relativistic with single component

Kikuchi, Tsumura and TK, PRC92 (2015);

Quantum and relativistic with multiple reactive species

Kikuchi, Tsumura and TK, PLA380 (2016), 2075 and arXiv:1604.07458;

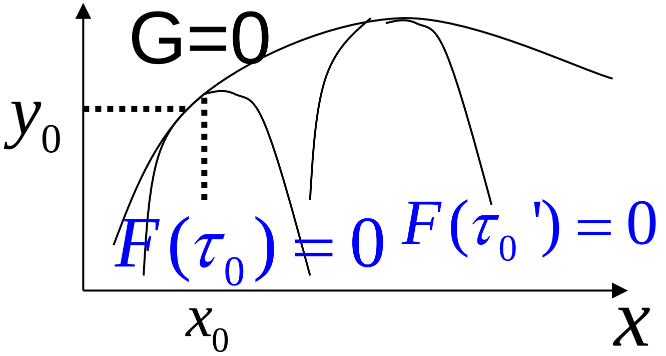
Quantum and non-rel with application to cold fermionic gas.

Tsumura and T.K., EPJA 48 (2012), 162 : A Review

Use of envelopes of a family of curves/surfaces: -- RG eq. as the envelope eq.--

T.K. ('95)

Let $\{C_\tau\}_\tau$ be a family of curves parametrized by τ in the x-y plane;



$$C_\tau : F(x,y,\tau,\mathbf{C}(\tau))=0 \;?$$

E: The envelope of C_τ $G(x,y)=0.$

$$F_{\tau_0}(x_0,y_0,\tau_0,C(\tau_0)) \equiv \frac{\partial F(x_0,y_0,\tau_0)}{\partial \tau_0} + \frac{\partial C}{\partial \tau_0}\frac{\partial F(x_0,y_0,\tau_0,C(\tau_0))}{\partial C} = 0.$$

The envelop equation: $dF/d\tau_0 = 0$ ⟺ RG eq.

the solution is inserted to F with the condition

$\tau_0 = x_0$ ← the tangent point

⇒ $G(x,y) = F(x,y,\mathbf{C}(x))$

Resummation of seemingly divergent pert. series and extracting slow dynamics by the envelope/RG eq.

T.K. (’95

A simple example: the dumped oscillator!

$$\frac{d^2x}{dt^2} + \epsilon\frac{dx}{dt} + x = 0,$$

$$x(t) = \bar{A}\exp(-\frac{\epsilon}{2}t)\sin(\sqrt{1-\frac{\epsilon^2}{4}}t + \bar{\theta}),$$

$$x(t,t_0) = x_0(t,t_0) + \epsilon x_1(t,t_0) + \epsilon^2 x_2(t,t_0) + \dots,$$

$$\ddot{x}_0 + x_0 = 0, \qquad \ddot{x}_{n+1} + x_{n+1} = -\dot{x}_n.$$

$$x(t_0,t_0) = W(t_0).$$

$$W(t_0) = W_0(t_0) + \epsilon W_1(t_0) + \epsilon^2 W_2(t_0) + \dots,$$

$$x_0(t,t_0) = A(t_0)\sin(t+\theta(t_0)), \quad W_0(t_0) = x_0(t_0,t_0) = A(t_0)\sin(t_0+\theta(t_0)).$$

$$x_1(t,t_0) = -\frac{A}{2}\cdot(t-t_0)\sin(t+\theta), \qquad W_1(t_0) = 0$$

a secular term appears, invalidating P.T.

$$x_2(t) = \frac{A}{8}\{(t-t_0)^2\sin(t+\theta) - (t-t_0)\cos(t+\theta)\}, \qquad W_2(t_0) = 0$$

Secular terms appear again!

Collecting the terms, we have

$$\boxed{\begin{aligned} x(t,t_0) &= A\sin(t+\theta) - \epsilon\frac{A}{2}(t-t_0)\sin(t+\theta) \\ &\quad + \epsilon^2\frac{A}{8}\{(t-t_0)^2\sin(t+\theta) - (t-t_0)\cos(t+\theta)\} \end{aligned}}$$

With I.C.: $W(t_0) = W_0(t_0) = A(t_0)\sin(t_0 + \theta(t_0))$

; parameterized by the functions,

$$A(t_0), \phi(t_0) \equiv t_0 + \theta(t_0)$$

The secular terms invalidate the pert. theory, like the log-divergence in QFT!

$\{C_{t_0}\}_{t_0}$: $\{x(t,t_0)\}_{t_0}$ $\qquad x_E(t) = x(t,t) = W(t).$

$\frac{dx(t,t_0)}{dt_0} = 0,$ $t_0 = t.$ $\Longrightarrow$ $A(t_0)$ and $\theta(t_0)$

$$x_2(t) = \frac{A}{8}\{(t-t_0)^2 \sin(t+\theta) - (t-t_0)\cos(t+\theta)\}, \qquad W_2(t_0) = 0$$

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With I.C.: $W(t_0) = W_0(t_0) = A(t_0)\sin(t_0 + \theta(t_0))$

; parameterized by the functions,

$$A(t_0), \phi(t_0) \equiv t_0 + \theta(t_0)$$

Let us try to construct the envelope function of the set of locally divergent functions, parameterized by t_0 !

$\{C_{t_0}\}_{t_0}$: $\{x(t,t_0)\}_{t_0}$ $x_{E}(t) = x(t,t) = W(t).$

$\frac{dx(t,t_0)}{dt_0} = 0,$ $t_0 = t.$ → $A(t_0)$ and $\theta(t_0)$

$$\frac{dA}{dt_0} + \epsilon A = 0, \quad \frac{d\theta}{dt_0} + \frac{\epsilon^2}{8} = 0,$$

$$A(t_0) = \bar{A} e^{-\epsilon t_0/2}, \quad \theta(t_0) = -\frac{\epsilon^2}{8} t_0 + \bar{\theta},$$

Extracted the amplitude and phase equations, separately!

$$x_{_E}(t) = x(t,t) = W_0(t) = \bar{A} \exp(-\frac{\epsilon}{2}t) \sin((1 - \frac{\epsilon^2}{8})t + \bar{\theta}),$$

$$\sqrt{1 - \epsilon^2/4} = 1 - \epsilon^2/8 + O(\epsilon^4)$$

The envelop function $x_{_E}(t) = W_0(t)$ an approximate but **global solution** in contrast to the pertubative solutions which have secular terms and valid only in local domains.

Notice also the resummed nature!

RG analysis of Van der Pol eq. with a limit cycle

$$\ddot{x} + x = \epsilon(1 - x^2)\dot{x}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 1\right)\tilde{x}_0 = 0, \quad \tilde{x}_0(t;t_0) = A(t_0)\cos(t + \theta(t_0)) \qquad \tilde{x}_0(t_0;t_0) = A(t_0)\cos(t_0 + \theta(t_0))$$

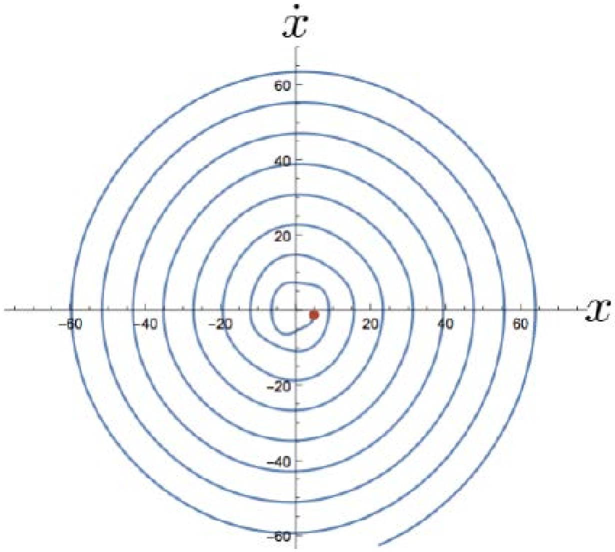
$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + 1\right)\tilde{x}_1 = -A(t_0)\left(1 - \frac{A(t_0)^2}{4}\right)\sin(t + \theta(t_0)) + \frac{A(t_0)^3}{4}\sin 3(t + \theta(t_0))$$

$$\tilde{x}_1(t;t_0) = (t - t_0)\frac{A(t_0)}{2}\left(1 - \frac{A(t_0)^2}{4}\right)\sin(t + \theta(t_0)) - \frac{A(t_0)^3}{32}\sin 3(t + \theta(t_0))$$

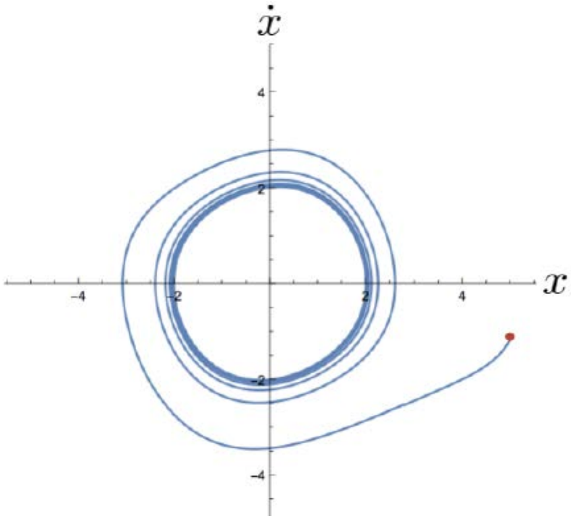
$$\tilde{x}_1(t_0;t_0) = -\frac{A(t_0)^3}{32}\sin 3(t_0 + \theta(t_0))$$

$$\left.\frac{\mathrm{d}\tilde{x}}{\mathrm{d}t_0}\right|_{t_0=t} = 0$$

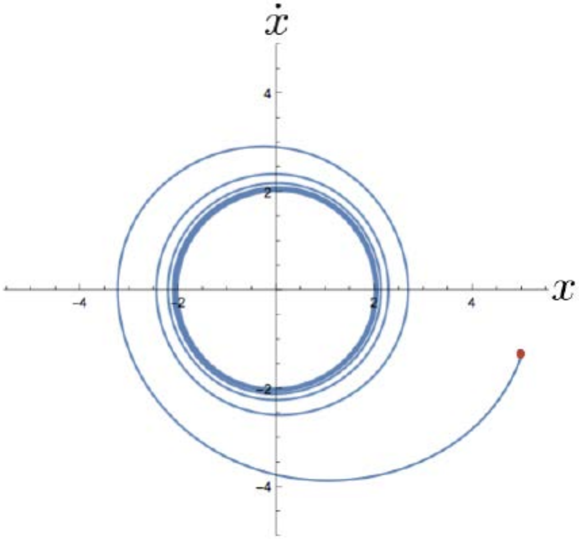
$$\dot{A} = \epsilon\frac{A}{2}\left(1 - \frac{A^2}{4}\right), \qquad \dot{\theta} = 0.$$



(a) perturbative solution



`Exact' numerical solution



RG improved solution
in 1st order perturbation

From Mater thesis by Y. Kikuchi (2015)

A foundation of the RG method a la ERG.

