13/02/12

Role of complementarity on Entanglement detection

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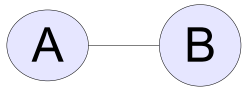
普遍性と創発性から紡ぐ
次世代物理学
フロンティア開拓のための
自立的人材養成

求職中

•Quantum entanglement



•Inseparability (entangled state)

Form of density operators

$$\rho_{AB} \neq \sum_i p_i \rho_{Ai} \otimes \rho_{Bi}$$

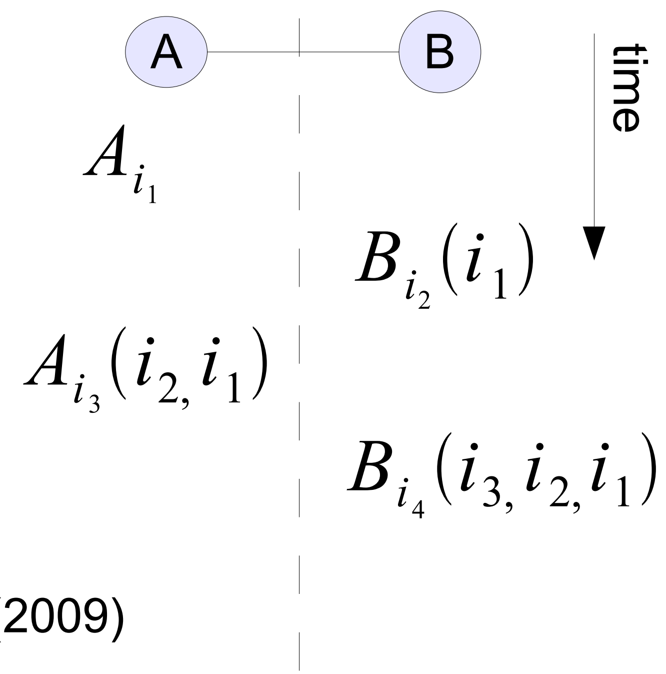
$$e.g.,\ |\psi\rangle_{AB} = \sum_i a_i |i\rangle_A |i\rangle_B \qquad \langle i|j\rangle = \delta_{i,j}$$

LOCC:
Local quantum
Operation &
Classical
Communication

•Entanglement measure (LOCC monotone)
Quantify the strength of quantum correlation

$$M(\rho_{AB}) \geq M(L_{LOCC}(\rho_{AB}))$$

$$L_{LOCC}(\rho_{AB}) = \sum_i B_{i_4} A_{i_3} B_{i_2} A_{i_1} \rho_{AB} A_{i_1}^\dagger B_{i_2}^\dagger A_{i_3}^\dagger B_{i_4}^\dagger$$
$$= \sum_i (K_{Ai} \otimes L_{Bi}) \rho_{AB} (K_{Ai} \otimes L_{Bi})^\dagger$$



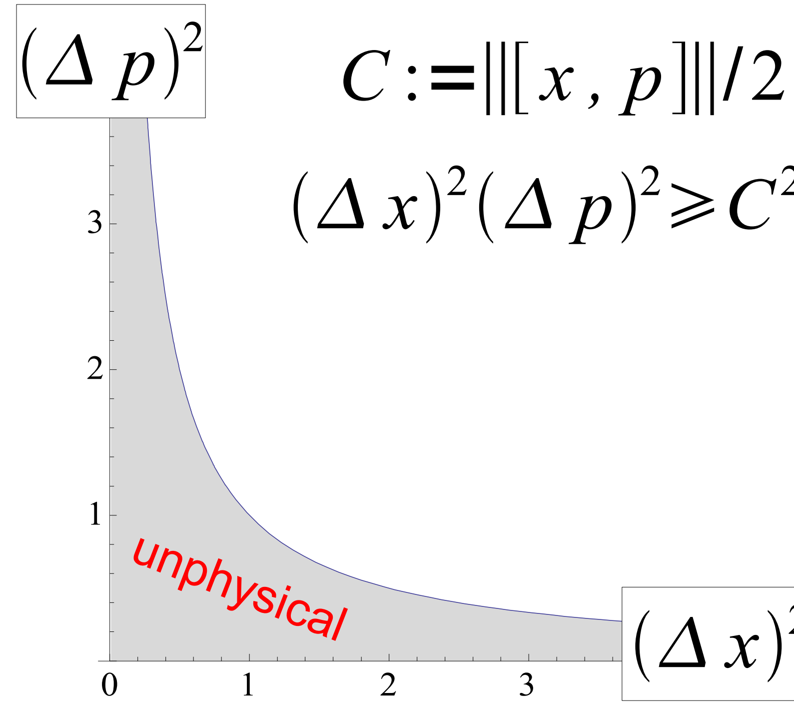
Guhne&Toth, Phys. Rep. 474, 1 (2009)
Horodeckis, Rev. Mod. Phys. 81, 865, (2009)

Basic concepts on Quantum mechanics

- <u>Canonical uncertainty relation</u>

Trade off

$$\langle \Delta^2 \hat{x} \rangle \langle \Delta^2 \hat{p} \rangle \geq 1/4 \, \| [x, p] \|$$

