Polyakov loop effects on the phase diagram in strong-coupling lattice QCD

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(Dated: October 31, 2016)

We investigate the Polyakov loop effects on the QCD phase diagram by using the strong-coupling $(1/g^2)$ expansion of the lattice QCD (SC-LQCD) with one species of unrooted staggered quark, including $O(1/g^2)$ effects. We take account of the effects of Polyakov loop fluctuations in Weiss mean-field approximation (MFA), and compare the results with those in the Haar-measure MFA (no fluctuation from the mean-field). The Polyakov loops strongly suppress the chiral transition temperature in the second-order/crossover region at small chemical potential $(\mu)$, while they give a minor modification of the first-order phase boundary at larger $\mu$. The Polyakov loops also account for a drastic increase of the interaction measure near the chiral phase transition. The chiral and Polyakov loop susceptibilities $(\chi_{\sigma}, \chi_{\ell})$ have their peaks close to each other in the second-order/crossover region. In particular in Weiss MFA, there is no indication of the separated deconfinement transition boundary from the chiral phase boundary at any $\mu$. We discuss the interplay between the chiral and deconfinement dynamics via the bare quark mass dependence of susceptibilities $\chi_{\sigma,\ell}$.

PACS numbers: 11.15.Me, 12.38.Gc, 11.10.Wx, 25.75.Nq

I. INTRODUCTION

The phase diagram of Quantum Chromodynamics (QCD) at finite temperature $(T)$ and/or quark chemical potential $(\mu)$ [1, 2] provides a deep insight into the Universe. At the few microsecond after the big-bang, a quark-gluon plasma (QGP) is supposed to undergo the QCD phase transition/crossover, which results in confinement of color degrees of freedom and the dynamical mass generation of hadrons. In fact, the first principle calculations based on lattice QCD Monte Carlo simulations (LQCD-MC) indicates the crossover around $T_c = 145 - 195$ (MeV) [3]. In compact star cores, a cold-dense system would appear, where various interesting phases are expected [4–8].

The QCD phase transition can be investigated in the laboratory experiments [9]: Circumstantial experimental evidence at the Relativistic Heavy-Ion Collider (RHIC) in Brookhaven National Laboratory together with theoretical arguments implies that the QGP is created in heavy-ion collisions at $\sqrt{s_{NN}} = 200$ GeV, and recent experiments at the Large Hadron Collider (LHC) in CERN give stronger evidence. Probing the phase diagram at finite $\mu$, in particular the critical point (CP) [10], is a central topic in the on-going and future heavy-ion collision experiments at the Facility for Antiproton and Ion Research (FAIR) at GSI, the Nuclotron-based Ion Collider fAcility (NICA) at JINR, and the beam energy scan program at RHIC [11]. Unfortunately, the first principle studies by LQCD-MC loses the robustness at finite $\mu$ due to the notorious sign problem [1, 12–15]. Many interesting subjects, for example, the location of CP, the equation of state (EOS) at high density, are still under debate.

The QCD phase diagram may be characterized by two underlying dynamics, the chiral and deconfinement transitions, which are associated with the spontaneous breaking of the chiral symmetry in the chiral limit and the $Z_{N_c}$ center symmetry of the color SU$(N_c)$ gauge group in the heavy quark mass limit, respectively. The order parameter is the chiral condensate $(\sigma)$/Polyakov loop $(\ell)$ for the chiral/deconfinement transition. Although the $Z_{N_c}$ symmetry is explicitly broken by the quark sector (with a finite or vanishing mass), the Polyakov loops are still important degrees of freedom to be responsible for the thermal excitation of quarks near the chiral phase transition. The interplay between the $\sigma$ and $\ell$ is under active scrutiny; the LQCD-MC reports that the chiral and Polyakov loop susceptibilities show their peaks at almost the same temperatures for $\mu = 0$, and the separation of two dynamics is proposed at finite $\mu$ in several models [2].

We investigate the QCD phase diagram by using the strong-coupling expansion in the lattice QCD (SC-LQCD), which provides a lattice-based and well-suited framework for the chiral and deconfinement transitions without a serious contamination by the sign problem. The SC-LQCD has been successful since the beginning of the lattice gauge theory [16–19], and revisited after the QGP discovery at RHIC as an instructive guide to the QCD phase diagram [20–33]. It is remarkable that a promising phase diagram structure has been obtained even in the strong-coupling limit $(\beta = 2N_c/g^2 \rightarrow \infty)$ with mean-field approximation (MFA) [20, 22, 24], and exactly determined based on the Monomer-Dimer-Polymer (MDP) formulation [30] and the Auxiliary Field Monte Carlo simulation [33]. The MFA results are then shown to be capturing the essential feature of the exact phase diagram.
In Fig. 1, we summarize the SC-LQCD studies on the color SU(3) QCD phase diagram using MFA. Based on the success in the strong-coupling limit (top in the second column), we have investigated the phase diagram [25, 27] by taking account of the next-to-leading order (NLO, $O(1/g^2)$, middle in the second column) and the next-to-next-to-leading order (NNLO, $O(1/g^4)$, bottom in the second column) of the strong-coupling expansion. The chiral phase transition temperature $T_c$ is strongly suppressed by the NLO effects, and the phase diagram evolves into the empirical shape with increasing lattice coupling $\beta = 2N_c/g^2$, while the NNLO effects give much milder corrections.

In the works mentioned above (listed in the second column of Fig. 1), the main focus was put on the chiral dynamics, rather than the $Z_N$ deconfinement dynamics, which is another important dynamics described by the Polyakov loops $\ell$ of the pure-gluonic sector. The SC-LQCD has been well-suited to include both dynamics at the strong-coupling limit [34–36] (top in third and fourth columns in Fig. 1); the strong-coupling limit for the quark sector is combined with the leading-order effect of the Polyakov loops in the pure-gluonic sector and the quark determinant term provides the lattice-based derivation of the $\sigma - \ell$ coupling. It is intriguing to include the higher-order of the strong-coupling expansion, which has been carried out in our previous work [28] (middle and bottom lines in third and last columns in Fig. 1); we have shown that the Polyakov loop effects combined with finite lattice couplings $\beta$ further suppresses the chiral transition temperature $T_c$, which reproduces the results of LQCD-MC simulations [38–40] at $\mu = 0$ in the certain lattice coupling range $\beta \sim 4$. Thus, the long-standing problem of the SC-LQCD - too large $T_c$ - is greatly relaxed by the Polyakov loops. Moreover, the Polyakov loop sector at the chiral phase transition $\sim O([1/g^2]^{1/T_c})$ is found to be comparable with the quark sector with NLO $O(1/g^2)$ and NNLO $O(1/g^4)$ at $T_c(\beta \sim 4) \sim 0.5 – 0.6$ (in lattice units); the Polyakov loop effects are necessary to evaluate $T_c$ with respect to the order counting of the strong-coupling expansion.