T.K. (1998); Ei, Fujii and T.K.('00)

Let us take the following n-dimensional equation;

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{X}, t), \tag{B·11}$$

where n may be infinity. Let $\boldsymbol{X}(t) = \boldsymbol{W}(t)$ be an yet unknown exact solution to Eq.(B·11), and we try to solve the equation with the initial condition at $t = \forall t_0$;

$$\boldsymbol{X}(t = t_0) = \boldsymbol{W}(t_0). \tag{B·12}$$

Then, the solution may be written as $\boldsymbol{X}(t; t_0, \boldsymbol{W}(t_0))$.

$$\boldsymbol{X}(t; t_0, \boldsymbol{W}(t_0)) = \boldsymbol{X}(t; t_0', \boldsymbol{W}(t_0')).$$

Taking the limit $t_0' \to t_0$, we have

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t_0} = \frac{\partial \boldsymbol{X}}{\partial t_0} + \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{W}}\frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}t_0} = \boldsymbol{0}.$$

Pert. Theory:

$\boldsymbol{X}(t; t_0, \boldsymbol{W}(t_0))$ and $\boldsymbol{X}(t; t_0', \boldsymbol{W}(t_0'))$ may be valid only for $t \sim t_0$ and $t \sim t_0'$,

$t_0 < t < t_0'$ (or $t_0' < t < t_0$)

$$\left.\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t_0}\right|_{t_0=t} = \left.\frac{\partial \boldsymbol{X}}{\partial t_0}\right|_{t_0=t} + \left.\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{W}}\frac{\mathrm{d}\boldsymbol{W}}{\mathrm{d}t_0}\right|_{t_0=t} = \boldsymbol{0},$$ with $t_0 = t$

RG equation!

Let $\boldsymbol{X}(t;t_0)$ is an approximate solution to Eq.(B·11) around $t \sim t_0$;

$$\frac{\mathrm{d}\boldsymbol{X}(t;t_0)}{\mathrm{d}t} \simeq \boldsymbol{F}(\boldsymbol{X}(t;t_0),t).$$

Then, we have

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{W}(t)}{\mathrm{d}t} &= \left.\frac{\partial \boldsymbol{X}(t;t_0)}{\partial t}\right|_{t_0=t} + \left.\frac{\partial \boldsymbol{X}(t;t_0)}{\partial t_0}\right|_{t_0=t} \\
&= \left.\frac{\partial \boldsymbol{X}(t;t_0)}{\partial t}\right|_{t_0=t} \\
&\simeq \boldsymbol{F}(\boldsymbol{X}(t;t_0),\, t)|_{t_0=t}, \\
&= \boldsymbol{F}(\boldsymbol{W}(t),t),
\end{aligned}$$

showing that our envelope function satisfies the original equation (B.11) in the global domain uniformly.

Eg. RG reduction of a generic equation with zero modes

S.Ei, K. Fujii & T.K. Ann. Phys. 280('00)

$$\partial_t \mathbf{u} = A\mathbf{u} + \epsilon \mathbf{F}(\mathbf{u}) \qquad \mathrm{dim}\,\mathbf{u} = n \quad |\epsilon| < 1$$

$$A\mathbf{U}_i = 0, \quad (i = 1, 2, \ldots, m).$$

We suppose that other eigenvalues have negative real parts;

$$A\mathbf{U}_\alpha = \lambda_\alpha \mathbf{U}_\alpha, \quad (\alpha = m+1, m+2, \cdots, n),$$

where $\mathrm{Re}\lambda_\alpha < 0$. One may assume without loss of generality that $\mathbf{U}_i$'s and $\mathbf{U}_\alpha$'s are linearly independent.

P **the projection onto the kernel** $\ker A$ $\qquad P + Q = 1$

Perturbative expansion around arbitrary time t_0 in the asymp. regime

$$\mathbf{u}(t;t_0) = \mathbf{u}_0(t;t_0) + \epsilon \mathbf{u}_1(t;t_0) + \epsilon^2 \mathbf{u}_2(t;t_0) + \cdots$$

With the initial value at t_0:

$$\begin{aligned}\mathbf{W}(t_0) &= \mathbf{W}_0(t_0) + \epsilon \mathbf{W}_1(t_0) + \epsilon^2 \mathbf{W}_2(t_0) + \cdots, \\ &= \mathbf{W}_0(t_0) + \boldsymbol{\rho}(t_0),\end{aligned}$$

$$\begin{aligned}(\partial_t - A)\mathbf{u}_0 &= 0, \\ (\partial_t - A)\mathbf{u}_1 &= \mathbf{F}(\mathbf{u}_0), \\ (\partial_t - A)\mathbf{u}_2 &= \mathbf{F}'(\mathbf{u}_0)\mathbf{u}_1,\end{aligned} \quad (\mathbf{F}'(\mathbf{u}_0)\mathbf{u}_1)_i = \sum_{j=1}^{n} \{\partial(F'(\mathbf{u}_0))_i / \partial(u_0)_j\} (u_1)_j$$

Since we are interested in the asymptotic state as $t \to \infty$, we may assume that the lowest-order initial value belongs to $\ker A$:

Unperturbed manifold

$$\mathbf{W}_0(t_0) = \sum_{i=1}^{m} C_i(t_0)\mathbf{U}_i = \mathbf{W}_0[C]. \quad \longleftrightarrow \quad \mathrm{M}_0$$

$$\mathbf{u}_0(t;t_0) \;=\; \mathrm{e}^{(t-t_0)A}\mathbf{W}_0(t_0) \;=\; \textstyle\sum_{i=1}^{m} C_i(t_0)\mathbf{U}_i.$$

Parameterized with n variables, instead of m ! $\quad C = {}^t(C_1, C_2, \cdots, C_m)$

1st-order solution reads

$$\begin{aligned}\mathbf{u}_1(t;t_0) = {} & \mathrm{e}^{(t-t_0)A}[\mathbf{W}_1(t_0) + A^{-1}Q\mathbf{F}(\mathbf{W}_0(t_0))] \\ & +(t-t_0)P\mathbf{F}(\mathbf{W}_0(t_0)) - A^{-1}Q\mathbf{F}(\mathbf{W}_0(t_0)).\end{aligned}$$

The would-be rapidly changing terms can be eliminated by the choice;

$$\mathbf{W}_1(t_0) = -A^{-1}Q\mathbf{F}(\mathbf{W}_0(t_0)), \qquad P\mathbf{W}_1(t_0) = 0$$

Then, the secular term appears only in the P space;

$$\mathbf{u}_1(t;t_0) = (t-t_0)P\mathbf{F} - A^{-1}Q\mathbf{F}.$$

a deformation of the manifold ρ

Deformed (invariant) slow manifold: $\mathrm{M}_1 = \{\mathbf{u} | \mathbf{u} = \mathbf{W}_0 - \epsilon A^{-1} Q\mathbf{F}(\mathbf{W}_0)\}$

$$\mathbf{u}(t; t_0) = \mathbf{W}_0 + \epsilon\{(t - t_0)P\mathbf{F} - A^{-1}Q\mathbf{F}\}$$

A set of locally divergent functions parameterized by t_0 !

The RG/E equation $\partial \mathbf{u} / \partial t_0 \big|_{t_0 = t} = \mathbf{0}$ gives the envelope, which is globally valid:

T.K., PTP (1995), (1997)

$$\dot{\mathbf{W}}_0(t) = \epsilon P\mathbf{F}(\mathbf{W}_0(t)),$$

which is reduced to an m-dimensional coupled equation,

$$\dot{C}_i(t) = \epsilon \langle \tilde{\mathbf{U}}_i, \mathbf{F}(\mathbf{W}_0[C]) \rangle, \quad (i = 1, 2, \cdots, m).$$

The global solution (the invariant manifod):

$$\mathbf{u}(t) = \mathbf{u}(t; t_0 = t) = \sum_{i=1}^{m} C_i(t)\mathbf{U}_i - \epsilon A^{-1}Q\mathbf{F}(\mathbf{W}_0[C]).$$

We have derived the invariant manifold and the **slow dynamics** on the manifold by the RG method.

It can be shown that the so-constructed global sol. satisfies the original eq. in a global domain up to the order with which the local sol.'s are constructed.

T.K. PTP(1995)

Extensions

a) A is not semi-simple.with Jordan cell

S. Ei, K. Fujii and T.K. , Ann.Phys.('00)

b) Higher orders.

c) PD equations;

Layered pulse dynamics for TDGL and Non-lin.Schroedinger.

See also, T.K.,Jpn. J. Ind. Appl. Math. 14 ('97), 51

d) Reduction of stochastic equation with several variables;

Liouville to Boltzmann, Langevin to Focker-Plank:

Further reduction of F-P with hierarchy of time scales.

Y. Hatta and T.K. Ann. Phys. (2002)

e) Discrete systems T.K. and J. Matsukidaira, Phys. Rev. E57 ('98), 4817

f) Derivation of hydrodynamic limit of Boltzmann eq. in classical/quantum (non) relativistic (reactive multicomponent) systems

Remark

The (arbitrary) initial value (in the asymptotic region) play an essential role in the RG method. An intimate similarity of the method with the holographic AdS/CFT method is indicated; see for example,

Yu Nakayama, PRD88, 105006 (2013).

Basics about Rel. Hydrodynamics

1. The fluid dynamic equations as conservation (balance) equations

$\partial_\mu N_i^\mu \equiv 0\,,\quad i = 1,\ldots,n\;,$ local conservation of charges

$\partial_\mu T^{\mu\nu} \equiv 0\,,\quad \nu = 0,\ldots,3\;.$ local conservation of energy-mom.

2.Tensor decomposition and choice of frame

u^μ ; **arbitrary** normalized **time-like** vector $u \cdot u = 1$

Def. **space-like** **projection** $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu\,,\ \Delta^{\mu\nu} u_\nu = 0\,,\ \Delta^{\mu\alpha}\Delta^\nu_\alpha = \Delta^{\mu\nu}$

space-like vector

space-like traceless tensor

$$N_i^\mu = n_i\, u^\mu + \nu_i^\mu\,,$$

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu - p\,\Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu}$$

$n_i \equiv N_i \cdot u$; net density of charge *i* in the **Local Rest Frame**

$\nu_i^\mu \equiv \Delta^\mu_\nu N_i^\nu$; net flow in LRF

$\epsilon \equiv u_\mu T^{\mu\nu} u_\nu$; energy density in LRF $p \equiv -\frac{1}{3} T^{\mu\nu}\Delta_{\mu\nu}$; isotropic pressure in LRF

$q^\mu \equiv \Delta^{\mu\alpha} T_{\alpha\beta} u^\beta$; heat flow in LRF

$\pi^{\mu\nu} \equiv \left[\frac{1}{2}\left(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right] T^{\alpha\beta}$; stress tensor in LRF

Define u^{μ} so that it has a physical meaning.