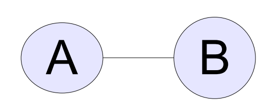


$$C := \| [x, p] \| / 2$$

$$(\Delta x)^2 (\Delta p)^2 \geqslant C^2$$

- <u>Quantum entanglement</u>

- Inseparability $\quad \rho_{AB} \neq \sum_i p_i \rho_{Ai} \otimes \rho_{Bi}$



$$e.g., |\psi\rangle_{AB} = \sum_i a_i |i\rangle_A |i\rangle_B \qquad \langle i | j \rangle = \delta_{i,j}$$

- Entanglement measure

$$M(\rho_{AB}) \geq M(L_{LOCC}(\rho_{AB}))$$

Einstein-Podolsky-Rosen (EPR) state

$$|\psi\rangle_{AB} := \int dx\, |x\rangle_A |x\rangle_B / \sqrt{2\pi}$$

Simultaneous eigenstate of

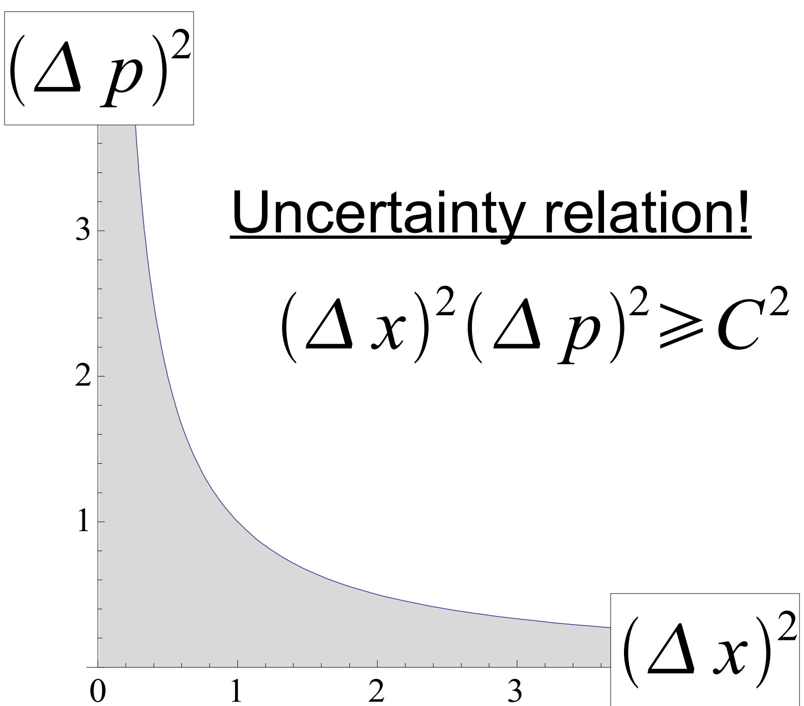
$$\hat{p}_A + \hat{p}_B,\ \hat{x}_A - \hat{x}_B$$

Positions are correlated and
Momentums are anti-correlated

$$\langle \Delta^2(\hat{p}_A + \hat{p}_B)\rangle \sim 0;\ \langle \Delta^2(\hat{x}_A - \hat{x}_B)\rangle \sim 0$$

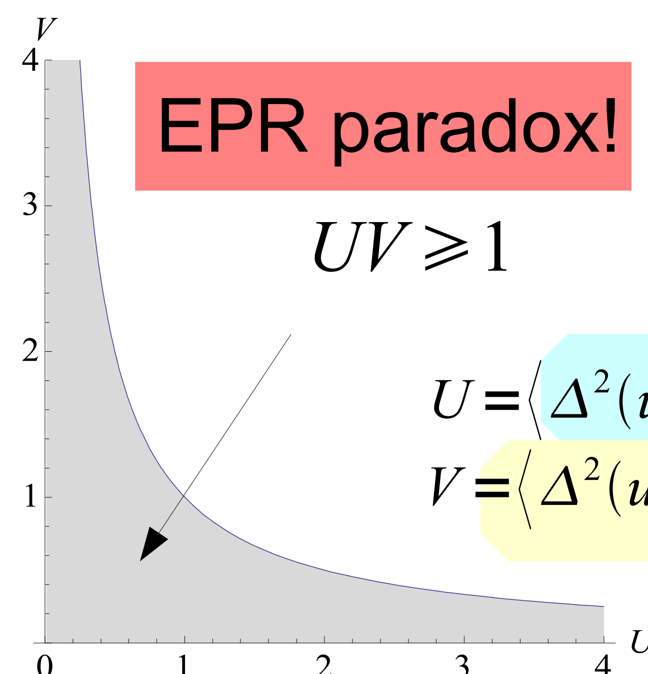
A ... B

$|x_1\rangle$ .... $|x_1\rangle$
$|x_2\rangle$ .... $|x_2\rangle$
$|x_3\rangle$ .... $|x_3\rangle$
⋮
$|p_1\rangle$ .... $|-p_1\rangle$
$|p_2\rangle$ .... $|-p_2\rangle$
$|p_3\rangle$ .... $|-p_3\rangle$
⋮



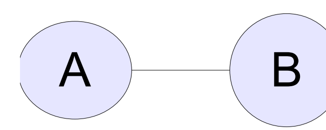
Uncertainty relation!

$$(\Delta x)^2 (\Delta p)^2 \geqslant C^2$$



EPR paradox!

$$UV \geqslant 1$$



$$u^2 + v^2 = 1$$

$$U = \langle \Delta^2(u\,\hat{x}_A - v\,\hat{x}_B)\rangle / C$$

$$V = \langle \Delta^2(u\,\hat{p}_A + v\,\hat{p}_B)\rangle / C$$

Entanglement detection via EPR paradox

Product criterion for entanglement

$$\left\langle \Delta^2 (u\hat{x}_A - v\hat{x}_B) \right\rangle \left\langle \Delta^2 (u\hat{p}_A + v\hat{p}_B) \right\rangle < C^2$$

$$C := \left| [x, p] \right| / 2$$

$$\left\langle \Delta^2 (u\hat{x}_A - v\hat{x}_B) \right\rangle \left\langle \Delta^2 (u\hat{p}_A - v\hat{p}_B) \right\rangle \geqslant C^2$$

Uncertainty relation!