In our previous paper [28], however, the analysis was limited at vanishing chemical potential $\mu = 0$, while the finite $\mu$ region receives a growing interest by the forthcoming experiments focusing the CP and high density phase. The purpose of the present paper is to extend our previous work [28] to the finite $\mu$ region, and to investigate the Polyakov loop effects on the whole region of the QCD phase diagram as indicated by red-solid characters in Fig. 1. We adopt two approximation schemes for the Polyakov loops, a simple mean-field treatment (Haar-measure MFA) and an improved treatment with fluctuation effects (Weiss MFA). Through the various comparisons indicated by the arrows in Fig. 1, we elucidate the effect of the Polyakov loop itself, either the effects of the Polyakov loop fluctuations, as well as the higher-order (NNLO) effects of the strong-coupling expansion. In particular, we focus on thermodynamic quantities, which is of great interest in the study of the equation of state for quark matter but has been challenging in SC-LQCD. Moreover, we discuss the interplay between the chiral and deconfinement dynamics at finite $\mu$ via the bare quark mass dependence of susceptibilities $\chi_{\sigma,\ell}$.

We employ one species (unrooted) of staggered fermion, which has a $U_3(1)$ chiral symmetry in the strong-coupling region and becomes the four flavor QCD with degenerate masses in the continuum limit. We investigate the $U_3(1)$ chiral phase transition/crossover at finite $T$ and $\mu$ in color SU($N_c = 3$) gauge group in the $3 + 1$ dimension ($d = 3$). Our focus is not necessarily put on quantitative prediction of the realistic phase diagram, but we attempt to clarify which effects make the SC-LQCD phase diagram being closer to realistic one. Such lattice based arguments would be instructive to future LQCD-MC studies on the QCD phase diagram, even though the flavor-chiral structure in the present study is different from the real-life QCD with 2+1 flavors.

This paper is organized as follows: In Sec. II, we explain the effective potential in strong-coupling lattice QCD with Polyakov loop effects. In Sec. III, we investigate the phase diagram and related quantities by using the effective potential. In Sec. IV, we summarize our work and give a future perspective. Appendix A is devoted to the review of the effective potential derivation.
II. STRONG-COUPLING LATTICE QCD WITH POLYAKOV LOOP EFFECTS

We explain the effective potential of the strong-coupling lattice QCD including the Polyakov loop effects. The derivation has been detailed in our previous work [28], and recapitulated in Appendix A in this paper. Here we explain the essential property of the effective potential. We will work on lattice units $a = 1$ in color $SU(N_c = 3)$ gauge and $3 + 1$ dimension ($d = 3$). The parameters in the effective potential are the lattice coupling $\beta = 2N_c/g^2$, lattice bare quark mass $m_0$, lattice temperature $T = 1/N_t$ ($N_t$ = temporal lattice extension), and quark chemical potential $\mu$.

The effective potential $F_{\text{eff}}^{\text{H/W}}$ involves the plaquette-driven Polyakov loop sector $F_{\text{p}}^{\text{H/W}}$ and the quark sector $F_{\text{q}}^{\text{H/W}}$,

$$F_{\text{eff}}^{\text{H/W}}(\Phi, \ell, \bar{\ell}; \beta, m_0, T, \mu) = F_{\text{p}}^{\text{H/W}}(\ell, \bar{\ell}, \beta, T) + F_{\text{q}}^{\text{H/W}}(\Phi, \beta, m_0, T, \mu) + \mathcal{O}(1/g^6, 1/g^{2(N_c+2)}, 1/\sqrt{d}).$$  \hspace{1cm} (1)

The $F_{\text{p}}^{\text{H/W}}$ is responsible for the Polyakov loop effects

$$L_p = N_c^{-1} \prod_\tau U_{0,\tau \bar{\tau}}, \quad U_0 = \text{temporal link variable},$$  \hspace{1cm} (2)

which result from the integral over the spatial link variables for the plaquettes wrapping around the temporal direction. Such Polyakov loops are dubbed “plaquette-driven”, and purely gluonic. The effects of $L_p$ is investigated in two MFA scheme: the Haar measure and Weiss MFA - as indicated by the suffixes “H” and “W”. In the former, the Polyakov loop $L_p$ is simply replaced with its constant mean-field $\ell$, while in the latter, the mean-field $\ell$ is introduced via the extended Hubbard-Stratonovich transformation [25] and the fluctuations from the mean-field is taken account in the integral over the $U_0$. The Polyakov loop effective potential of Haar measure MFA is well-known since 1980’s [41].

$$F_{\text{p}}^{\text{H}}(\ell, \bar{\ell}, \beta, T) = -2TdN_c \left( \frac{1}{g^2 N_c} \right)^{1/T} \bar{\ell} - T \log \mathcal{R}_{\text{Haar}},$$ \hspace{1cm} (3)

$$\mathcal{R}_{\text{Haar}} \equiv 1 - 6\bar{\ell} - 3(\bar{\ell}^2) + 4(\bar{\ell}N_c + \bar{\ell}N_\ell),$$ \hspace{1cm} (4)

where the Haar measure in the $U_0$ path integral leads to the $Z_3$ symmetric term $\mathcal{R}_{\text{Haar}}$. Since the $\mathcal{R}_{\text{Haar}}$ does not couple to the dynamical quarks, the $Z_3$ symmetry affects the phase diagram separately from the chiral dynamics in Haar measure MFA. In sharp contrast to this, there is no counterpart in Weiss MFA [28].

$$F_{\text{p}}^{\text{W}}(\ell, \bar{\ell}, \beta, T) = 2TdN_c \left( \frac{1}{g^2 N_c} \right)^{1/T} \bar{\ell}.$$ \hspace{1cm} (5)

The Polyakov loop effects other than the quadratic term (5) are entangled to the dynamical quarks in the quark determinant as explained in the followings. Thus, the $Z_3$ dynamics is totally spoiled by the dynamical quarks in Weiss MFA.