A. Particle frame (Eckart frame)

$$u_{\rm E}^{\mu} \equiv \frac{N_i^{\mu}}{\sqrt{N_i \cdot N_i}}$$; parallel to particle current of i $\longrightarrow$ $0 = N_i^{\mu}\Delta_{\mu\nu} = \nu_i^{\mu}$

space-like

B. Energy frame (Landau-Lifshitz frame)

$$u_{\rm L}^{\mu} \equiv \frac{T^{\mu}_{\nu} u_{\rm L}^{\nu}}{\sqrt{u_{\rm L}^{\alpha} T_{\alpha}^{\beta} T_{\beta\gamma} u_{\rm L}^{\beta}}}$$; flow of the energy-momentum density

$$\longrightarrow \quad q^{\mu} = 0$$

$$T^{\mu}_{\nu} u^{\nu} = \varepsilon u^{\mu} + q^{\mu}$$

$$N_i^{\mu} = n_i\, u^{\mu} + \nu_i^{\mu} \ ,$$

$$T^{\mu\nu} = \epsilon\, u^{\mu} u^{\nu} - p\, \Delta^{\mu\nu} + q^{\mu} u^{\nu} + q^{\nu} u^{\mu} + \pi^{\mu\nu}$$

Typical hydrodynamic equations for a viscous fluid
--- Choice of the frame and ambiguities in the form ---

Fluid dynamics = a system of balance equations

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\mu N^\mu = 0.$$

energy−momentum : $T^{\mu\nu}$ **number :** N^μ

$$T^{\mu\nu} \equiv \epsilon\, u^\mu u^\nu - p\,\Delta^{\mu\nu} + \delta T^{\mu\nu} \qquad N^\mu \equiv n\, u^\mu + \delta N^\mu$$

Dissipative part

Eckart eq.

no dissipation in the number flow; ⇨ Describing the flow of matter

$$\begin{aligned}\delta T^{\mu\nu} &= u^\mu T\lambda\left(\frac{1}{T}\nabla^\nu T - Du^\nu\right) + u^\nu T\lambda\left(\frac{1}{T}\nabla^\mu T - Du^\mu\right)\\ &\quad + 2\eta\frac{1}{2}\left(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla\cdot u\right) + \zeta\,\Delta^{\mu\nu}\nabla\cdot u\\ \delta N^\mu &= 0.\end{aligned}$$

with $D \equiv u^\mu \partial_\mu \quad \nabla^\mu \equiv \Delta^{\mu\nu}\partial_\nu$

$$\boldsymbol{\Delta}_p^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \equiv \Delta^{\mu\nu},$$

--- Involving time-like derivative ---.

Landau−Lifshits

no dissipation in energy flow ⇨ describing the energy flow.

$$\begin{aligned}\delta T^{\mu\nu} &= 2\eta\frac{1}{2}\left(\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3}\Delta^{\mu\nu}\nabla\cdot u\right) + \zeta\,\Delta^{\mu\nu}\nabla\cdot u,\\ \delta N^\mu &= -\lambda\frac{nT}{\epsilon+p}\left(\frac{1}{T}\nabla^\mu T - \frac{1}{\epsilon+p}\nabla^\mu p\right)\end{aligned}$$

--- Involving only space-like derivatives ---

⇔

$\delta T^{\mu\nu} u_\nu = 0,$ No dissipative energy-density nor energy-flo

$u_\mu \delta N^\mu = 0$ No dissipative particle density

with transport coefficients:

ς ; Bulk viscocity, η; Shear viscocity

λ ;Heat conductivity

Acausality problem

P. Romatschke, arXiv:0902.3636v3[hep-ph]

Fluctuations around the equilibrium:

$$\epsilon = \epsilon_0 + \delta\epsilon(t,x) \qquad u^\mu = (1,\vec{0}) + \delta u^\mu(t,x)$$

Linearized equation;

$$(\epsilon+p)Du^y - \nabla^y p + \Delta^y_\nu \partial_\mu \Pi^{\mu\nu} = (\epsilon_0+p_0)\partial_t \delta u^y + \partial_x \Pi^{xy}$$

$$\Pi^{xy} = \eta\left(\nabla^x u^y + \nabla^y u^x\right) + \left(\zeta - \frac{2}{3}\eta\right)\Delta^{xy}\nabla_\alpha u^\alpha = -\eta_0\, \partial_x \delta u^y$$

$$\Downarrow$$

$$\partial_t \delta u^y - \frac{\eta_0}{\epsilon_0+p_0}\partial_x^2 \delta u^y = 0$$

Diffusion equation!

The signal runs with an infinite speed.

Non-local thermodynamics (Maxwell-Cattaneo) → Mueller-Israel-Stewart

P. Romatschke, arXiv:0902.3636v3[hep-ph]

$$\tau_\pi \partial_t \Pi^{xy} + \Pi^{xy} = -\eta_0 \partial_x \delta u^y$$

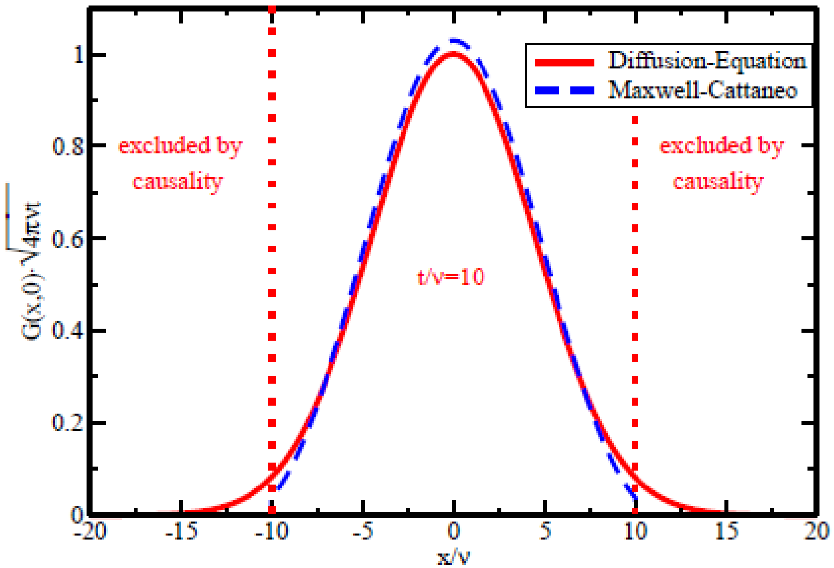
↓ Telegrapher’s equation

$$\partial_t \delta u^y + \frac{1}{\epsilon_0 + p_0} \partial_x \pi^{xy} = 0, \quad \tau_\pi \partial_t \pi^{xy} + \pi^{xy} = -\eta_0 \partial_x \delta u^y$$

$$\left[\partial_t^2 + \frac{\partial_t}{\tau_\pi} - \frac{\nu}{\tau_\pi}\partial_x^2\right] G(\mathbf{x},\mathbf{x}') = \frac{1}{\tau_\pi}\delta^2(\mathbf{x}-\mathbf{x}')$$

$$G(\mathbf{x},\mathbf{x}') = \theta(t-t')\,\theta\left(\frac{(t-t')^2\nu}{\tau_\pi} - (x-x')^2\right)\frac{e^{-\frac{t-t'}{2\tau_\pi}}}{\sqrt{4\nu\tau_\pi}} I_0\left(\sqrt{\frac{(t-t')^2}{4\tau_\pi^2} - \frac{(x-x')^2}{4\nu\tau_\pi}}\right)$$

Diffusion Eq. vs. Maxwell-Cattaneo



Compatibility of the definition of the flow and the LRF

In the kinetic approach, one needs a matching condition.

Seemingly plausible ansatz are;

$$\epsilon \equiv u_\mu T^{\mu\nu} u_\nu = \epsilon_0 \equiv u_\mu T_0^{\mu\nu} u_\nu$$
$$n \equiv u \cdot N = n_0 \equiv u \cdot N_0$$

Is this always correct, irrespective of the frames?

Particle frame is the same local equilibrium state as the energy frame?

Note that the entropy density S(x) and the pressure P(x) etc can be quite Different from those in the equilibrium.

Eg. $\exists$ the bulk viscosity

Local equilibrium → No dissipation!

D. H. Rischke, nucl-th/9809044

Distribution function in LRF:

$$f_0(k,x) = \frac{g}{(2\pi)^3} \left[\exp\{y_0(k,x)\} \pm 1\right]^{-1} \qquad y_0(k,x) \equiv [k \cdot u(x) - \mu(x)]/T(x).$$

Non-local distribution function;

$$f(k,x) \equiv \frac{g}{(2\pi)^3} \left[\exp\{y(k,x)\} \pm 1\right]^{-1}$$
$$y(k,x) \simeq y_0(k,x) + \varepsilon_1(x) + k \cdot \varepsilon_2(x) + k_\mu k_\nu\, \varepsilon_3^{\mu\nu}(x)$$

The problem of causality:

$$C_v \partial T / \partial t = -\partial q / \partial x$$

Fourier's law;

$$q = -\lambda \partial T / \partial x$$

Then

$$C_v \partial T / \partial t = \lambda \nabla^2 T$$

Causality is broken; the signal propagate with an infinite speed.

Modification;

$$\tau_q \frac{\partial}{\partial t} q(t\,,\,x) + q(t\,,\,x) = -\lambda \frac{\partial}{\partial x} T(t\,,\,x)$$

Extended thermodynamics

Nonlocal thermodynamics

$$q(t\,,\,x) = \int \mathrm{d}s \left[\theta(t-s)\,\frac{1}{\tau_q}\,\mathrm{e}^{-\frac{1}{\tau_q}(t-s)}\,\lambda\right]\left[-\frac{\partial}{\partial x} T(s\,,\,x)\right]$$

Memory effects; i.e., non-Markovian

Derivation(Israel-Stewart) : Grad's 14-moments method

+ ansats so that Landau/Eckart eq.'s are derived.

Problematic

The problems:

- Foundation of Grad's 14 moments method
- ad-hoc constraints on $\delta T^{\mu\nu}$ and δN^{μ} consistent with the underlying dynamics?

