How to Reconcile Maxwell's Demon with the Second Law of Thermodynamics?

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Outline

• Introduction

• Information and Entropy

• Second Law with Quantum Feedback

• Second Law with Quantum Measurement

• Conclusion & Discussions
Thermodynamics in the Fluctuating World

**Thermodynamics of small systems**

Thermodynamic quantities are fluctuating!

- Second law
- Nonequilibrium thermodynamics

Greiner et al., Nature **462**, 74-77 (2009)

From NEC website
Maxwell’s Demon

Fundamental problem on thermodynamics and statistical mechanics since the 19th century

Observe the velocities of each molecules, and open or close the door...

Create a temperature difference?

Second law

Maxwell’s demon

J. C. Maxwell (1831-1879)
Demon from the Modern Viewpoint

Feedback control at the level of thermal fluctuations

Thermodynamics of information processing

- Foundation of the second law of thermodynamics
- Application to nanomachines and nanodevices
Szilard Engine (1929)

Free energy: \( F = E - TS \)

Work: \( \kappa_B T \ln 2 \)

Decrease by feedback

Can control physical entropy by using information

L. Szilard, Z. Phys. 53, 840 (1929)
Information Heat Engine

- System (Working engine)
- Memory (Controller)

Information Feedback

Free energy / work

Entropic cost

✓ Can increase the system’s free energy even if there is no energy flow between the system and the controller
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Shannon and von Neumann Entropies

Classical probability distribution \( \{p_k\} \)

Shannon entropy:
\[
H = - \sum_k p_k \ln p_k
\]
Randomness of the distribution

Quantum version:

Von Neumann entropy:
\[
S(\rho) = -\text{tr}\left[\rho \ln \rho\right]
\]
\( \rho \) : density operator
Entropy Production

System $S$

Heat bath $B$ (inverse temperature $\beta$)

Heat $Q$

Work $W$

Entropy production in the total system:

$$\Delta S_{SB} = \Delta S_S - \beta Q$$

Change in the von Neumann entropy of $S$

If the initial and the final states are canonical distributions:

$$\Delta S_{SB} = \beta(W - \Delta F)$$

Free-energy difference
Second Law of Thermodynamics

\[ \Delta S_{SB} \geq 0 \]

Holds true for nonequilibrium initial and final states

A lot of “derivations” have been known
(Positivity of the relative entropy, Its monotonicity,
Fluctuation theorem & Jarzynski equality, ...)

Review article: T. Sagawa, arXiv:1202.0983

If the initial state is the canonical distribution: \[ W \geq \Delta F \]
Mutual Information

System \( S \) \( \rightarrow \) Memory \( M \)

Measurement with stochastic errors

\[
I(S : M) \equiv H(S) + H(M) - H(SM)
\]

\[
0 \leq I \leq H(M)
\]

No information

No error

Ex. Binary symmetric channel

\[
I = \ln 2 + \varepsilon \ln \varepsilon + (1 - \varepsilon) \ln(1 - \varepsilon)
\]
Quantum Measurement

**Projection measurement** (error-free)

- Observable: \( A = \sum_k \alpha_k P_k \)
- Projection operators: \( \{ P_k \} \)
- Probability: \( p_k = \text{tr}(\rho P_k) \)
- Post-measurement state: \( \frac{1}{p_k} P_k \rho P_k \)

**General measurement**

- Kraus operators: \( \{ M_{k,i} \} \)
- \( k \) : measurement outcome
- POVM: \( \{ E_k \} \)
  - \( E_k = \sum_i M_{k,i}^\dagger M_{k,i} \)
  - \( \sum_k E_k = I \)
- Probability: \( p_k = \text{tr}(\rho E_k) \)
- Post-measurement state: \( \frac{1}{p_k} \sum_i M_{k,i} \rho M_{k,i}^\dagger \)

Assume that \( E_k = M_{k}^\dagger M_k \) (a generalization is a future problem)
QC-mutual Information (1)

\[ I_{QC} = H + S(\rho) + \sum_y \text{tr}\left( \sqrt{E_y} \rho \sqrt{E_y} \ln \sqrt{E_y} \rho \sqrt{E_y} \right) \]

\[ = S(\rho) - \sum_y p(y) S\left( \sqrt{E_y} \rho \sqrt{E_y} / p(y) \right) \]

\( \rho \) : measured state \hspace{1cm} E_y : POVM

\[ p(y) = \text{tr}[\rho E_y] \] : probability of obtaining outcome \( y \)

\[ H = -\sum_y p(y) \ln p(y) \] : Shannon information of the outcomes

\[ S(\rho) = -\text{tr}(\rho \ln \rho) \] : von Neumann entropy of the measured state