In both Haar measure and Weiss MFA cases, the order counting of the strong-coupling expansion reads,

$$F_{\text{p}}^{\text{H/W}} \sim \mathcal{O}(1/g^2 N_t = 1/T),$$ \hspace{1cm} (6)

and thus depends on the lattice temperature $T = 1/N_t$, which is subject to the integer value $N_t$. However in this paper, we regard $T$ as a continuous valued given number, which naturally follows in the lattice Matsubara formalism [42]. Around the chiral transition/crossover temperature $T_c$, we will show that the $F_{\text{p}}^{\text{H/W}}$ becomes comparable to the NLO or NNLO effects: $\mathcal{O}(1/g^2 T_c^2) \sim \mathcal{O}(1/g^2 - 4)$. The quark sector $F_{\text{q}}^{\text{H/W}}$ in Eq. (1) is derived by integrating out the staggered quarks with link/plaquette variables in each order of the strong-coupling expansion. In this paper, we consider the LO, NLO, and NNLO effects;

$$F_{\text{q}}^{\text{H/W}} \supset \mathcal{O}(1/g^0), \mathcal{O}(1/g^2), \mathcal{O}(1/g^4).$$ \hspace{1cm} (7)

The integral is evaluated by introducing several auxiliary fields $\Phi$, which includes the chiral condensate $\sigma$, the order parameter of the $U_q(1)$ chiral symmetry, as well as other fields,

$$\Phi = \{ \sigma, \psi, \bar{\psi}, \bar{\psi}, \psi, \bar{\psi}, \bar{\psi}, \bar{\psi}, \bar{\psi}, \bar{\psi}, \bar{\psi} \},$$ \hspace{1cm} (8)

whose physical meanings are summarized in Tables I and II in the Appendix A. The coefficients of the effective potential terms are solely characterized by $(\beta, N_c, \mu)$ and $\mathcal{O}(1/g^{d-4})$ (see Table III). The total quark sector $F_{\text{q}}^{\text{H/W}}$ is then divided into the auxiliary field part $F_{\text{x}}$ and the quark determinant part $F_{\text{det}}^{\text{H/W}}$. As shown in Eq. (A17) in Appendix A, the $F_{\text{x}}$ is composed of the quadratic terms of the auxiliary fields $\Phi$.

The quark determinant term $F_{\text{det}}^{\text{H/W}}$ is responsible for the dynamical quark effects, and includes the quark hoppings with link variables $U_0$ wrapping around the temporal direction, which give rise to the “quark-driven” Polyakov loops. In Haar measure MFA, the quark determinant part becomes similar to that in the Polyakov-loop-extended Nambu-Jona-Lasinio (PNJL) model [37, 43] [51] and the Polyakov-loop-extended Quark-Meson (PQM) model [44]:

$$F_{\text{det}}^{\text{H/W}} = -N_c E_q - N_c \log \sqrt{Z_+ Z_-} - T \left( \log \mathcal{R}_q(E_q - \bar{\mu}, \ell, \bar{\ell}) + \log \mathcal{R}_q(E_q + \bar{\mu}, \ell, \bar{\ell}) \right),$$ \hspace{1cm} (9)

$$\mathcal{R}_q(x, y, \bar{y}) \equiv 1 + N_c (ye^{-x/T} + \bar{y}e^{-2x/T}) + e^{-3x/T}.$$ \hspace{1cm} (10)

See Table IV for the quark excitation energy $E_q$, the shifted quark chemical potential $\bar{\mu}$, and the wave function renormalization factor $\sqrt{Z_+ Z_-}$. In Weiss MFA, the plaquette-driven and quark-driven Polyakov loops are combined in the quark determinant, and the $U_0$ path integral accounts for the Polyakov loop fluctuations. Then we obtain the following expression,

$$F_{\text{det}}^{\text{W}} = -N_c \log \sqrt{Z_+ Z_-} - T \log \sum_l \mathcal{Q}^l(\Phi) \mathcal{P}^l(\ell, \bar{\ell}),$$ \hspace{1cm} (11)

$$\mathcal{P}^l(\ell, \bar{\ell}) = \sum_{n = -\infty}^{\infty} \sqrt{\ell/\bar{\ell}}^{-N_c n + \bar{N}_d} \mathcal{P}_n^l(\sqrt{\ell/\bar{\ell}}),$$ \hspace{1cm} (12)
where the thermal excitations of a quark and its composite $Q_c^4$, the thermal excitation of Polyakov loops $P_{\mu}^L$, and the baryon number index $N_B$ are summarized in Table V in Appendix A. In the heavy quark limit $m_q \to \infty$, Eq. (11) recovers the $Z_3$ symmetry as shown in Appendix A.

The auxiliary fields $\{\Phi, \ell, \bar{\ell}\}$ at equilibrium are determined as a function of $(\beta, m_0, T, \mu)$ via the saddle point search of the effective potential $F_{\text{eff}}^{\mu/\nu}$. In particular, the important quantities to probe the phase diagram are the chiral condensates $\sigma \in \Phi$, Polyakov loops $(\ell, \bar{\ell})$, and their (dimensionless) susceptibilities $(\chi_\sigma, \chi_\ell)$. In the present mean-field framework, the susceptibilities are evaluated as follows: We consider the curvature matrix $C$ of the effective potential at equilibrium,

$$C_{ij} = \frac{1}{T^4} \frac{\partial^2 F_{\text{eff}}^{\mu/\nu}}{\partial \phi_i \partial \phi_j} \big|_{\text{equilibrium}},$$

where the field $\phi_i$ represents the dimensionless auxiliary fields normalized by $T$ and $N_c$,

$$\phi_i \in \left\{ \frac{\sigma}{T^3 N_c^3}, \frac{\psi_\tau}{T^6 N_c^6}, \frac{\bar{\psi}_\tau}{T^6 N_c^6}, \frac{\psi_s}{T^6 N_c^6}, \frac{\bar{\psi}_s}{T^6 N_c^6}, \frac{\psi_{\bar{\tau}}}{T^6 N_c^6}, \frac{\bar{\psi}_{\bar{\tau}}}{T^6 N_c^6}, \frac{\psi_{\bar{\tau}}}{T^6 N_c^6}, \frac{\bar{\psi}_{\bar{\tau}}}{T^6 N_c^6}, \frac{\ell}{N_c}, \frac{\bar{\ell}}{N_c} \right\}.$$ (14)

Then the chiral and Polyakov loop susceptibilities are given by

$$\chi_\sigma = (C^{-1})_{ij=\sigma \sigma}, \quad \chi_\ell = (C^{-1})_{ij=\ell \bar{\ell}}.$$ (15)

In addition, we investigate thermodynamic quantities, a pressure $p$, quark number density $\rho_q$, and interaction measure $\Delta$,

$$p = -(F_{\text{eff}}^{\mu/\nu}(T, \mu) - F_{\text{eff}}^{\mu/\nu}(0, 0)),$$

$$\rho_q = \frac{\partial p}{\partial \mu},$$

$$\Delta = \frac{\epsilon - 3p}{T^4},$$

where $\epsilon = -p + Ts + \mu \rho_q$ represents an internal energy with $s = \partial p/\partial T$ being an entropy.