Relativistic Boltzmann equation

$$p^\mu \partial_\mu f_p(x) = C[f]_p(x),$$

Collision integral

$$C[f]_p(x) \equiv \frac{1}{2!}\int \mathrm{d}p_1 \mathrm{d}p_2 \mathrm{d}p_3 \omega(p,p_1|p_2,p_3)$$
$$\times ((1+af_p(x))(1+af_{p_1}(x))f_{p_2}(x)f_{p_3}(x)$$
$$- f_p(x)f_{p_1}(x)(1+af_{p_2}(x))(1+af_{p_3}(x))),$$

a=+1 boson
=-1 fermion
=0 classical

$$\mathrm{d}p \equiv \mathrm{d}^3p/[(2\pi)^3 p^0]$$

Symm. property of the transition probability:

$$\omega(p\,,\,p_1|p_2\,,\,p_3) = \omega(p_2\,,\,p_3|p\,,\,p_1) = \omega(p_1\,,\,p|p_3\,,\,p_2) = \omega(p_3\,,\,p_2|p_1\,,\,p) \quad \text{--- (1)}$$

Energy-mom. conservation; $\omega(p\,,\,p_1|p_2\,,\,p_3) \propto \delta^4(p+p_1-p_2-p_3)$ --- (2)

Owing to (1),

$$\int \mathrm{d}p \varphi_p C[f]_p = \frac{1}{2!}\frac{1}{4}\int \mathrm{d}p \mathrm{d}p_1 \mathrm{d}p_2 \mathrm{d}p_3\, \omega(p\,,\,p_1|p_2\,,\,p_3)$$
$$\times (\varphi_p + \varphi_{p_1} - \varphi_{p_2} - \varphi_{p_3})$$
$$\times ((1+af_p)(1+af_{p_1})f_{p_2}f_{p_3} - f_p f_{p_1}(1+af_{p_2})(1+af_{p_3})) \quad \text{--- (3)}$$

Collision Invariant $\varphi_p(x)$ **:** $\int \mathrm{d}p\frac{1}{p^0}C[f]_p = 0, \quad \int \mathrm{d}p\frac{1}{p^0}p^\mu C[f]_p = 0$

Eq.'s (3) and (2) tell us that
the general form of a collision invariant; $\varphi_p(x) = \alpha(x) + p^\mu \beta_\mu(x),$

which can be x-dependent!

Local equilibrium distribution

The entropy current: $S^\mu \equiv -\int \mathrm{d}p\, p^\mu \left[f_p \ln f_p - \frac{(1+af_p)\ln(1+af_p)}{a} \right]$

$$\partial_\mu S^\mu = -\int \mathrm{d}p\, C[f]_p \ln\left[\frac{f_p}{1+af_p}\right]$$

Conservation of entropy ⟶ $\ln(f_p/(1+af_p)) = \alpha(x) + p^\mu \beta_\mu(x).$

$$f_p(x) = \frac{1}{\mathrm{e}^{(p^\mu u_\mu - \mu)/T} - a} \equiv f_p^{\mathrm{eq}}$$

i.e., **the local equilibrium distribution fn;**

Remark:

Owing to the energy-momentum conservation,
the collision integral also vanishes for the local equilibrium distribution fn.;

$$C[f_p^{eq}](x) = 0.$$

Previous attempts to derive the dissipative hydrodynamics as a reduction of the dynamics

N.G. van Kampen, J. Stat. Phys. 46(1987), 709

unique but non-covariant form and hence not Landau either Eckart!

Cf. Chapman-Enskog method to derive Landau and Eckart eq.'s; see, eg, de Groot et al ('80)

Here,

In the covariant formalism,
in a unified way and systematically
derive dissipative rel. hydrodynamics at once!

Introduction of the macroscopic frame vector

K. Tsumura, T.K. K. Ohnishi,PLB646(2007)134

Ansatz of the origin of the dissipation= the spatial inhomogeneity,
leading to Navier-Stokes in the non-rel. case .

$\boldsymbol{a}_p^\mu$ would become a macro flow-velocity ← **Coarse graining of space-time**

and will be identified with u^μ

Difference of the time scales of kinetic and hydrodynamics.

$$\tau \equiv \boldsymbol{a}_p^\mu \, x_\mu , \qquad \sigma^\mu \equiv \left(g^{\mu\nu} - \frac{\boldsymbol{a}_p^\mu \boldsymbol{a}_p^\nu}{\boldsymbol{a}_p^2} \right) x_\nu \equiv \boldsymbol{\Delta}_p^{\mu\nu} \, x_\nu \qquad x^\mu \Longrightarrow \tau \quad \sigma^\mu$$

$$\frac{\partial}{\partial \tau} = \frac{1}{\boldsymbol{a}_p^2} \boldsymbol{a}_p^\mu \, \partial_\mu \equiv \boldsymbol{D},$$ time-like derivative

$$\boldsymbol{\Delta}_p^{\mu\nu} \frac{\partial}{\partial \sigma^\nu} = \boldsymbol{\Delta}_p^{\mu\nu} \, \partial_\nu \equiv \boldsymbol{\nabla}^\mu$$ space-like derivative

Rewrite the Boltzmann equation as,

$$\Longrightarrow \quad \frac{\partial}{\partial \tau} f_p(\tau\,,\,\sigma) = \frac{1}{p \cdot \boldsymbol{a}_p} C[f]_p(\tau\,,\,\sigma) - \frac{1}{p \cdot \boldsymbol{a}_p} p \cdot \boldsymbol{\nabla} f_p(\tau\,,\,\sigma)$$

perturbation

Only spatial inhomogeneity leads to dissipation.

Solution by the perturbation theory

0th

$$\frac{\partial}{\partial\tau}\tilde{f}_p^{(0)} = \frac{1}{p\cdot\boldsymbol{a}_p}C[f]_p\Big|_{f=\tilde{f}^{(0)}}$$

We seek for a slow solution in the asymptotic regime:

$$\frac{\partial}{\partial\tau}\tilde{f}_p^{(0)} = 0 \quad \Rightarrow \quad \frac{1}{p\cdot\boldsymbol{a}_p}C[f]_p\Big|_{f=\tilde{f}^{(0)}} = 0$$

$$\Rightarrow \quad \tilde{f}_p^{(0)}(\tau,\sigma;\tau_0) = f_p^{\mathrm{eq}}(\sigma;\tau_0) = \frac{1}{\mathrm{e}^{[p^\mu u_\mu(\sigma;\tau_0)-\mu(\sigma;\tau_0)]/T(\sigma;\tau_0)} - a}$$

written in terms of the hydrodynamic variables.
Asymptotically, the solution can be written solely in terms of the hydrodynamic variables.

- Five conserved quantities $T(\sigma\,;\,\tau_0),\ \ \mu(\sigma\,;\,\tau_0),\ \ u_\mu(\sigma\,;\,\tau_0)$

reduced degrees of freedom m = 5 $u^\mu(\sigma\,;\,\tau_0)\,u_\mu(\sigma\,;\,\tau_0) = 1$

- 0th invariant manifold $f_p^{(0)}(\tau_0\,,\,\sigma) = f_p^{\mathrm{eq}}(\sigma\,;\,\tau_0)$

$$\Rightarrow \quad f^{(0)}(\tau_0) \quad = \quad f^{\mathrm{eq}}$$

↑

Local equilibrium

1st

$$\frac{\partial}{\partial\tau}\tilde{f}^{(1)}(\tau) = f^{\rm eq}\bar{f}^{\rm eq}\hat{L}(f^{\rm eq}\bar{f}^{\rm eq})^{-1}\tilde{f}^{(1)}(\tau) + f^{\rm eq}\bar{f}^{\rm eq}F_0$$

Lin. collision operator

$$\hat{L}_{pq} \equiv (f^{\rm eq}_p\bar{f}^{\rm eq}_p)^{-1}\frac{1}{p\cdot u}\frac{\delta}{\delta f_q}C[f]_p\Big|_{f=f^{\rm eq}} f^{\rm eq}_q\bar{f}^{\rm eq}_q$$

with $\bar{f}^{\rm eq}_p(\sigma;\tau_0) \equiv 1 + a f^{\rm eq}_p(\sigma;\tau_0)$

$\downarrow$ $a \to 0$ classical limit

1

Inhomogeneous term

$$F_{0p} \equiv -(f^{\rm eq}_p\bar{f}^{\rm eq}_p)^{-1}\frac{1}{p\cdot u}p\cdot\nabla f^{\rm eq}_p$$

The lin. op. L has good properties:

Def. **inner product:** $\langle\psi,\chi\rangle \equiv \int \mathrm{d}p\,(p\cdot u)f^{\rm eq}_p\underline{\bar{f}^{\rm eq}_p}\psi_p\chi_p$ ← Quantum effect

1. $\langle\,\varphi\,,\,L\,\psi\,\rangle = \langle\,L\,\varphi\,,\,\psi\,\rangle$ Self−adjoint

2. $\langle\,\varphi\,,\,L\,\varphi\,\rangle \leq 0$ for all φ Semi−negative definite

3. $L\,\varphi^\alpha_0 = 0 \Longrightarrow \varphi^\alpha_{0p} = \begin{cases} p^\mu & \alpha=\mu, \\ m & \alpha=4 \end{cases}$

L has 5 **zero modes** and other eigenvalues are negative.

Def. Projection operators:

$$\begin{cases} \left[P\psi\right]_p \equiv \sum_{\alpha\beta} \varphi^{\alpha}_{0p}\,\eta^{-1}_{\alpha\beta}\,\langle\,\varphi^{\beta}_0\,,\,\psi\,\rangle, \\ Q \equiv 1 - P. \end{cases}$$

$$\eta^{-1}_{\alpha\beta}\;;\;\sum_{\gamma}\eta^{\alpha\gamma}\,\eta^{-1}_{\gamma\beta} = \delta^{\alpha}_{\beta}$$

Metric is given in terms of
The zero modes:

$$\eta^{\alpha\beta} \equiv \langle\,\varphi^{\alpha}_0\,,\,\varphi^{\beta}_0\,\rangle$$

1st order solution:

would produce a fast motion

$$\tilde{f}^{(1)}(\tau,\sigma;\tau_0) = f^{\mathrm{eq}}\bar{f}^{\mathrm{eq}}\Big[\underline{\mathrm{e}^{\hat{L}(\tau-\tau_0)}\Psi} + (\tau-\tau_0)P_0F_0 + \underline{(\mathrm{e}^{\hat{L}(\tau-\tau_0)}-1)\hat{L}^{-1}Q_0F_0}\Big]$$

With the initial value: $f^{(1)}(\sigma;\tau_0) = \tilde{f}^{(1)}(\tau=\tau_0,\sigma;\tau_0) = f^{\mathrm{eq}}\bar{f}^{\mathrm{eq}}\Psi$

which is yet to be determined.

In the case of the 1st-order (N-S) equation:

$\mathrm{e}^{\hat{L}(\tau-\tau_0)}\,\hat{L}^{-1}Q_0F_0$ can be cancelled out by a choice of the initial value $\mathrm{e}^{\hat{L}(\tau-\tau_0)}\Psi$

$\Psi = -L^{-1}Q_0F_0$ → 1st-order (Landau) equation

Tsumura, Ohnishi and TK, PLB46(2007);
Tsumura, Kikuchi and TK, PRD92 (2015).

Envelope/RG eq. $\left.\dfrac{\mathrm{d}\tilde{f}(\tau;\tau_0)}{\mathrm{d}\tau_0}\right|_{\tau_0=\tau} = 0$

Note: we can assume that $P_0\Psi = 0,$

because possible zero modes can be renormalized into the zero-th sol.