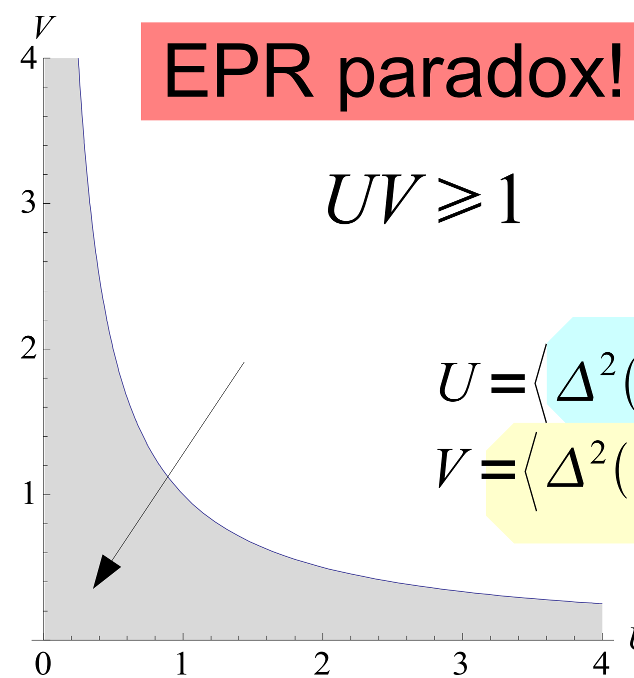
V. Giovannetti et al., Phys. Rev. A 67, 022320 (2003)

Stronger Correlation beyond the uncertainty limit

- Quantum entanglement

$$\rho_{AB} \neq \sum_i p_i \rho_{Ai} \otimes \rho_{Bi}$$

Give a constraint on the form
Of the density operator



EPR paradox!

$$UV \geqslant 1$$

$$u^2 + v^2 = 1$$

$$U = \left\langle \Delta^2 (u\hat{x}_A - v\hat{x}_B) \right\rangle / C$$

$$V = \left\langle \Delta^2 (u\hat{p}_A + v\hat{p}_B) \right\rangle / C$$

Complementary correlations for entanglement

- Maximally entangled state (of two qubits)

$$Z|0\rangle=|0\rangle \qquad X|\bar{0}\rangle=|\bar{0}\rangle$$
$$Z|1\rangle=-|1\rangle \qquad X|\bar{1}\rangle=-|\bar{1}\rangle$$

$$|\Phi_0\rangle=(|0\,0\rangle+|1\,1\rangle)/\sqrt{2}$$
$$=(|\bar{0}\,\bar{0}\rangle+|\bar{1}\,\bar{1}\rangle)/\sqrt{2}$$

$$|\bar{0}\rangle=(|0\rangle+|1\rangle)/\sqrt{2}$$
$$|\bar{1}\rangle=(|0\rangle-|1\rangle)/\sqrt{2}$$

Simultaneous eigenstate of product Pauli operators:

$$\hat{X}_A\hat{X}_B,\ \ \hat{Z}_A\hat{Z}_B$$

$$\hat{Z}:=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

Strong correlations on the conjugate variables

$$\hat{X}:=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$\langle\hat{Z}_A-\hat{Z}_B\rangle=0 \qquad \langle\hat{X}_A-\hat{X}_B\rangle=0$$

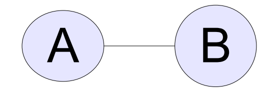
An average correlation of the Z-basis bits and X-basis bits exceeds 75%

$$\frac{1}{2}\sum_{j=0,1}\langle |j\rangle\langle j|\otimes|j\rangle\langle j| + |\bar{j}\rangle\langle\bar{j}|\otimes|\bar{j}\rangle\langle\bar{j}|\rangle > \frac{3}{4} \iff \langle\hat{X}_A\hat{X}_B+\hat{Z}_A\hat{Z}_B\rangle>1$$

$\Rightarrow$ the state is entangled:

$$\rho_{AB}\neq\sum_i p_i\,\rho_{Ai}\otimes\rho_{Bi}$$

Uncertainty relations and entanglement



Continuous-variable systems
Continuous-variable entanglement

$$\left\langle \Delta^2 (u\,\hat{x}_A - v\,\hat{x}_B) \right\rangle \left\langle \Delta^2 (u\,\hat{p}_A + v\,\hat{p}_B) \right\rangle < C^2$$

$$C := |[x, p]|/2$$

Pair of two-levels systems
Qubit-Qubit entanglement

$$\left\langle \hat{X}_A \hat{X}_B + \hat{Z}_A \hat{Z}_B \right\rangle > 1$$

$$\frac{1}{2} \sum_{j=0,1} \langle |j\rangle\langle j| \otimes |j\rangle\langle j| + |\bar{j}\rangle\langle \bar{j}| \otimes |\bar{j}\rangle\langle \bar{j}| \rangle > \frac{3}{4}$$

Pair of d-level systems
Qudit-Qudit entanglement

Strength of measured correlations

Uncertainty relations?