### III. RESULTS

We investigate the QCD phase diagram based on the effective potential explained in the previous section. We show the phase diagram and related quantities obtained in Haar measure MFA at next-to-leading order (NLO) in Subsec. III A. Weiss MFA at NLO in Subsec. III B for the fixed lattice coupling $\beta = 4$ in the chiral limit ($m_0 = 0$). We extend our study to include the finite bare quark mass $m_0 > 0$ in Subsec. III C with a particular focus on the chiral and Polyakov loop susceptibilities. Then, in Subsec. III D, we show the phase diagram evolution for various $\beta$. Finally, in Subsec. III E, we study the next-to-next-to-leading order (NNLO) effects in the phase diagram.

#### A. Haar Measure MFA at NLO

We consider the NLO Haar measure MFA, where the NNLO $O(1/g^4)$ terms in the coupling coefficients shown in Table III and the Polyakov loop fluctuations are ignored. We concentrate on the chiral limit case $m_0 = 0$. We take the lattice coupling $\beta = 4.0$ as a typical value, for which the chiral transition temperature at vanishing quark chemical potential $T_{c,\mu=0}$ becomes close to the LQCD-MC result [38] (For details on the comparison, see Refs. [28, 31]). The effects ignored or restricted here will be investigated in later subsections. The phase diagram in the Haar measure MFA is partly studied in our previous work [26], and we provide more complete analyses in the followings.

![FIG. 2](Color online) Upper: The chiral condensates $\sigma$, Polyakov loops $(\ell, \bar{\ell})$, in NLO Haar Measure MFA as a function of $T$ at $(\beta, m_0, \mu) = (4.0, 0.0, 0.4)$. Lower: The chiral and Polyakov loop susceptibilities $(\chi_\sigma, \chi_\ell)$ in the same condition as the upper panel. For a comparison, the $\chi_\sigma$ is multiplied by 1/1000.

In the upper panel of Fig. 2, we show the chiral condensates $(\sigma/N_c)$ and Polyakov loops $(\ell, \bar{\ell})$ at finite quark chemical potential $\mu = 0.4$ as a function of temperature $T$ for $(\beta, m_0) = (4.0, 0.0)$. In the low $T$ region, the chiral broken $(\sigma \neq 0)$ and confined $(\ell \sim 0)$ phase appears. As $T$ increases, we observe the second-order chiral phase transition $(\sigma \to 0)$ at $T_c \simeq 0.44$ and the large increase of the Polyakov loops.
\( (\ell \to \mathcal{O}(1)) \). These results are similar to the zero chemical potential case shown in the previous study [28].

We find that the Polyakov loop is smaller than the anti-Polyakov loop \( (\ell < \bar{\ell}) \) in the chiral broken phase. This is understood from a quark screening effect at high density: A finite \( \mu \) leads to a net quark number density at equilibrium, where putting additional quarks into the system would give a larger energy cost than anti-quarks. Therefore the free energy of the quark gets larger than that of the anti-quark \( F_q > \bar{F}_q \), which attributes to our observation \( \ell < \bar{\ell} \) through the relation \( (\ell, \bar{\ell}) \propto (e^{-F_q/T}, e^{-\bar{F}_q/T}) \).

In the lower panel of Fig. 2, we compare the temperature dependence of the chiral and Polyakov loop susceptibilities \( (\chi_\sigma, \chi_q) \) which are defined in Eq. (15) in the same condition as the upper panel. The Polyakov loop susceptibility has two peaks with a relatively wide width. We note that the action in the present SC-LQCD (A2) has the \( U_\chi(1) \) chiral symmetry, which governs the dynamics of the system. Since the first peak is found in the vicinity of the chiral phase transition, it should be associated with the chiral dynamics. The second peak (or bump) is found in the chiral restored phase \( T \simeq 0.53 > T_c \), and interpreted as the remnant of the \( \mathcal{Z}_3 \) deconfinement dynamics as discussed in the subsection III C.

In the upper panel of Fig. 3, we show the chiral condensates \( \sigma/N_c \) as a function of chemical potential \( \mu \) for three fixed temperatures \( T = 0.15, 0.20, 0.25 \). At \( T = 0.25 \) (red-solid line), we find the second-order phase transition. At lower \( T \sim 0.20 \) (blue-dashed line), the chiral symmetry is partially restored with the first-order phase transition as \( \mu \) increases, and gets completely restored with the second-order phase transition at larger \( \mu \). As shown in the previous study [25], the partial chiral restoration (PCR) emerges due to the self-consistent evaluation of the finite \( \beta \) effects for the chemical potential: The effective chemical potential appears as an implicit function of \( \sigma, \mu \rightarrow \tilde{\mu}(\sigma, \beta) = \mu - \delta \mu(\sigma, \beta) \) (see, Table IV), which allows a stable equilibrium satisfying \( \sigma \sim \tilde{\mu}(\sigma) \), leading to the PCR. Our finding in the present study is that the PCR is not spoiled by the Polyakov loop effects, but still exists. As \( T \) decreases, the PCR disappears and the first-order chiral transition dominates as indicated by the \( T = 0.15 \) case (dashed-dotted black line).

In the lower panel of Fig. 3, we pick up the \( T = 0.25 \) case from the upper panel and show the \( \mu \) dependence of \( \sigma/N_c \) in a wider range. The Polyakov loops \( (\ell, \bar{\ell}) \) and the quark number density \( (\rho_q/N_c \) defined by Eq. (17)) are also displayed. The Polyakov loops increase in the chiral broken phase \( \mu < \mu_c \simeq 0.59 \), and the increasing rate stays quite small compared with the finite \( T \) transition case. In contrast, the quark number density rapidly increases in the vicinity of the chiral phase transition. After the transition \( (\mu \geq \mu_c) \), we observe a high density system \( (\rho_q \sim N_c) \) with a little quark excitation \( (\ell \ll 1) \). This property as well as the possibility of two sequential transitions associated with the PCR would be reminiscent of the original idea of the quarkyonic phase [5].

In the symmetric phase, the Polyakov loops \( (\ell, \bar{\ell}) \) start decreasing with the relation \( \ell < \bar{\ell} \) as \( \mu \) increases. This would be a saturation artifact on the lattice: As we explained above, the chiral symmetry restoration leads to a high density system \( \rho_q > N_c/2 \) so that more than half of the lattice sites are filled by quarks. Then the holes - sites without quarks - behave like anti-quarks, and the system with the quark number density \( \rho_q > N_c/2 \) would be identical to the system with the anti-quark number density \( \rho_q^\prime = (N_c - \rho_q) < N_c/2 \). Therefore, the excitation property of quarks and anti-quarks becomes opposite \( (F_q < F_{\bar{q}}) \) as illustrated in Fig 4, and thus \( \ell < \bar{\ell} \) holds. As \( \mu \) becomes larger after the half-filling, the number of holes...
decreases and the degrees of freedom get frozen. Hence the excitations of both quarks and anti-quarks are suppressed at larger $\mu$, which results in the decreasing trend of $(\ell, \bar{\ell})$ as functions of $\mu$.