Resultant 1st-order Hydrodynamic equation

Tsumura, Ohnishi, TK(2007);
Tsumura, Kikuchi, TK (2015)

$$\partial_\mu J^{\mu\alpha}_{\text{hydro}} = 0, \qquad J^{\mu\alpha}_{\text{hydro}} \equiv \int \mathrm{d}p\, p^\mu \varphi^\alpha_{0p} f^{\text{eq}}_p (1 + \bar{f}^{\text{eq}}_p \Psi_p)$$

$$= \begin{cases} e u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}, & \alpha = \nu, \\ n u^\mu + J^\mu, & \alpha = 4. \end{cases}$$

$$\Pi = -\zeta\theta \qquad J^\mu = \lambda \frac{T^2}{h^2} \nabla^\mu \frac{\mu}{T} \qquad \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$$

$$\zeta = -\frac{1}{T}\langle \hat{\Pi}, \hat{L}^{-1}\hat{\Pi}\rangle \qquad \lambda = \frac{1}{3T^2}\langle \hat{J}^\mu, \hat{L}^{-1}\hat{J}_\mu\rangle \qquad \eta = -\frac{1}{10T}\langle \hat{\pi}^{\mu\nu}, \hat{L}^{-1}\hat{\pi}_{\mu\nu}\rangle$$

$$= \frac{1}{T}\int_0^\infty \mathrm{d}s \langle \hat{\Pi}(0), \hat{\Pi}(s)\rangle \qquad = -\frac{1}{3T^2}\int_0^\infty \mathrm{d}s \langle \hat{J}^\mu(0), \hat{J}_\mu(s)\rangle \qquad = \frac{1}{10T}\int_0^\infty \mathrm{d}s \langle \hat{\pi}^{\mu\nu}(0), \hat{\pi}_{\mu\nu}(s)\rangle$$

$$(\hat{\Pi}_p(s), \hat{J}^\mu_p(s), \hat{\pi}^{\mu\nu}_p(s)) \equiv \int \mathrm{d}q \left[\mathrm{e}^{s\hat{L}}\right]_{pq} (\hat{\Pi}_q, \hat{J}^\mu_q, \hat{\pi}^{\mu\nu}_q)$$

$$(\hat{\Pi}_p, \hat{J}^\mu_p, \hat{\pi}^{\mu\nu}_p) = \frac{1}{p\cdot u}(\Pi_p, J^\mu_p, \pi^{\mu\nu}_p)$$

$$\Pi_p \equiv (p\cdot u)^2 \left[\frac{1}{3} - \left.\frac{\partial P}{\partial e}\right|_n\right] + (p\cdot u)\left.\frac{\partial P}{\partial n}\right|_e - \frac{1}{3}m^2,$$

$$J^\mu_p \equiv -\Delta^{\mu\nu} p_\nu((p\cdot u) - h) \qquad \pi^{\mu\nu}_p \equiv \Delta^{\mu\nu\rho\sigma} p_\rho p_\sigma \qquad h \equiv (e + P)/n$$

$$\Delta^{\mu\nu\rho\sigma} \equiv 1/2(\Delta^{\mu\rho}\Delta^{\nu\sigma} + \Delta^{\mu\sigma}\Delta^{\nu\rho} - 2/3\Delta^{\mu\nu}\Delta^{\rho\sigma})$$

Transport coefficients for a recative flow

First-order transport coefficients

$$\zeta = \frac{1}{T}\int_0^{\infty} \mathrm{d}s \langle \hat{\Pi}(0), \hat{\Pi}(s) \rangle$$

$$\eta = \frac{1}{10T}\int_0^{\infty} \mathrm{d}s \langle \hat{\pi}^{\mu\nu}(0), \hat{\pi}_{\mu\nu}(s) \rangle$$

$$\lambda_{AB} = -\frac{1}{3T^2}\int_0^{\infty} \mathrm{d}s \langle \hat{J}^{\mu}_{A}(0), \hat{J}_{B,\mu}(s) \rangle$$

- First-order transport coefficients have the same expressions as those of Chapman-Enskog method and consistent with the field theoretic calculation based on Green-Kubo formula.

Jeon, PRD 52, 3591 (1995);
Hidaka and TK, PRD 83, 076004 (2011)

$$\langle \psi, \chi \rangle \equiv \sum_{k=1}^{N} \int \mathrm{d}p_k (p_k \cdot u) f^{\mathrm{eq}}_{k,p_k} \bar{f}^{\mathrm{eq}}_{k,p_k} \psi_{k,p_k} \chi_{k,p_k}$$

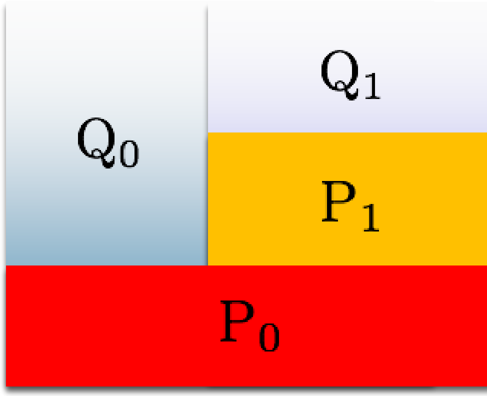
$$(\hat{\Pi}(s), \hat{J}^{\mu}_{A}(s), \hat{\pi}^{\mu\nu}(s))_{k,p_k} \equiv \sum_{m=1}^{M} \int \mathrm{d}q \left[\mathrm{e}^{sL} \right]_{k,p_k;m,q_m} (\hat{\Pi}, \hat{J}^{\mu}_{A}, \hat{\pi}^{\mu\nu})_{m,q_m}$$

Derivation of 2nd-order solution: How to deform the invariant manifold so as to incorporate excited modes

$$\tilde{f}^{(1)}(\tau,\sigma;\tau_0) \simeq f^{\rm eq}\bar{f}^{\rm eq}\left[\Psi + (\tau-\tau_0)\hat{L}\Psi + (\tau-\tau_0)P_0F_0 \; + (\tau-\tau_0)Q_0F_0\right]$$

$\tau - \tau_0$: small

- $\hat{L}\Psi$ and Q_0F_0 should belong to a common vector space.
- The P_1 space is spanned by independent components of $\hat{L}\Psi$ and Ψ.



Ψ and $\hat{L}^{-1}Q_0F_0$ should belong to the same vector subspace.

Now an explicit calculations give

$$\left[\hat{L}^{-1}Q_0F_0\right]_p = \frac{1}{T}\left[[\hat{L}^{-1}\hat{\Pi}]_p(-\nabla\cdot u) - [\hat{L}^{-1}\hat{J}^\mu]_p\frac{T}{h}\nabla_\mu\frac{\mu}{T} \; + [\hat{L}^{-1}\hat{\pi}^{\mu\nu}]_p\Delta_{\mu\nu\rho\sigma}\nabla^\rho u^\sigma\right]$$

Here the microscopic dissipative currents are given by

$$(\hat{\Pi}_p, \hat{J}^\mu_p, \hat{\pi}^{\mu\nu}_p) = \frac{1}{p\cdot u}(\Pi_p, J^\mu_p, \pi^{\mu\nu}_p)$$

$$h \equiv (e+P)/n,$$

$$\Delta^{\mu\nu\rho\sigma} \equiv 1/2(\Delta^{\mu\rho}\Delta^{\nu\sigma} + \Delta^{\mu\sigma}\Delta^{\nu\rho} - 2/3\Delta^{\mu\nu}\Delta^{\rho\sigma})$$

$$\Pi_p \equiv (p\cdot u)^2\left[\frac{1}{3} - \left.\frac{\partial P}{\partial e}\right|_n\right] + (p\cdot u)\left.\frac{\partial P}{\partial n}\right|_e - \frac{1}{3}m^2,$$

$$J^\mu_p \equiv -\Delta^{\mu\nu}\, p_\nu((p\cdot u) - h),$$

$$\pi^{\mu\nu}_p \equiv \Delta^{\mu\nu\rho\sigma}\, p_\rho\, p_\sigma .$$

$$\Longrightarrow \quad \Psi_p = \left[\frac{[\hat{L}^{-1}\hat{\Pi}]_p}{\langle\hat{\Pi}, \hat{L}^{-1}\hat{\Pi}\rangle}\right]\Pi + \left[\frac{h[\hat{L}^{-1}\hat{J}^\mu]_p}{\frac{1}{3}\langle\hat{J}^\nu, \hat{L}^{-1}\hat{J}_\nu\rangle}\right]J_\mu + \left[\frac{[\hat{L}^{-1}\hat{\pi}^{\mu\nu}]_p}{\frac{1}{5}\langle\hat{\pi}^{\rho\sigma}, \hat{L}^{-1}\hat{\pi}_{\rho\sigma}\rangle}\right]\pi_{\mu\nu}$$

Second-order perturbative eq.

$$\frac{\partial}{\partial\tau}\tilde{f}^{(2)}(\tau) = f^{\mathrm{eq}}\bar{f}^{\mathrm{eq}}L(f^{\mathrm{eq}}\bar{f}^{\mathrm{eq}})^{-1}\tilde{f}^{(2)}(\tau) + f^{\mathrm{eq}}\bar{f}^{\mathrm{eq}}K(\tau-\tau_0)$$

$$K(\tau-\tau_0) \equiv F^{(1)}(\tau) + \frac{1}{2}B[\tilde{f}^{(1)},\tilde{f}^{(1)}](\tau),$$

$$B[\chi,\psi]_{k,p_k;m,q_m;n,r_n} \equiv -(f^{\mathrm{eq}}_{k,p_k}\bar{f}^{\mathrm{eq}}_{k,p_k})^{-1} \times \frac{\delta^2}{\delta f_{m,q_m}\delta f_{n,r_n}}\left(\frac{1}{p_k\cdot u}\sum_{l=1}^{N}C_{kl}[f]_{k,p_k}\right)\Bigg|_{f=f^{\mathrm{eq}}}\chi^{(1)}_{m,q_m}\psi^{(1)}_{n,r_n}$$

Second-order perturbative solution

$$\tilde{f}^{(2)}(\tau) = f^{\mathrm{eq}}\bar{f}^{\mathrm{eq}}\Big[(\tau-\tau_0)P_0 + (\tau-\tau_0)\mathcal{G}(s)^{-1}P_1\mathcal{G}(s)Q_0 - (1+(\tau-\tau_0)\partial/\partial s)Q_1\mathcal{G}(s)Q_0\Big]\,K(s)\Big|_{s=0}$$

``Initial" value at arbitrary time $\tau = \tau_0$

$$\mathcal{G}(s) \equiv (L - \partial/\partial s)^{-1}$$

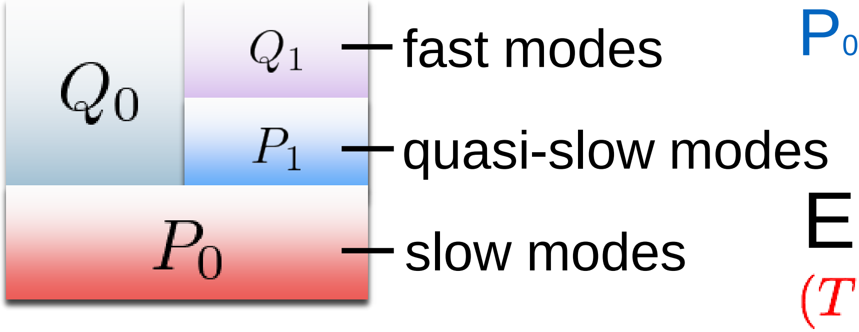
$$f^{(2)}(\tau_0) = -f^{\mathrm{eq}}\bar{f}^{\mathrm{eq}}Q_1\mathcal{G}(s)Q_0\,K(s)\Big|_{s=0}$$

The perturbative calculation finished.