TS and M. Ueda, PRL 100, 080403 (2008).
QC-mutual Information (2)

\[ 0 \leq I_{QC} \leq H \]

- No information
- Error-free & classical

\[ E_y = p(y)I_d \]

Identity operator

For any \( y \quad [\rho, E_y] = 0 \)

\( E_y \) is a projection

Classical measurement:

For any \( y \quad [\rho, E_y] = 0 \) \( \rightarrow \) \( I_{QC} \) reduces to the classical mutual information

If the measured state is a pure state:

\[ I_{QC} = 0 \]
The QC-mutual information gives an upper bound of the accessible classical information $x$ encoded in $\rho$


$$\rho = \sum_x q(x) \rho_x : \text{an arbitrary decomposition}$$

$\rho_x$ are not necessarily orthogonal

$$p(x, y) = \text{tr}[E_y \rho_x] q(x) \quad p(y) = \text{tr}[\rho E_y] = \sum_x p(x, y)$$

$$I = \sum_{xy} p(x, y) \ln \frac{p(x, y)}{q(x) p(y)} \quad I_{QC} = S(\rho) - \sum_y p(y) S\left(\sqrt{E_y \rho} \sqrt{E_y} / p(y)\right)$$

Theorem: $I \leq I_{QC}$ A variant (a “dual”) of the Holevo bound
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Feedback: Control protocol depends on the measurement outcome
Generalized Second Law: Entropic Balance

\[ \Delta S_{SB} \geq -I_{QC} \]

TS and M. Ueda, PRL 100, 080403 (2008)
Generalized Second Law: Energetics

\[ \Delta S_{SB} \geq -I_{QC} \]

\[ W_{ext} \leq -\Delta F + k_B T I_{QC} \]

The upper bound of the work extracted by the demon is bounded by the QC-mutual information.

The equality can be achieved:

K. Jacobs, PRA 80, 012322 (2009)
T. Sagawa & M. Ueda, PRE 85, 021104 (2012)
Information Heat Engine

Conventional heat engine:
Heat $\rightarrow$ Work

Heat efficiency

$$e \equiv \frac{W_{\text{ext}}}{Q_{H}} \leq 1 - \frac{T_{L}}{T_{H}}$$

Carnot cycle

Information heat engine:
Mutual information $\rightarrow$ Work and Free energy

$$W_{\text{ext}} + \Delta F \leq k_{B} T I_{\text{QC}}$$

Szilard engine
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What about thermodynamics for measurement processes?
Generalized Second Law: Entropic Balance

TS and M. Ueda, PRL 102, 250602 (2009); 106, 189901(E) (2011).
Details of Memory

Standard state “0” of the memory with free energy $F_0$

The memory stores measurement outcome “k” with probability $p_k$.

Free-energy difference: $\Delta F \equiv \sum_k p_k F_k - F_0$

Symmetric memory $\Delta F = 0$

Asymmetric memory $\Delta F \neq 0$
Generalized Second Law: Energetics

$$\Delta S_{MB} \geq + I_{QC}$$

$$W_{\text{meas}} \geq -k_B TH + \Delta F + k_B TI_{QC}$$

$$H = -\sum_k p_k \ln p_k : \text{Shannon entropy of outcomes}$$

$$\Delta F = 0 \text{ (symmetric memory)} \& H = I_{QC} \text{ (classical and error-free measurement)}$$

$$W_{\text{meas}} \geq 0 \text{ (Bennett’s result)}$$

TS and M. Ueda, PRL 102, 250602 (2009); 106, 189901(E) (2011).
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Generalized Second Law: Summary

\[ \Delta S_{SB} \geq -I_{QC} \]

\[ \Delta S_{MB} \geq +I_{QC} \]

\[ \Delta S_{\text{total}} = \Delta S_{SB} + \Delta S_{MB} \geq 0 \]
“Duality” between Measurement and Feedback

Time-reversal transformation
Swap system and memory

Measurement becomes feedback (and vice versa)
Resolving the Paradox of Maxwell’s Demon

Sagawa-Ueda:

What compensates for the gain (negative entropy production) here?

The loss (positive entropy production) here compensates for it during measurement!

\[ \Delta S_{SB} \geq -I_{QC} \]

\[ \Delta S_{MB} \geq +I_{QC} \]

The additional entropy production (accompanied by an excess energy cost) during the measurement compensates for the negative entropy production during the feedback!

TS and M. Ueda, PRL 100, 080403 (2008)
TS and M. Ueda, PRL 102, 250602 (2009); 106, 189901(E) (2011)
Summary and Future Plans

✓ Generalized second law of thermodynamics with quantum information processing
✓ Corollary: Resolved Maxwell’s demon paradox

• Ultracold atoms
• Superconducting qubit
• Etc...

Thank you for your attentions!