The phase diagram in the Haar measure MFA is similar to that in PQM [44]: When the $\mu$ dependence is absent in the Polyakov loop potential in PQM, the derivative of the Polyakov mean-field in terms of $T$ at finite $\mu$ has double peaks, which is analogous to our result shown in the lower panel of Fig. 2 as well as in our previous study [26]. We will revisit this subject in Weiss MFA case in the next subsection.

The lower panel of Fig. 5 shows the difference of the Polyakov loop and anti-Polyakov loop $(\ell - \bar{\ell})$ in the $T - \mu$ plane. The relation $\ell < \bar{\ell}$ holds in the whole $T; \mu > 0$ region in the chiral broken phase as shown by the blue color. The saturation effect $\ell > \bar{\ell}$ is observed as a general tendency at large $\mu$ region in the chiral restored phase as indicated by the red color.

As shown in Eq. (6), the plaquette-driven Polyakov loop action includes the $O(1/g^2/T)$ correction. At the chiral phase boundary, this effect gives $O(1/g^2/Tc) \lesssim O(1/g^4)$. For the consistency of the strong coupling expansion, we have to take account of the NNLO $1/g^4$ effects for the quark sector, which will be discussed in the later subsection.

B. Weiss MFA at NLO

We investigate the phase diagram of NLO Weiss MFA, where the Polyakov loop fluctuations from the mean fields $(\ell, \bar{\ell})$ are considered, while the NNLO effects $O(1/g^4)$ in the coupling coefficients shown in Table III are ignored. We compare the Weiss MFA results with the Haar measure MFA to clarify the effects of the Polyakov loop fluctuations to the phase diagram. We choose the same parameter set as the Haar measure MFA case, $(\beta, m_0) = (4.0, 0.0)$.

As shown in Fig. 6, $T$ or $\mu$ dependence of $(\sigma, \ell, \bar{\ell}, \rho_1)$ is qualitatively the same as the Haar measure MFA results. In the following, we concentrate on the results which are characteristic of the Weiss MFA.

In Fig. 7, we show the chiral and Polyakov loop susceptibilities $(\chi_\sigma, \chi_\ell)$ at finite chemical potential $\mu = 0.4$ as a function of temperature $T$. Two peaks are almost degenerated, and the width of $\chi_\ell$ is sharper than the Haar measure MFA case. We do not see the second ($Z_3$ associated) peak in the chiral symmetric phase in sharp contrast to the Haar measure MFA case.

In Fig. 8, we show the phase diagram of NLO Weiss MFA with $(\beta, m_0) = (4.0, 0.0)$. We find two qualitative differences between the NLO Weiss MFA and NLO Haar measure MFA results: First, the peak of $\chi_\ell$ (green-band showing the width of $\chi_\ell$ at 90% of the peak height) is more strongly locked to the chiral phase boundary in Weiss MFA than the Haar measure MFA case. Second, the remnant of the $Z_3$ dynamics such as the yellow band in Fig. 5 does not appear at any $\mu$ in the Weiss MFA case. As explained after Eq. (5) in the previous section, the plaquette-driven Polyakov loops are combined into the quark determinant and coupled to the dynamical quark effects via the $U_0$ path integral. Then, the Weiss MFA does not admit the remnant of the $Z_3$ symmetry in sharp contrast to the Haar measure MFA and many other chiral effective models [37, 43, 44]. It is sometimes argued that the chiral and deconfinement dynamics might be separated at finite $\mu$ [2], but the Weiss MFA does not support the isolated deconfinement
Here, we comment on the recent phase diagram study by the PQM model [44]. In this model, a dependence was assumed in the Polyakov loop effective potential based on the phenomenological insights to describe the back reaction of the quark-matter to the Polyakov loops at finite density. This prescription led to a stronger locking between the peak of $d\ell/dT$ and the chiral crossover line, and the double peak structure of $d\ell/dT$ disappeared. These phenomena would be analogous to our findings in the Weiss MFA. We stress that the Weiss MFA effective potential directly results from the path integral in the lattice QCD without additional assumptions. This would be the advantage of the SC-LQCD based effective potential.

We shall consider the formal limit $(\ell, \bar{\ell}) \to 0$ in the effective potential of Weiss MFA $F_{\text{eff}}^{W}$. In the second line of Eq. (11), the thermal excitations (see Table V) carrying a baryon number $0$ ($I = \text{MMM, MQQ}$) and $\pm 3$ ($I = \text{B, B}$) remains and the $F_{\text{eff}}^{W}$ reduces into the effective potential which has been derived in our previous study [25]. We express the reduced effective potential as $F_{\text{eff}}^{\text{NLO}}$, and the results obtained by using $F_{\text{eff}}^{\text{NLO}}$ will be referred to as NLO without Polyakov loops in the later discussions. See Eq. (A24) for the expression of $F_{\text{eff}}^{\text{NLO}}$. Needless to say, the $F_{\text{eff}}^{\text{NLO}}$ does not implement the Polyakov loop dynamics. By comparing the Weiss NLO MFA and the NLO without Polyakov loops, the Polyakov loop effects become more transparent.

In Fig. 8, we compare the chiral phase boundary of the NLO Weiss MFA and the NLO without Polyakov loops. The second-order phase boundary of the NLO Weiss MFA (blue-dashed line) is found in lower $T$ region than that of NLO without Polyakov loop (magenta-dotted line). As $\mu$ becomes larger, two phase boundaries get closer to each other and degenerate in the vicinity of the TCP. The first-order phase boundary is almost independent of the Polyakov loop effects. This is understood as follows. As explained in the previous section, the plaquette-driven Polyakov loops gives the contribution of $O(1/g^2)/T_c(\mu)$. At larger $\mu$, this factor decreases because the $T_c(\mu)$ does, and thus the Polyakov loop effects becomes higher order effects of the strong-coupling expansion,

---

**FIG. 6:** (Color online) Upper: The chiral condensates $\sigma$, Polyakov loops $(\ell, \bar{\ell})$, in NLO Weiss MFA as a function of $T$ at $(\beta, m_0, \mu) = (4.0, 0.0, 0.4)$. Lower: The chiral condensates $\sigma$, Polyakov loops $(\ell, \bar{\ell})$, and quark number density $\rho_q/N_c$ in NLO Weiss MFA as a function of $\mu$ at $(\beta, m_0, T) = (4.0, 0.0, 0.5)$.