Hydrodynamic eq. through the RG equation

$$\tilde{f}(\tau=\tau_0;\tau_0)=f^{\mathrm{G}}(\tau_0) \longrightarrow f^{\mathrm{G}}(\tau)$$

$$\tilde{f}(\tau;\tau_0) \xrightarrow{\text{RG eq.}} \left.\frac{\mathrm{d}\tilde{f}(\tau;\tau_0)}{\mathrm{d}\tau_0}\right|_{\tau_0=\tau}=0 \quad \left(\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}t}=\boldsymbol{G}(\boldsymbol{C})\right)$$



Projection onto P_0-space

Eq. of continuity

$$(T(\tau), \mu_A(\tau), u^\mu(\tau))$$

Projection onto P_1-space

Eq. of relaxation

$$(\Pi(\tau), J^\mu_A(\tau), \pi^{\mu\nu}(\tau))$$

Second−order hydrodynamics

K. Tsumura and T. Kunihiro, Eur.Phys.J.A48, 162 (2012)
K. Tsumura Y. Kikuchi and T K (2013) arXiv:1311.7059
K. Tsumura, Y.Kikuchi and T K, PRD (2015)

Second-order multi-component quantum hydrodynamic Equation

Hydrodynamic equation

Y.Kikuchi, K. Tsumura, and T. K (2015), PRC92 (2015)

$$\partial_\mu T^{\mu\nu} = 0$$

$$\partial_\mu N^\mu_A = 0$$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - (P - \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$N^\mu_A = n_A u^\mu + J^\mu_A$$

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$$

$$\nabla_\mu = \Delta^\nu_\mu \partial_\nu$$

$$\Delta^{\mu\nu\rho\sigma} = \frac{1}{2}\left(\Delta^{\mu\rho}\Delta^{\nu\sigma} + \Delta^{\mu\sigma}\Delta^{\nu\rho} - \frac{2}{3}\Delta^{\mu\nu}\Delta^{\rho\sigma}\right)$$

$$\theta = \nabla \cdot u$$

$$\sigma^{\mu\nu} = \Delta^{\mu\nu\rho\sigma}\nabla_\rho u_\sigma$$

$$\begin{aligned}\Pi = &-\zeta\theta - \boxed{\tau_\Pi \frac{\partial}{\partial\tau}\Pi} - \sum_{a=1}^{M} \ell^a_{\Pi J}\nabla\cdot J_A \\ &+ \kappa_{\Pi\Pi}\Pi\theta + \sum_{A=1}^{M}\kappa^{(1)A}_{\Pi J}J_{A,\rho}\nabla^\rho T + \sum_{A,B=1}^{M}\kappa^{(2)BA}_{\Pi J}J_{A,\rho}\nabla^\rho\frac{\mu_B}{T} + \kappa_{\Pi\pi}\pi_{\rho\sigma}\sigma^{\rho\sigma} \\ &+ b_{\Pi\Pi\Pi}\Pi^2 + \sum_{A,B=1}^{M} b^{AB}_{\Pi JJ}J^\rho_A J_{B,\rho} + b_{\Pi\pi\pi}\pi^{\rho\sigma}\pi_{\rho\sigma},\end{aligned}$$

$$\begin{aligned}J^\mu_A = &\sum_{B=1}^{M}\lambda_{AB}\frac{T^2}{h^2}\nabla^\mu\frac{\mu_B}{T} - \boxed{\sum_{B=1}^{M}\tau^{AB}_J\Delta^{\mu\rho}\frac{\partial}{\partial\tau}J_{B,\rho}} - \ell^A_{J\Pi}\nabla^\mu\Pi - \ell^A_{J\pi}\Delta^{\mu\rho}\nabla_\nu\pi^\nu{}_\rho \\ &+ \kappa^{(1)A}_{J\Pi}\Pi\nabla^\mu T + \sum_{B=1}^{M}\kappa^{(2)AB}_{J\Pi}\Pi\nabla^\mu\frac{\mu_B}{T} + \sum_{B=1}^{M}\kappa^{(1)AB}_{JJ}J^\mu_B\theta + \sum_{B=1}^{M}\kappa^{(2)AB}_{JJ}J_{B,\rho}\sigma^{\mu\rho} \\ &+ \kappa^{(3)AB}_{JJ}J_{B,\rho}\omega^{\mu\rho} + \kappa^{(1)A}_{J\pi}\pi^{\mu\rho}\nabla_\rho T + \sum_{B=1}^{M}\kappa^{(2)AB}_{J\pi}\pi^{\mu\rho}\nabla_\rho\frac{\mu_B}{T} \\ &+ \sum_{B=1}^{M} b^{AB}_{J\Pi J}\Pi J^\mu_B + \sum_{B=1}^{M} b^{AB}_{JJ\pi}J_{B,\rho}\pi^{\rho\mu},\end{aligned}$$

$$\begin{aligned}\pi^{\mu\nu} = &\,2\eta\sigma^{\mu\nu} - \boxed{\tau_\pi\Delta^{\mu\nu\rho\sigma}\frac{\partial}{\partial\tau}\pi_{\rho\sigma}} - \sum_{a=1}^{M}\ell^a_{\pi J}\nabla^{\langle\mu}J^{\nu\rangle}_a \\ &+ \kappa_{\pi\Pi}\Pi\sigma^{\mu\nu} + \sum_{A=1}^{M}\kappa^{(1)A}_{\pi J}J^{\langle\mu}_A\nabla^{\nu\rangle}T + \sum_{A,B=1}^{M}\kappa^{(2)BA}_{\pi J}J^{\langle\mu}_A\nabla^{\nu\rangle}\frac{\mu_B}{T} \\ &+ \kappa^{(1)}_{\pi\pi}\pi^{\mu\nu}\theta + \kappa^{(2)}_{\pi\pi}\pi^{\lambda\langle\mu}\sigma^{\nu\rangle}{}_\lambda + \kappa^{(3)}_{\pi\pi}\pi^{\lambda\langle\mu}\omega^{\nu\rangle}{}_\lambda \\ &+ b_{\pi\Pi\pi}\Pi\pi^{\mu\nu} + \sum_{A,B=1}^{M} b^{AB}_{\pi JJ}J^{\langle\mu}_A J^{\nu\rangle}_B + b_{\pi\pi\pi}\pi^{\lambda\langle\mu}\pi^{\nu\rangle}{}_\lambda\end{aligned}$$

Dissipative relaxation times

$$\tau_\Pi = \frac{\int_0^\infty \mathrm{d}s\, s\langle\hat{\Pi}(0),\hat{\Pi}(s)\rangle}{\int_0^\infty \mathrm{d}s\langle\hat{\Pi}(0),\hat{\Pi}(s)\rangle} = \frac{\int_0^\infty \mathrm{d}s\, sR_\Pi(s)}{\int_0^\infty \mathrm{d}sR_\Pi(s)}$$

$$\tau_\pi = \frac{\int_0^\infty \mathrm{d}s\, s\langle\hat{\pi}^{\mu\nu}(0),\hat{\pi}_{\mu\nu}(s)\rangle}{\int_0^\infty \mathrm{d}s\langle\hat{\pi}^{\rho\sigma}(0),\hat{\pi}_{\rho\sigma}(s)\rangle} = \frac{\int_0^\infty \mathrm{d}s\, sR_J(s)}{\int_0^\infty \mathrm{d}sR_J(s)}$$

$$\tau_\pi = -\frac{\langle\hat{\pi}^{\mu\nu},\hat{L}^{-2}\hat{\pi}_{\mu\nu}\rangle}{\langle\hat{\pi}^{\rho\sigma},\hat{L}^{-1}\hat{\pi}_{\rho\sigma}\rangle} = \frac{\int_0^\infty \mathrm{d}s\, sR_\pi(s)}{\int_0^\infty \mathrm{d}sR_\pi(s)}$$

$$R_\Pi(s) \equiv \frac{1}{T}\langle\hat{\Pi}(0),\hat{\Pi}(s)\rangle,$$

$$R_J(s) \equiv -\frac{1}{3T^2}\langle\hat{J}^\mu(0),\hat{J}_\mu(s)\rangle,$$

$$R_\pi(s) = \frac{1}{10T}\langle\hat{\pi}^{\mu\nu}(0),\hat{\pi}_{\mu\nu}(s)\rangle.$$

For a reactive case,

$$\tau_J^{AB} = \sum_{C=1}^{M}\left(\int_0^\infty \mathrm{d}s\, s\langle\hat{J}^\mu(0),\hat{J}_\mu(s)\rangle\right)_{AC}\left(\int_0^\infty \mathrm{d}s\langle\hat{J}^\mu(0),\hat{J}_\mu(s)\rangle\right)_{CB}^{-1}$$

Kikuchi, Tsumura, TK , PRC(2015)

Correlation times!, which are different
for the (respective) dissipative quantities.

c.f. Israel–Stewart 14 moment formulae:

$$\tau_\Pi^{\mathrm{IS}} \equiv -\frac{\langle\Pi,\Pi\rangle}{\langle\Pi,\hat{L}\Pi\rangle} \quad \tau_J^{\mathrm{IS}} \equiv -\frac{\langle J^\mu, J_\mu\rangle}{\langle J^\rho,\hat{L}J_\rho\rangle} \quad \tau_\pi^{\mathrm{IS}} \equiv -\frac{\langle\pi^{\mu\nu},\pi_{\mu\nu}\rangle}{\langle\pi^{\rho\sigma},\hat{L}\pi_{\rho\sigma}\rangle}$$

G. Denicol, H. Niemi, E. Molnar, D. Rischke,
PRD 85 (2012);An elaborated moments expansion
with **41** moments.

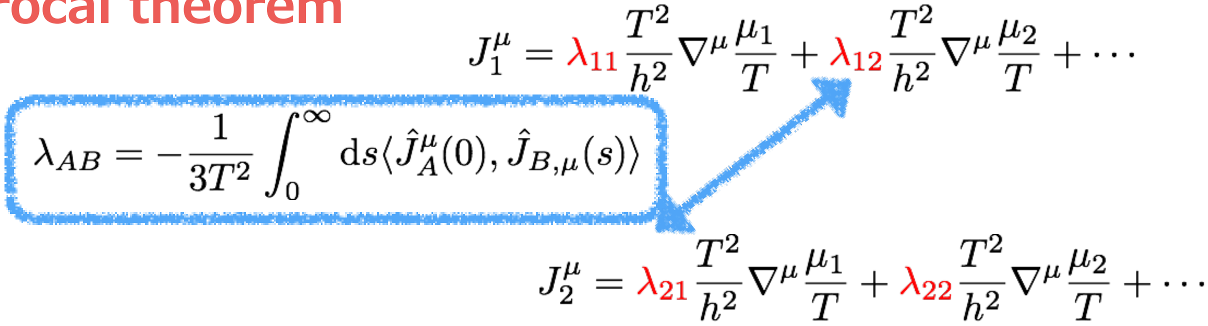
Comparison with other methods:

	RG	Israel-Stewart	Denicol et al.
$\eta\ [T/\sigma_T]$	1.27	1.2	1.267
$\tau_\pi\ [1/n\sigma_T]$	1.66	1.8	2

Properties of resultant hydrodynamic eq,

Following properties are proved: Y.Kikuchi, K. Tsumura, and T. K , PRC92 (2015)

- ☑ **Causality** (Propagating velocities of fluctuation of hydrodynamic variables do not exceed the light speed)
- ☑ **Stability** (Equilibrium state is stable for any perturbation)
- ☑ **Positive definiteness of the entropy production rate**
- ☑ **Onsager's reciprocal theorem**



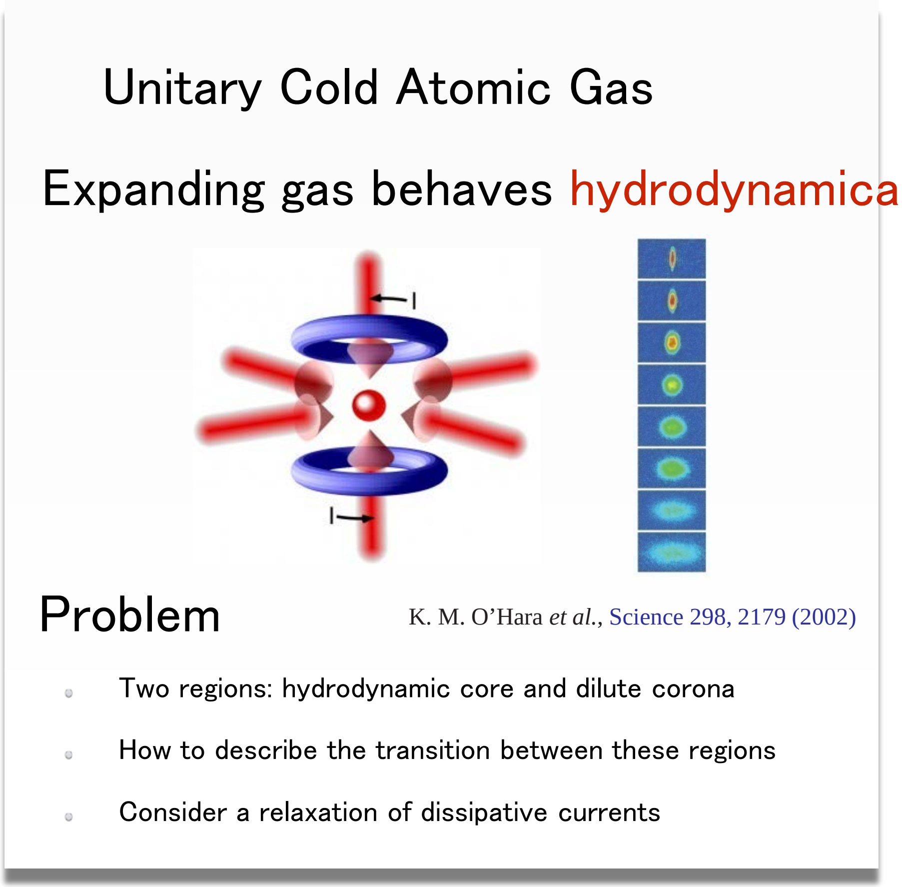
$$J_1^\mu = \lambda_{11}\frac{T^2}{h^2}\nabla^\mu\frac{\mu_1}{T} + \lambda_{12}\frac{T^2}{h^2}\nabla^\mu\frac{\mu_2}{T} + \cdots$$

$$\boxed{\lambda_{AB} = -\frac{1}{3T^2}\int_0^\infty \mathrm{d}s\langle \hat{J}^\mu_A(0), \hat{J}_{B,\mu}(s)\rangle}$$

$$J_2^\mu = \lambda_{21}\frac{T^2}{h^2}\nabla^\mu\frac{\mu_1}{T} + \lambda_{22}\frac{T^2}{h^2}\nabla^\mu\frac{\mu_2}{T} + \cdots$$

Indicating that our way of solution respect the fundamental property of Boltzmann equation that the microscopic process is time-reversal invariant..

Derivation of Second-order hydrodynamic equations: Similarity of rel H-I with (unitary) cold atomic gass



Unitary Cold Atomic Gas

Expanding gas behaves hydrodynamically.

K. M. O'Hara *et al.*, Science 298, 2179 (2002)

Problem

- Two regions: hydrodynamic core and dilute corona
- How to describe the transition between these regions
- Consider a relaxation of dissipative currents

Strongly correlated quantum fluid P. K. Kovtun *et al.*, Phys. Rev. Lett. 94, 111601 (2005

nearly perfect fluid $\frac{\eta}{s} \gtrsim \frac{\hbar}{4\pi k_B}$

From Boltzmann eq. with mean field to hydrodynamic eq.

Y. Kikuchi, K. Tsumura and T.K. Phys. Lett. A380 (2016),2075.

Boltzmann eq.

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\boldsymbol{\nabla}+\boldsymbol{F}\cdot\boldsymbol{\nabla}_p\right)f_p(t,\boldsymbol{x})=C[f]_p(t,\boldsymbol{x})$$

$$\boldsymbol{F}=-\boldsymbol{\nabla}E_p(\boldsymbol{x})=-\boldsymbol{\nabla}V(\boldsymbol{x})$$

ϵ : measure of the inhomogeneity of fluid

⬇

$$\left(\frac{\partial}{\partial t}+\epsilon\boldsymbol{v}\cdot\boldsymbol{\nabla}+\epsilon\boldsymbol{F}\cdot\boldsymbol{\nabla}_p\right)f_p(t,\boldsymbol{x})=C[f]_p(t,\boldsymbol{x})$$

To which the RG method is applied to obtain the 2nd–order hydrodynamic equations, together with the microscopic expressions of the transport coefficients and relaxation times

Y. Kikuchi, K. Tsumura and T.K. Phys. Lett. A380 (2016),2075.

Shear viscosity

$$f^{\mathrm{eq}}_{\mathrm{classical}} = \mathrm{e}^{-(p^2/2m-\mu)/T}, \qquad f^{\mathrm{eq}}_{\mathrm{Fermi}} = \frac{1}{\mathrm{e}^{(p^2/2m-\mu)/T}+1}$$

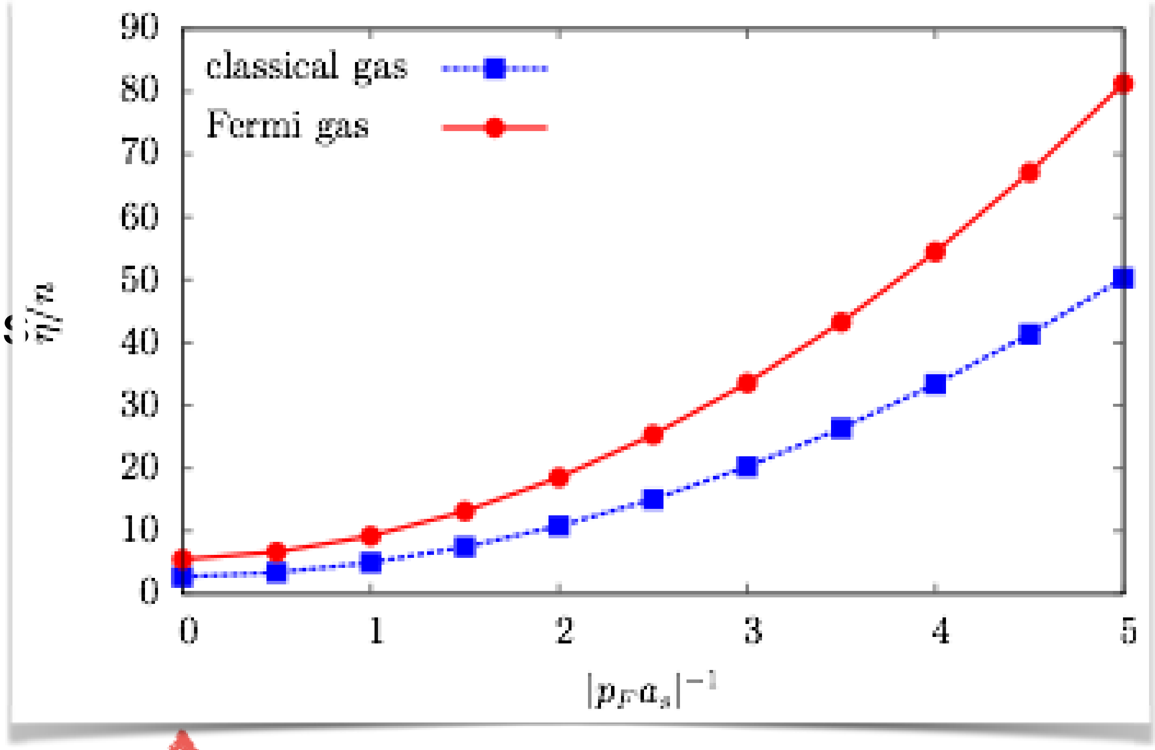
S-wave scattering

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{(1/\underline{a_s})^2 + \underline{q}^2}$$

scattering length

relative momentum

scattering length dependence



unitary limit

Microscopic expressions

$$\eta = \frac{1}{10T}\int_0^\infty \mathrm{d}s \langle \hat{\pi}^{ij}(0), \hat{\pi}^{ij}(s)\rangle$$

$$= -\frac{1}{10T}\langle \hat{\pi}^{ij}, L^{-1}\hat{\pi}^{ij}\rangle$$

$$L_{pq} \equiv \frac{\delta}{\delta f_q} C[f]_p(t)\Big|_{f=f^{\mathrm{eq}}}$$

$$\langle \psi, \chi \rangle \equiv \int_p f^{\mathrm{eq}}_p (1 + a f^{\mathrm{eq}}_p)\psi_p \chi_p$$

$$\hat{\pi}^{ij}_p(s) \equiv [\mathrm{e}^{sL}\hat{\pi}^{ij}]_p$$

Y. Kikuchi, K. Tsumura and T.K. Phys. Lett. A380 (2016),2075.

Viscous relaxation times from kinetic theory

Boltzmann eq. $\left(\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} + \boldsymbol{F} \cdot \boldsymbol{\nabla}_p\right) f_p(t, \boldsymbol{x}) = C[f]_p(t, \boldsymbol{x})$ $\quad L_{pq} \equiv \frac{\delta}{\delta f_q} C[f]_p(t)\Big|_{f=f^{\mathrm{eq}}}$