- Fourier-based uncertainty relations
- Generalized Pauli-operators on *d-level* systems

R. Namiki and Y. Tokunaga, Phys. Rev. Lett. 108, 230503 (2012)

Complementary elements and Uncertainty Relations

Conjugate bases: on *d*-level system

Z-basis $\{|0\rangle, |1\rangle, \cdots, |d-1\rangle\}$

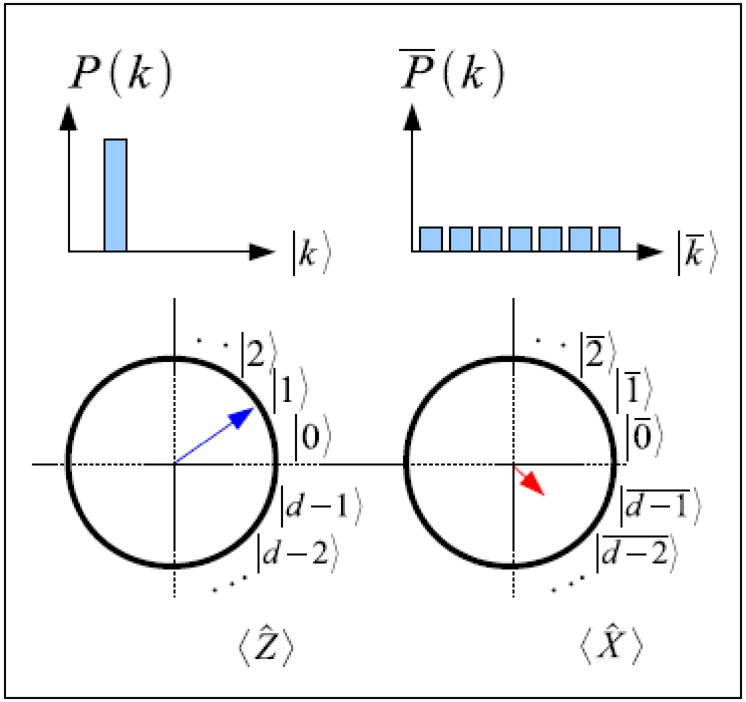
X-basis $|\bar{j}\rangle := \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{ik\omega j} |k\rangle$

Two Fourier distributions $P(j) := \langle j|\rho|j\rangle \quad \bar{P}(j) := \langle \bar{j}|\rho|\bar{j}\rangle$

Trade-off

Cannot have sharp peaks simultaneously

Never coexist on the unit circle (At least one is inside)



Two (generalized) Pauli operators

$$\hat{Z} := \sum_{j=0}^{d-1} e^{i\omega j} |j\rangle\langle j|, \quad \hat{X} := \sum_{j=0}^{d-1} |j+1\rangle\langle j|$$

$$\langle Z\rangle = \sum_j P(j) e^{i\omega j} \qquad \omega := 2\pi/d \qquad = \sum_{j=0}^{d-1} e^{i\omega j} |\bar{j}\rangle\langle\bar{j}|$$

Complementary elements and Uncertainty Relations

Conjugate bases: on *d*-level system

Z-basis $\{|0\rangle, |1\rangle, \cdots, |d-1\rangle\}$

X-basis $|\bar{j}\rangle := \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{ik\omega j} |k\rangle$

Discrete Fourier-based Uncertainty relations

$$\sum_{j=0}^{d-1} (P^2(j) + \bar{P}^2(j)) \le 1 + \frac{1}{d}.$$

U. Larsen, J. Phys. A: Math. Gen. 23, 1041 (1990).

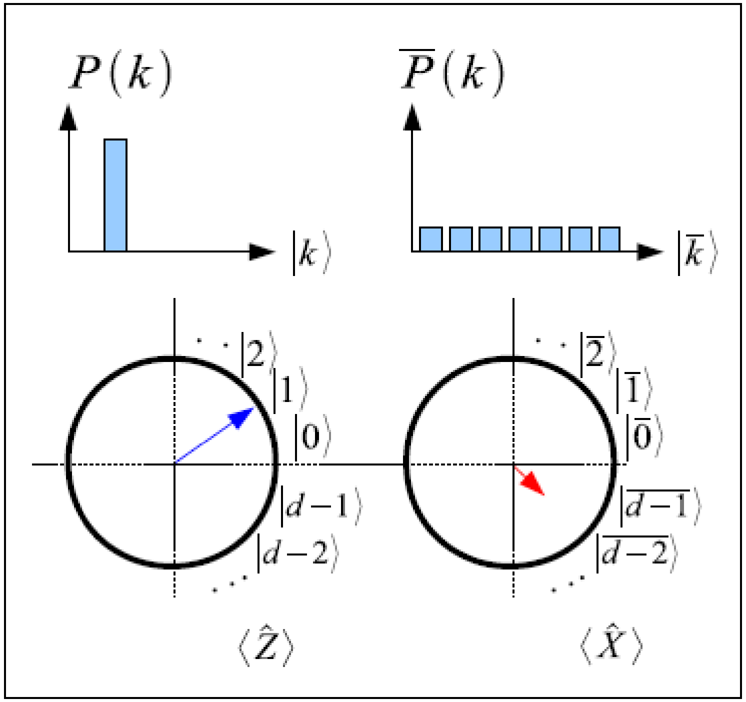
$$|\langle Z\rangle| \cos\theta + |\langle X\rangle| \sin\theta$$