**FIG. 7:** (Color online) The chiral and Polyakov loop susceptibilities $(\chi_\sigma, \chi_\ell)$ in NLO Weiss MFA as a function of $T$ at $(\beta, m_0, \mu) = (4.0, 0.0, 0.4)$. For a comparison, the chiral susceptibility $\chi_\sigma$ has been multiplied by $1/200$.

**FIG. 8:** (Color online) The phase diagram at $(\beta, m_0) = (4.0, 0.0)$ in NLO with Weiss MFA. See texts for details.
and thereby suppressed.

Compared with the Haar measure MFA, the transition temperature \(T_c(\mu)\) in the Weiss MFA becomes somewhat larger. Then, the effect of the plaquette-driven Polyakov loops for \(\beta = 4.0\) is maximally \(\mathcal{O}(\langle 1/g^2 \rangle^{1/T_c(\mu=0)}) = \mathcal{O}(1/g^{4.3})\), which is larger than the NNLO effects \(1/g^4\). Thus, the present NLO approximation for the quark sector is consistent with respect to the order counting of the strong coupling expansion, at least for \(\beta \lesssim 4.0\).

Next, we investigate the thermodynamic quantities in the Weiss MFA. In the upper panel of Fig. 9, we show the normalized pressure \(p/T^4\) as a function of \(T\) at chemical potential \(\mu = 0.4\), the same condition as Fig. 7. In NLO Weiss MFA, the \(p/T^4\) (red-solid line) becomes significantly larger at \(T \gtrsim T_c \approx 0.507\) and closer to the Stefan-Boltzmann result

\[
\frac{p}{T^4} = \frac{N_f N_c}{6} \left[ 7\pi^2 \frac{\mu^2}{T^2} + \frac{1}{2\pi^2} \frac{\mu^4}{T^4} + \frac{(N_c^2 - 1)\pi^2}{45} \right].
\]  

(19)

We do not see such a large enhancement of \(p/T^4\) in the case of NLO without Polyakov loops (blue-dashed line). Thus, the Polyakov loop plays an essential role to realize the pressure enhancement which is expected in the QGP phase at high \(T\). More specifically, the pressure enhancement is attributed to the increase of Polyakov loop thermal excitations \(P_{\ell}^\ell(\sqrt{\ell})\) (see Table V) included in the Weiss MFA effective potential (11)-(12). This result should be compared with that in the PQM model, where the pressure is rather suppressed by Polyakov loops [44]. The different role of Polyakov loops is understood as follows. Considering the Polyakov loop effects in SC-LQCD means introducing deconfinement (quark thermal excitation) degrees of freedom as explained above. In contrast, the Polyakov loops act as confinement degrees of freedom in PQM and PNJL models.

In the middle panel of Fig. 9, we show the interaction measure \(I\) as a function of \(T\) at chemical potential \(\mu = 0.4\). In NLO Weiss MFA, the \(\Delta\) has a large peak in the vicinity of the chiral phase transition \(T \sim T_c\) as expected with regards to the increasing scale asymmetry in the strongly interacting quark-gluon plasma (sQGP). This should be compared with the result obtained in NLO without Polyakov loops (dashed-blue line) staying small and showing just a tiny bump structure at \(T \sim T_c\). In the lower panel of Fig. 9, we compare our results on the interaction measure at vanishing of chemical potential with those obtained in the Monte Carlo simulations (four flavor, the chiral limit is taken) [40]. The Monte Carlo results (green boxes) show the drastic increase in the vicinity of the chiral phase transition. This feature is qualitatively reproduced by the NLO Weiss result (red-solid line), but not in the NLO without Polyakov loops (blue-dashed line). Our results indicate that the Polyakov loops have a large contribution to the scale asymmetry expected in the sQGP.

### C. Quark mass dependence

In the previous subsections, we have studied the phase diagram in the chiral limit \(m_0 = 0\). In this subsection, we investigate the \(m_0\) dependence of the chiral and Polyakov loop susceptibilities. We choose the same parameter set of \(\beta = 4.0\) and \(\mu = 0.4\) as previous subsections.

In the upper panel of Fig. 10, we show the chiral suscepti-
The chiral susceptibility $\chi_c$ of the NLO Haar measure MFA as a function of $T$ for various bare quark mass $m_0$ at $(\beta, \mu) = (4.0, 0.4)$. The peak position defines the chiral crossover temperature at finite $m_0$. The chiral dynamics becomes weaker as indicated by the attenuating peak with increasing $m_0$. In the lower panel of Fig. 10, we show the Polyakov loop susceptibility $\chi_\sigma$ in the same condition as the upper panel. The double peak structure which we have shown in the chiral limit in Subsec III A evolves into a single peak with increasing $m_0$. The single peak grows up in the heavy mass region $m_0 = 0.9$, and comes to be responsible for the $Z_3$ crossover. Consistently, the chiral susceptibility does not show any signal there as shown in the upper panel.

We notice that the $Z_3$ peak of $\chi_\sigma$ at $m_0 = 0.9$ locates at the almost same temperature as the second peak appearing in the small mass region $m_0 \lesssim 0.01$. This implies that the second peak originates from the remnant of the $Z_3$ dynamics. In fact, the approximate $Z_3$ symmetry remains even in the chiral limit in the effective potential of the Haar measure MFA: The $Z_3$ symmetric (Haar measure) term $R_{\text{Haar}}$ in Eq. (4) has a large contribution and does not couple to the dynamical fermion effects $R_\ell$ in Eq. (A18), so that the former effect is not horribly spoiled by the latter. The result is consistent with our previous work [26].

This should be compared with NLO Weiss MFA results, Fig. 11. The chiral susceptibility $\chi_c$ (upper panel) is qualitatively the same as the Haar measure result, while the Polyakov loop susceptibility $\chi_\ell$ (lower panel) differs: The Weiss MFA does not lead to the double-peak structure in $\chi_\ell$ for any $m_0$. Thus, the scenario with the double-peak, or equivalently, the deconfinement seperated from the chiral phase boundary at high density would be less supported within the present approximation. To extract a definite conclusion on the relation between two susceptibilities $\chi_{c, \ell}$, we need to investigate the higher-order effects on the Polyakov loops.