Microscopic expressions

$$\tau_\pi^{\mathrm{exact}} = \frac{1}{10T\eta} \langle \hat{\pi}^{ij}, L^{-2} \hat{\pi}^{ij} \rangle$$

$$\tau_J^{\mathrm{exact}} = \frac{1}{3T^2\lambda} \langle \hat{J}^i, L^{-2} \hat{J}^i \rangle$$

$$\frac{\mathrm{D}n}{\mathrm{D}t} = -n \boldsymbol{\nabla} \cdot \boldsymbol{u},$$

$$mn \frac{\mathrm{D}u^i}{\mathrm{D}t} = -\nabla^i P + nF^i + \nabla^j \pi^{ij},$$

$$Tn \frac{\mathrm{D}s}{\mathrm{D}t} = \sigma^{ij} \pi^{ij} + \boldsymbol{\nabla} \cdot \boldsymbol{J}$$

$$\pi^{ij} = \eta \sigma^{ij} - \tau_\pi \frac{\mathrm{D}}{\mathrm{D}t} \pi^{ij} + \cdots$$

$$J^i = \lambda \nabla^i T - \tau_J \frac{\mathrm{D}}{\mathrm{D}t} J^i + \cdots$$

Test of reliability of the Relaxation-time approximation(RTA)

(BGK)

Y. Kikuchi, K. Tsumura and T.K. Phys. Lett. A380 (2016),2075.

$$\eta^{\rm exact} = -\frac{1}{10T}\langle \hat{\pi}^{ij}, L^{-1}\hat{\pi}^{ij}\rangle \quad \lambda^{\rm exact} = -\frac{1}{3T^2}\langle \hat{J}^i, L^{-1}\hat{J}^i\rangle$$

Our exact expressions

$$\tau_\pi^{\rm exact} = \frac{1}{10T\eta}\langle \hat{\pi}^{ij}, L^{-2}\hat{\pi}^{ij}\rangle$$

$$\tau_J^{\rm exact} = \frac{1}{3T^2\lambda}\langle \hat{J}^i, L^{-2}\hat{J}^i\rangle$$

(Improved) RTA

T. Schafer, Phys. Rev. A 90, 043633 (2014)

$$\tilde{\tau}_\pi^{\rm RTA} = \frac{\eta^{\rm exact}}{P}$$

$$\tilde{\tau}_J^{\rm RTA} = \frac{12mT\lambda^{\rm exact}}{(7Q - 75P^2/n)}$$

Relaxation-time approximation (RTA or BGK)

$$C[f]_p(t,\boldsymbol{x}) \simeq \frac{f(t,\boldsymbol{x}) - f^{\rm eq}(\boldsymbol{x})}{\tau}$$

$$\tau_\pi^{\rm RTA} = \tau_J^{\rm RTA} = \tau$$

$$\eta^{\rm RTA} = \tau P$$

G. M. Bruun and H. Smith, Phys. Rev. A **76**, 045602 (2007)

M. Braby, J. Chao, and T. Schafer, New J. Phys. **13**, 035014 (2011)

$$\lambda^{\rm RTA} = \frac{\tau}{12mT}\left(7Q - \frac{75P^2}{n}\right)$$

$$Q \equiv \frac{1}{m^2}\int_p \delta p^4 f_p^{\rm eq} \qquad \boldsymbol{\delta p} \equiv m(\boldsymbol{v} - \boldsymbol{u})$$

Viscous relaxation time of stress tensor

Kikuchi, Tsumura and T.K., PLA380(2016).

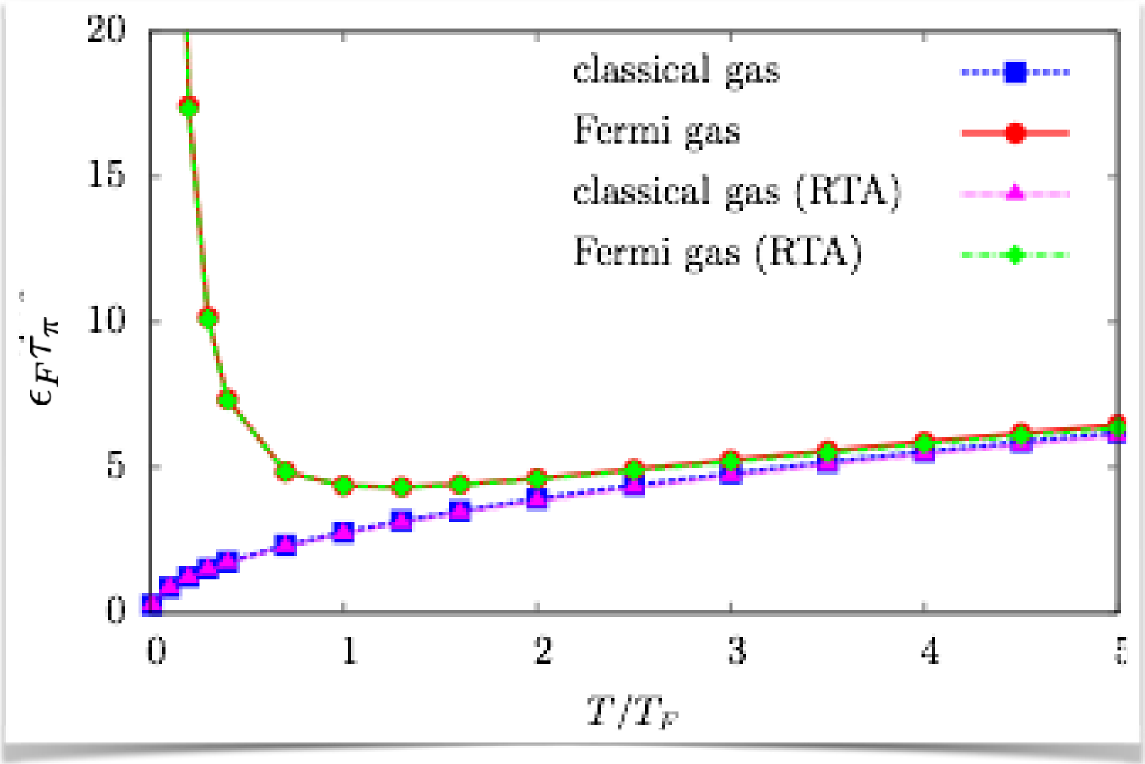
Our exact expressions

$$\tau_\pi^{\mathrm{exact}} = \frac{1}{10T\eta}\langle \hat{\pi}^{ij}, L^{-2}\hat{\pi}^{ij}\rangle$$

RTA

$$\tilde{\tau}_\pi^{\mathrm{RTA}} = \frac{\eta^{\mathrm{exact}}}{P}$$

temperature dependence



RTA well reproduces the exact results!!, which may also imply that our microscopic formulae of the relaxation times are correct!

Viscous relaxation time of heat conductivity

Kikuchi, Tsumura and T.K., PLA380(2016).

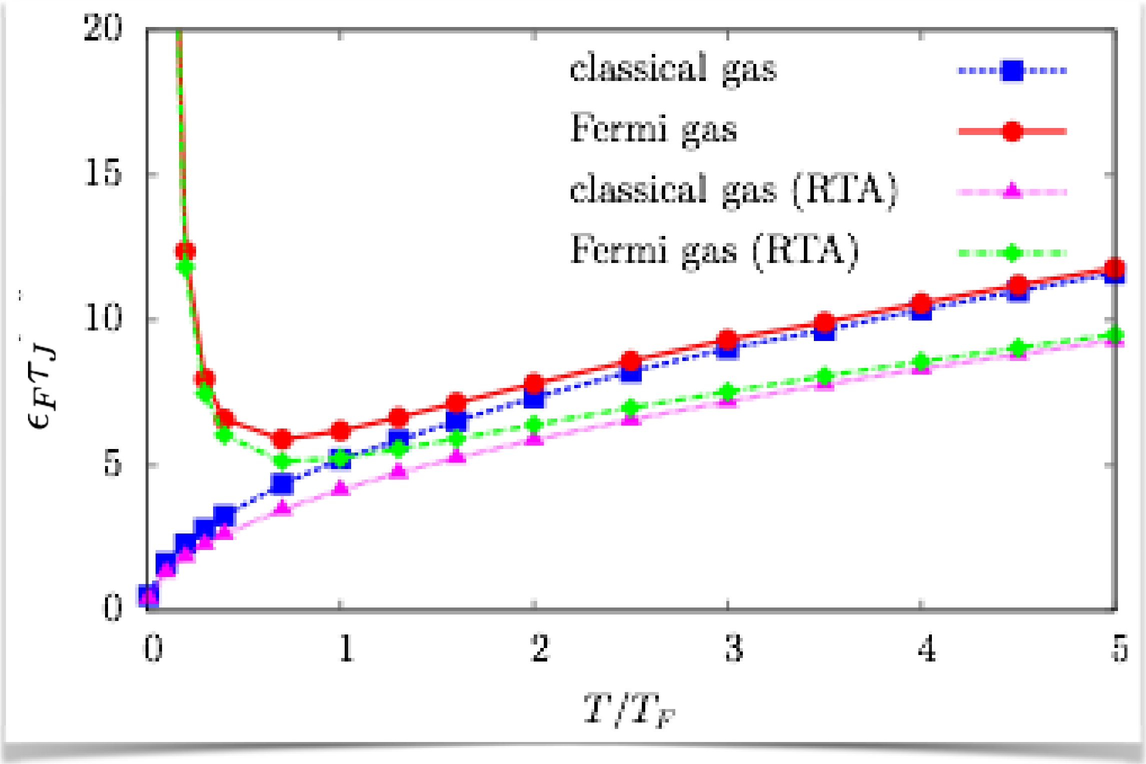
temperature dependence

Exact expressions

$$\tau_J^{\mathrm{exact}} = \frac{1}{3T^2\lambda}\langle \hat{J}^i, L^{-2}\hat{J}^i \rangle$$

RTA

$$\tilde{\tau}_J^{\mathrm{RTA}} = \frac{12mT\lambda^{\mathrm{exact}}}{(7Q - 75P^2/n)}$$



RTA reproduces the exact results
with a considerable error ..

Considerably violated in contrast to τ_π

$\tau_\pi^{\mathrm{RTA}} = \tau_J^{\mathrm{RTA}} = \tau$ is clearly invalid.

Summary

- ☑ A geometrical formulation of the reduction of the dynamics is given on the basis of the renormalization-group/envelope method, which may give a partial and intermediate resolution of Hilbert's 6th problem.➔ variational principle (Hilbert)?

- ☑ The microscopic expressions of the transport coefficients that coincide with those of Chapman-Enskog and **viscous relaxation times** are derived from the Boltzmann equation (quasi-particle approx.) by an adaptation of the RG method, and **numerical evaluations are perfomed without recourse to any approximation.**

- ☑ **Quantum statistics** makes significant contributions to the shear viscosity (and the others as well).

- ☑ We have numerically examined that the relation $\tau_\pi = \eta/P$,which is derived in the RTA, **is satisfied quite well.**

- ☑ The analogous relation for τ_J is **satisfied only approximately**.

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