$$\le \frac{1}{2} \|(Z + Z^\dagger) \cos\theta + (X + X^\dagger) \sin\theta\|$$

$$\|\hat{O}\| := \max_{\langle u|u\rangle=1} |\langle u| \hat{O} |u\rangle|.$$

S. Massar and P. Spindel, Phys. Rev. Lett. 100, 190401 (2008)

$P(j) := \langle j|\rho|j\rangle \quad \bar{P}(j) := \langle \bar{j}|\rho|\bar{j}\rangle$



$$\hat{Z} := \sum_{j=0}^{d-1} e^{i\omega j} |j\rangle\langle j|, \quad \hat{X} := \sum_{j=0}^{d-1} |j+1\rangle\langle j|$$

$$\omega := 2\pi/d \qquad = \sum_{j=0}^{d-1} e^{i\omega j} |\bar{j}\rangle\langle \bar{j}|$$

Theorem. *The state is entangled if it satisfies either of*

d × d level system

$$\left\langle \sum_{j=0}^{d-1} \left(|j\rangle\langle j| \otimes |j\rangle\langle j| + |\bar{j}\rangle\langle\bar{j}| \otimes |\overline{-j}\rangle\langle\overline{-j}| \right) \right\rangle > 1 + \frac{1}{d}.$$

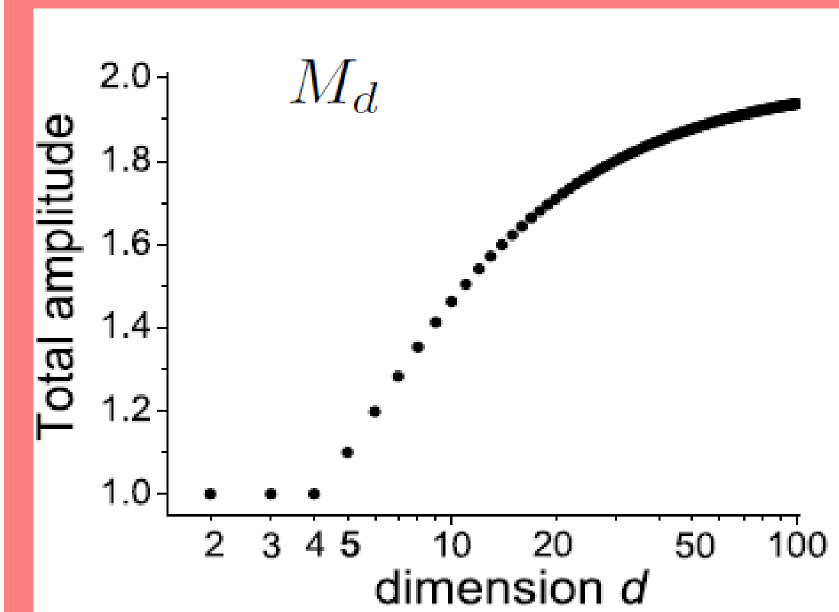
$$\langle \hat{Z}_A \hat{Z}_B^\dagger + \hat{Z}_A^\dagger \hat{Z}_B + \hat{X}_A \hat{X}_B + \hat{X}_A^\dagger \hat{X}_B^\dagger \rangle > 2M_d$$

2 qubits (d =2)

$$\frac{1}{2}\sum_{j=0,1} \langle |j\rangle\langle j| \otimes |j\rangle\langle j| + |\bar{j}\rangle\langle\bar{j}| \otimes |\bar{j}\rangle\langle\bar{j}| \rangle > \frac{3}{4}$$

$$\left\langle \hat{X}_A \hat{X}_B + \hat{Z}_A \hat{Z}_B \right\rangle > 1$$

R. Namiki and Y. Tokunaga,
Phys. Rev. Lett. 108, 230503 (2012)



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$$|\langle Z\rangle| \cos\theta + |\langle X\rangle| \sin\theta$$

$$\le \frac{1}{2} \|(Z + Z^\dagger)\cos\theta + (X + X^\dagger)\sin\theta\|$$

$$\|\hat{O}\| := \max_{\langle u|u\rangle=1} |\langle u|\hat{O}|u\rangle|.$$

S. Massar and P. Spindel, Phys. Rev. Lett. 100, 190401 (2008)

$$M_d = \max_\phi \left[|\langle\hat{Z}\rangle_\phi|^2 + |\langle\hat{X}\rangle_\phi|^2 \right]$$

$$= \left(\max_{0\le\theta\le\frac{\pi}{2}} \frac{1}{2} \|(Z + Z^\dagger)\cos\theta + (X + X^\dagger)\sin\theta\| \right)^2$$