It is worth mentioning that the Polyakov loop effective potential in the Haar measure MFA, Eq. (3) is similar to one of the popular choices of the potential in the PNJL model [37, 43] or PQM models. They could in principle contain the remnant of $Z_3$ dynamics as the Haar measure MFA does. As explained in the previous subsection, the recent work based on PQM assumed a certain $\mu$ dependence to the coefficients in the Polyakov loop effective potential [44]. This gives a phenomenological implementation of a back reaction from dynamical quark effects. The Weiss MFA effective potential
(especially Eq. (11)) proposes the lattice QCD based solution for the quark back reaction to the Polyakov loops, and opens a possibility to upgrade the PNJL and PQM models so that they account for the Polyakov loop and quark degrees of freedom more systematically. To invent such a model based on the Weiss MFA should be one of the future works.

D. Phase diagram evolution with increasing $\beta$

So far, we have studied the phase diagram at a fixed coupling, $\beta = 4.0$. In this subsection, we investigate the phase diagram for various lattice coupling ranging $0.0 \leq \beta \leq 6.0$, while we keep the vanishing bare quark mass $m_0 = 0$. For the chiral phase transition temperature at vanishing chemical potential $T_{c,\mu=0}$, the lattice MC data with one species of staggered fermion are available [30, 38–40] and are compared with $T_{c,\mu=0}$ evaluated in the strong-coupling expansion [25, 27, 28]. We extend our analyses up to $\beta = 6.0$, for which the physical scale of $(T_{c,\mu})$ can be extracted by utilizing the lattice spacing result in Ref. [45].

In the upper panel of Fig. 12, we show the phase diagram evolution with increasing $\beta$ in the case of NLO Haar measure MFA. In the whole range of $0.0 \leq \beta \leq 6.0$, the chiral phase transition is a first-order in the low temperature region, and it evolves into the second-order at higher $T$ via TCP. The transition temperature at $\mu = 0$ ($T_{c,\mu=0}$) acquires much larger modification with increasing $\beta$ than the transition chemical potential at $T = 0$ ($\mu_{c,T=0}$). Resultantly, the ratio $R = \mu_{c,T=0}/T_{c,\mu=0}$ which characterize the shape of the chiral phase boundary is greatly enhanced. For $\beta \geq 4$, the first-order transition line goes inside of the second-order transition line near the TCP, and the PCR explained in the previous subsection emerges between two lines.

In the lower panel of Fig. 12, we show the phase diagram evolution of the NLO Weiss MFA in the chiral limit $m_0 = 0$. The results are qualitatively same as the Haar Measure MFA case.

We compare the ratio $R = \mu_{c,T=0}/T_{c,\mu=0}$ of NLO Weiss MFA to that obtained in the “NLO without Polyakov loops”. At $\beta = 4.0$ (6.0), the former (red-solid line) in Fig. 13 becomes 1.38 (1.46) times larger than the latter (blue-dashed line). Thus the ratio $R$ becomes larger by the Polyakov loop effects. Next, we compare our $R$ with those obtained by the Monomer-Dimer-Polymer (MDP) simulation [31]. The MDP (green triangles in Fig. 13) gives a somewhat larger $R$ than our MFA result in the strong-coupling limit, and becomes closer to the NLO Weiss MFA at finite $\beta$. The increasing $R$ at larger $\beta$ is a common trend in both MFA and MDP, and preferable to be consistent with a realistic QCD phase diagram.

In both Haar measure MFA and Weiss MFAs, the TCP tends to go into low $T$ region with increasing $\beta$, and the second-order chiral phase boundary becomes dominant. This trend is also reported in the MDP simulations [31, 32] and supports the recent LQCD-MC results based on the critical surface analysis [46]. However, the trend is opposite to the anomaly based expectation for $N_f = 4 > 2$ [47]. This subject should be further discussed with a special care for the anomaly in the staggered fermion formalism.

Finally, we estimate the $(T_{c,\mu}c,\mu)$ in physical units by quoting the lattice spacing scale $a^{-1}(\beta = 0) = 440$ (MeV) and $a^{-1}(\beta = 6) = 524$ (MeV) from the zero temperature strong-coupling expansion [45]. In Haar measure MFA, we find $(T_{c,\mu=0},\mu_{c,T=0}) \simeq (550, 242)$ (MeV) in the strong-coupling limit, and $(T_{c,\mu=0},\mu_{c,T=0}) \simeq (200, 321)$ (MeV) at $\beta = 6$. In Weiss MFA, we find $(T_{c,\mu=0},\mu_{c,T=0}) \simeq (733, 242)$ (MeV) in the strong coupling limit, and $(T_{c,\mu=0},\mu_{c,T=0}) \simeq (229, 321)$ (MeV) at $\beta = 6.0$. Although the flavor-chiral structure of the present system differs from the real-life QCD, it is still interesting that the transition temperature of SC-LQCD gets closer.
Our finding here is that the locking of the Polyakov loop as a function of temperature at the two chemical potentials $\mu = 0.4$ and 0.7. At $\mu = 0.4$, the broken chiral symmetry leads to the dynamical quark mass and suppresses the thermal excitation of the quarks, while at 0.7, there is no suppression due to the symmetry restoration. Thus, the relatively large $\ell$ at low temperature can be a characteristic feature at high density phase. At higher $T$, $\ell$ at $\mu = 0.7$ becomes comparable with that at $\mu = 0.4$.

In Fig. 16, we show the temperature derivative of the chiral condensate and Polyakov loop as a function of temperature at $\mu = 0.4$ and 0.7. At $\mu = 0.4$ (solid lines), the chiral and deconfinement crossovers almost simultaneously take place as indicated by their peak positions. This property has been observed at $\mu = 0$ [28]. Our finding here is that the locking of the chiral and deconfinement crossovers remains intact at finite $\mu$ as long as the spontaneous symmetry breaking exists. At $\mu = 0.7$, $d(\sigma/N_c)/dT$ (blue-dashed line) shows no signal at any $T$ due to the absence of the chiral crossover, and the
investigated the whole structure of the SC-LQCD phase diagram with a special emphasis on the Polyakov loops effects.

In both Haar measure and Weiss MFAs, the first-order chiral phase boundary emerges in the low $T$ region and ends up with the tri-critical point (TCP), from which the second-order chiral phase boundary evolves to the smaller $\mu$ direction with increasing $T$ in the chiral limit ($m_0 = 0$). The Polyakov loop together with finite $\beta$ effects strongly suppresses the critical temperature $T_c$ in the second-order/crossover region at small $\mu$, while it gives a minor modification of the first-order phase boundary at larger $\mu$. As a result, the chiral phase boundary becomes much closer to the expected one in the real-life QCD as summarized in Fig. 12 (NLO case) and Fig. 17 (left: NNLO, right NLO). It is also remarkable that the NNLO effects are subdominant in whole region of the phase diagram.