R. Namiki and Y. Tokunaga, Phys. Rev. Lett. 108, 230503 (2012)

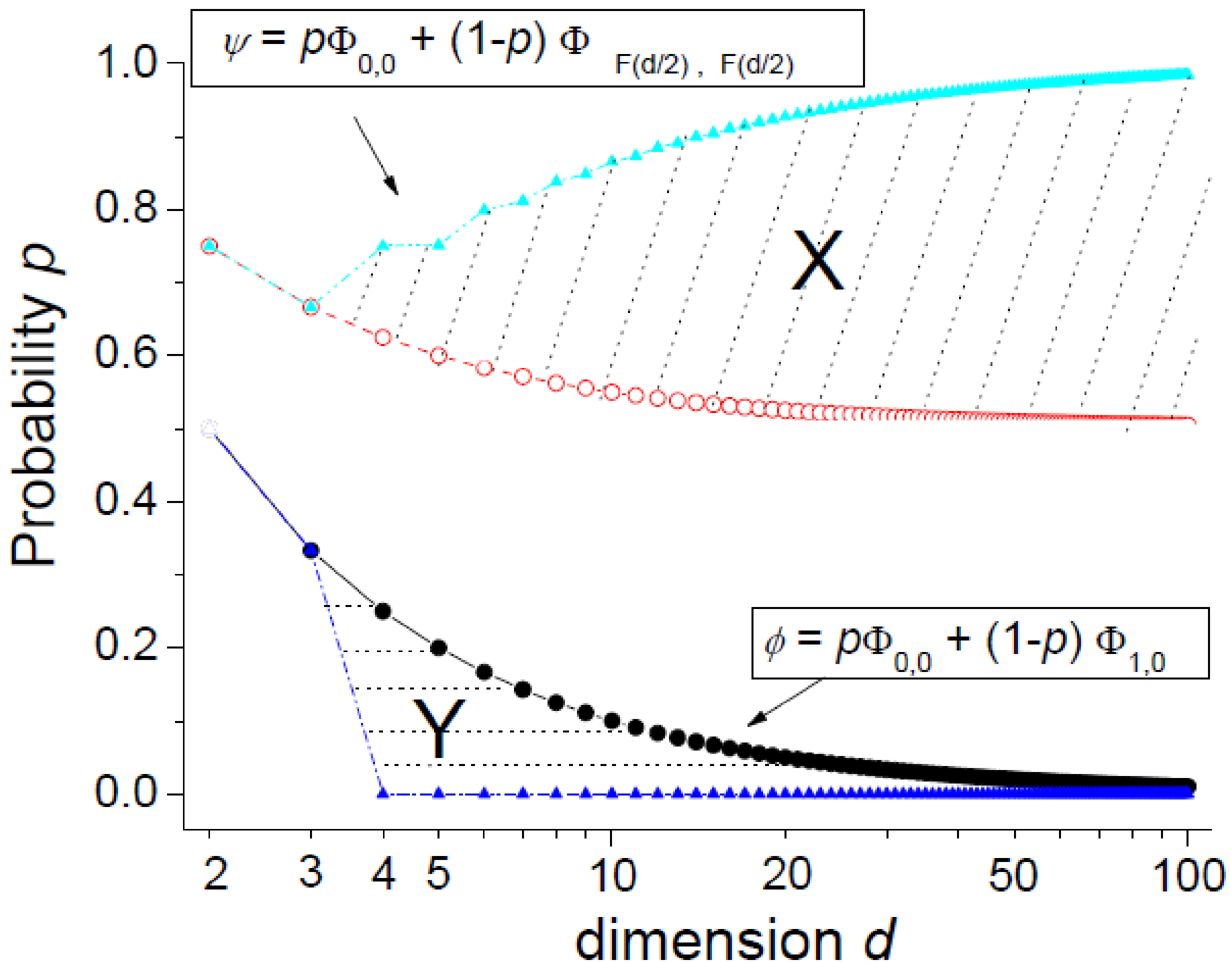
Theorem. *The state is entangled if it satisfies either of*

○ $$\left\langle \sum_{j=0}^{d-1} \left(|j\rangle\langle j| \otimes |j\rangle\langle j| + |\bar{j}\rangle\langle\bar{j}| \otimes |\overline{-j}\rangle\langle\overline{-j}| \right) \right\rangle > 1 + \frac{1}{d}.$$

△ $$\langle \hat{Z}_A \hat{Z}_B^\dagger + \hat{Z}_A^\dagger \hat{Z}_B + \hat{X}_A \hat{X}_B + \hat{X}_A^\dagger \hat{X}_B^\dagger \rangle > 2M_d$$

For d= 2,3 two conditions are equivalent.
For $d \geqq 4$ there are mutually exclusive subsets.

R. Namiki and Y. Tokunaga,
Phys. Rev. Lett. 108, 230503 (2012)



X: Verified to be entangled by the first condition ○.

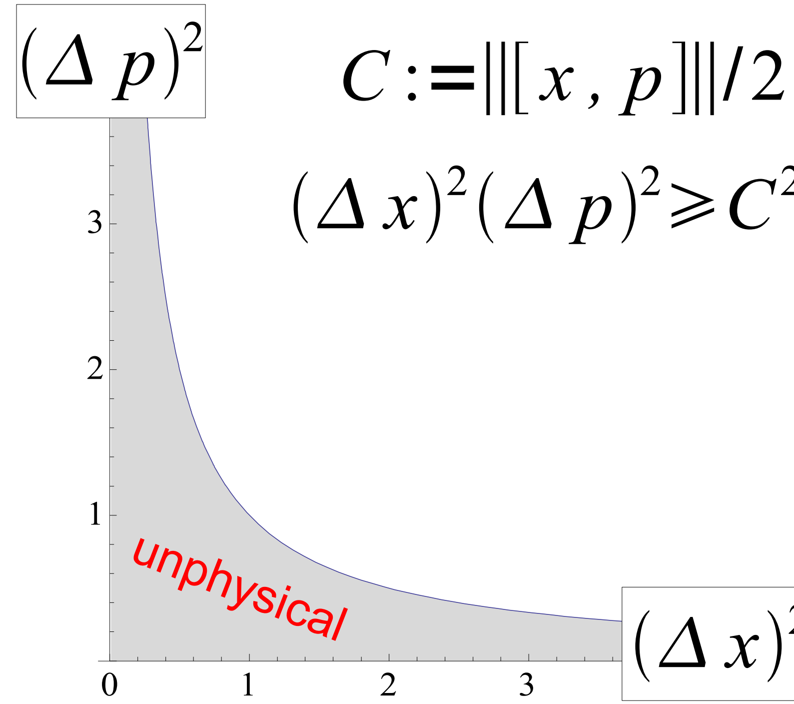
Y: Verified to be entangled by the first condition △.

$$|\Phi_{0,0}\rangle := \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_A |j\rangle_B$$

$$|\Phi_{l,m}\rangle = \hat{X}_A^l \hat{Z}_B^m |\Phi_{0,0}\rangle$$

$F(x)$: *Floor function*

Basic concepts on Quantum mechanics

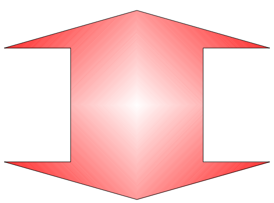


$$C := \| [x, p] \| / 2$$

$$(\Delta x)^2 (\Delta p)^2 \geqslant C^2$$

- <u>Canonical uncertainty relation</u>

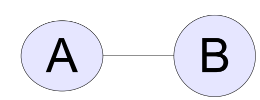
Trade off



$$\langle \Delta^2 \hat{x} \rangle \langle \Delta^2 \hat{p} \rangle \geq 1/4 \| [x, p] \|$$

- <u>Quantum entanglement</u>

- Inseparability $\quad \rho_{AB} \neq \sum_i p_i \rho_{Ai} \otimes \rho_{Bi}$



$$e.g., |\psi\rangle_{AB} = \sum_i a_i |i\rangle_A |i\rangle_B \qquad \langle i | j \rangle = \delta_{i,j}$$

- Entanglement measure

$$M(\rho_{AB}) \geq M(L_{LOCC}(\rho_{AB}))$$

Theorem. *Multi-level coherence*

$$\langle \hat{C}_d \rangle > 1 + \frac{k-1}{d},$$

$$\langle \hat{R}_d \rangle > \frac{(d-k+1)\cos\omega + (d+k-1)}{d}$$

$$(1 \le k \le d)$$

$$\Rightarrow \quad \rho_{AB} \neq \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

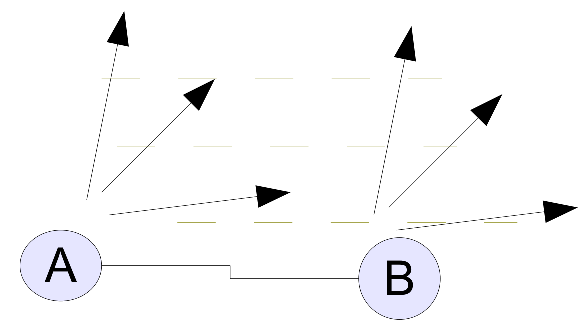
$$|\psi_i\rangle = \sum_{i=0}^{k-1} a_i |u_i\rangle_A \otimes |v_i\rangle_B$$

Total correlations

$$|\psi\rangle_{AB} \propto |0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle + \ldots..$$

$$\hat{C}_d := \sum_{j=0}^{d-1} (|j\rangle\langle j| \otimes |j\rangle\langle j| + |\bar{j}\rangle\langle\bar{j}| \otimes |\overline{-j}\rangle\langle\overline{-j}|).$$

$$\hat{R}_d := \tfrac{1}{2}(\hat{Z}_A \hat{Z}_B^\dagger + \hat{Z}_A^\dagger \hat{Z}_B) + \tfrac{1}{2}(\hat{X}_A \hat{X}_B + \hat{X}_A^\dagger \hat{X}_B^\dagger).$$



The state needs to include # of *k+1* coherent superposition of the product states

The figure ***k*** is called the Schmidt number which can quantify entanglement (entanglement monotone).

$$For\; k = 1$$

$$\rho_{AB} \neq \sum_i p_i \rho_{Ai} \otimes \rho_{Bi} = \sum_i p_i' |\phi_i\rangle\langle\phi_i| \otimes |\varphi_i\rangle\langle\varphi_i|$$

R. Namiki and Y. Tokunaga, Phys. Rev. Lett. 108, 230503 (2012)

Theorem. *Multi-level coherence of Quantum Gates*

$$F > F^{(k-1)} = \frac{1}{2}\left(1+\frac{k}{d}\right) \Rightarrow \rho' = E(\rho) \neq \sum_i A_i \rho A_i^\dagger \quad rank(A_i) \le k$$

Input-output correlation

$$F = \frac{1}{2d} \sum_i \left(\underbrace{\langle i | E(|i\rangle\langle i|) | i \rangle}_{\text{Z-basis}} + \underbrace{\langle \bar{i} | E(|\bar{i}\rangle\langle \bar{i}|) | \bar{i} \rangle}_{\text{X-basis}} \right)$$

Description by less-than rank-*k* Kraus operators is not admissible!

$$\exists A_i \; s.t., \; rank(A_i) > k$$

$\rho \rightarrow$ E $\rightarrow \rho'$

Trace-preserving

$$Tr\,\rho' = Tr\,\rho = 1$$

Ideal unitary gates

$$E_{ideal}(\rho) = U \rho U^\dagger \qquad U^\dagger U = 1 \qquad rank(U) = d$$

General physical maps

$$\rho' = E(\rho) = \sum_i A_i \rho A_i^\dagger \qquad \sum_i A_i^\dagger A_i = 1$$

$$rank(A) \le k$$

$$\rightarrow A_A |\psi\rangle \propto \sum_{i=0}^{k-1} a_i |u_i\rangle_A \otimes |v_i\rangle_B$$

Degrade Schmidt number Less-than *k*

R. Namiki and Y. Tokunaga, Phys. Rev. A 85, 010305(R) (2012).

Application for known experiments

$$F > F^{(k-1)} = \frac{1}{2}\left(1+\frac{k}{d}\right) \quad \Rightarrow \quad \rho' = E(\rho) \neq \sum_i A_i \rho A_i^{\dagger}, \quad rank(A_i) \leq k$$

Input-output correlation (Average fidelity)

$$F = \frac{1}{2d}\sum_i \left(\langle U_i | E(|i\rangle\langle i|) | U_i \rangle + \langle U_{\bar{i}} | E(|\bar{i}\rangle\langle \bar{i}|) | U_{\bar{i}} \rangle \right)$$

A basic elements of quantum computer: CNOT gate

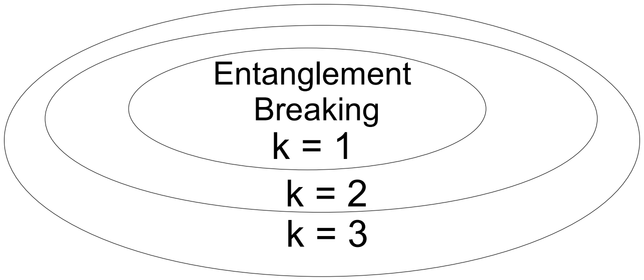
d = 4
C-Not Gate

$U_{C-NOT} : |i\rangle \rightarrow |U_i\rangle$

$|0\rangle|0\rangle \rightarrow |0\rangle|0\rangle$
$|0\rangle|1\rangle \rightarrow |0\rangle|1\rangle$
$|1\rangle|0\rangle \rightarrow |1\rangle|1\rangle$
$|1\rangle|1\rangle \rightarrow |1\rangle|0\rangle$

$|\bar{0}\rangle|\bar{0}\rangle \rightarrow |\bar{0}\rangle|\bar{0}\rangle$
$|\bar{0}\rangle|\bar{1}\rangle \rightarrow |\bar{1}\rangle|\bar{1}\rangle$
$|\bar{1}\rangle|\bar{0}\rangle \rightarrow |\bar{1}\rangle|\bar{0}\rangle$
$|\bar{1}\rangle|\bar{1}\rangle \rightarrow |\bar{0}\rangle|\bar{1}\rangle$

Schmidt number *k* (at least)	$F_{\epsilon} = \frac{1}{2}(F_Z + F_X)$
4	0.89 [24]
3	0.86 [23]
2	



$F^{(3)} = 0.875$

$F^{(2)} = 0.75$

$F^{(1)} = 0.625$

[19] S. Olmschenk et al., Science 323, 486 (2009).
[23] R. Okamoto et al., Phys. Rev. Lett. **95**, 210506 (2005).
[24] X. H. Bao et al., Phys. Rev. Lett. **98**, 170502 (2007).

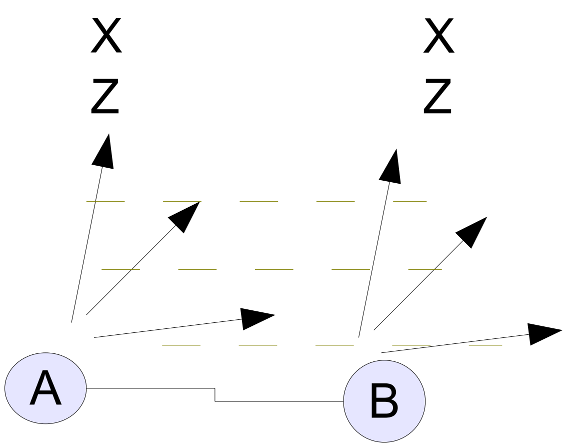
Summary

Role of Complementary on Entanglement detection

Quantum entanglement & Uncertainty relations
Simultaneous correlations on complementary observables

→ Inseparability of two-body density operators
→ Strength of quantum correlations
→(Coherence of Quantum Gates)

- Two measurement settings
- Multi-dimensional entanglement



→Detection of Non-Gaussian entanglement
uncertainty relations based on SU(2) and SU(1,1) generators

R. Namiki and Y. Tokunaga, Phys. Rev. Lett. 108, 230503 (2012).

(R. Namiki and Y. Tokunaga, Phys. Rev. A 85, 010305(R) (2012).)

R. Namiki, Phys. Rev. A 85, 062307 (2012).