In both Haar measure MFA and Weiss MFAs, the critical point (CP) tends to go into low $T$ region with increasing $\beta$, and the second-order chiral phase boundary becomes dominant. This trend is also reported in the MDP simulations [31, 32] and supports the recent MC results based on the critical surface analysis [46]. However, the trend is opposite to the anomaly based expectation for $N_f = 4 \geq 2$ [47]. The anomaly effects in the staggered fermion formalism should be further investigated in future.

We have investigated thermodynamic quantities, which is of great interest in the study of EOS of quark matter, which has however been challenging in SC-LQCD. Our findings are that a pressure and an interaction measure are drastically enhanced by Polyakov loop thermal excitations.

We have found some characteristic features of Polyakov loops at finite $\mu$. At finite $\mu$ in the broken phase, the Polyakov loop $\ell$ becomes larger than $\ell$, which is interpreted as a screening effect of quarks at equilibrium with net quark number density. In the chirally symmetric high density phase, the Polyakov loop becomes relatively large even at a small temperature, which can be understood from the absence of the dynamical quark mass in the symmetric phase.

We have shown that the chiral and Polyakov loop susceptibilities $\chi_\sigma$, $\chi_\ell$ have their peaks near to each other in the second-order transition or crossover region. In the vicinity of the critical point, the peak of the $\chi_\ell$ rapidly diminishes. We have found two qualitative differences between the Weiss and Haar measure MFA on the Polyakov loop susceptibilities: First, the peak of $\chi_\ell$ is more strongly locked to the chiral phase boundary in Weiss MFA than the Haar measure MFA case. Second, the $Z_3$ deconfinement dynamics artificially remains in the Haar measure MFA and disappears by taking account of the Polyakov loop fluctuations in Weiss MFA. Our findings are summarized in Fig. 8 (upper, Haar measure MFA result) and 8 (Weiss MFA result). The above difference results from the fact that the effective potential of Weiss MFA does not admit any remnant of the $Z_3$ symmetric structure in sharp contrast to the Haar measure MFA and many other chiral effective models [37, 43, 44]. Thus, the Weiss MFA does not support the isolated deconfinement transition/crossover from the chiral phase boundary at large $\mu$.

There are several future directions to be investigated. First,
it is important to evaluate the higher order terms of the strong-coupling expansion, and/or to invent a resummation technique to account for the higher orders. From this viewpoint, we find recent developments for the Polyakov loop effective potential [48]. Second, it is desirable to establish the exact evaluation of each order of the strong-coupling expansion beyond the mean-field approximation and $1/d$ expansion. This will be achieved by extending the MDP works [31, 32] to include the higher-order of expansions as well as the Polyakov loop effects. Another method to go beyond MFA is the Monte-Carlo simulations for the auxiliary field integrals at each order of the expansion [33]. Third, it is interesting to evaluate the complex phase effect of Polyakov loops; The susceptibilities associated with the phase may give a new probe of the QCD phase transition [49]. And finally, the Weiss MFA results, especially the quark and Polyakov loop thermal excitations summarized in Table V, may open a possibility to invent an upgraded version of the PNJL-type model which more reasonably describes the interplay between the chiral and deconfinement dynamics.

Acknowledgments

We thank Maria Paola Lombardo and Philippe de Forcrand for fruitful discussions. This work was supported in part by the Grants-in-Aid for Scientific Research from JSPS (Nos. 22-3314, 15K05079, 15H03663, 16K05350), for Young Scientists (B) No.15K17644 (Kohtaroh Miura), the Grants-in-Aid for Scientific Research on Innovative Areas from MEXT (Nos. 24105001, 24105008), and the Yukawa International Program for Quark-hadron Sciences (YIPQS). Kohtaroh Miura is supported by the OCEVU Labex (ANR-11-LABX-0060) and the A*MIDEX project (ANR-11-IDEX-0001-02), funded by the “Investissements d’Avenir” French government program and managed by the ANR.

Appendix A: Effective potential in strong-coupling lattice QCD

We briefly review the derivation of the effective potential Eq. (1) based on our previous papers [27, 28]. We start from the lattice QCD action with one species of staggered fermion ($\chi$) with a current quark mass ($m_0$) and chemical potential ($\mu$),

$$Z_{\text{LQCD}} = \int D[\chi, \bar{\chi}, U_{\nu}] e^{-S_{\text{LQCD}}[\chi, \bar{\chi}, U_{\nu}]} ,$$

(A1)

$$S_{\text{LQCD}} = S_F + S_G + m_0 \sum_{x} \bar{\chi}_x \chi_x ,$$

(A2)

where,

$$S_F = \frac{1}{2} \sum_{\nu, x} \left[ \bar{\eta}_{\nu, x} \chi_{\nu, x} U_{\nu, x} \chi_{x + \nu} \bar{\eta}_{\nu, x} - \bar{\eta}^{-1}_{\nu, x} (h.c.) \right] ,$$

(A3)

$$\eta_{\nu, x} = \exp(\mu \delta_{\nu, 0})(-1)^{x_0 + \cdots + x_{\nu - 1}} ,$$

(A4)

$$S_G = \beta \sum_{P} \left[ 1 - \frac{1}{2N_c} \left[ U_{P} + U_{P}^\dagger \right] \right] .$$

(A5)

We have employed lattice units $a = 1$. The $U_{\nu, x} \in SU(N_c)$ and $U_{P = \mu \nu, x} = \text{tr}_{c}[U_{\mu, x} U_{\nu, x + \mu} U_{\mu, x + \mu}^\dagger U_{\nu, x}]$ represent the link- and plaquette-variable, respectively. In the chiral limit ($m_0 \to 0$), the action has the $U_1(1)$ chiral symmetry, which is enhanced to $SU(N_f = 4)$ in the continuum limit.

There are four main steps to derive the effective potential from the lattice QCD action (A1) [28]: First, we carry out the strong-coupling expansion, and integrate out the spatial link variables in each order. The effective action is obtained as a function of various hadronic composites. For the composites including the staggered quarks ($\chi$, $\bar{\chi}$), we take account

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Composites</th>
</tr>
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<tbody>
<tr>
<td>$M_x$</td>
<td>$\chi_x \bar{\chi}_x$</td>
</tr>
<tr>
<td>$(V_+^x, V_-^x)$</td>
<td>$(\bar{\chi}<em>x e^{\mu U</em>{0,x} \chi_x + 0}, \bar{\chi}<em>x e^{-\mu U</em>{0,x} \chi_x})$</td>
</tr>
<tr>
<td>$(W_+^x, W_-^x)$</td>
<td>$(\bar{\chi}<em>x e^{2 \mu U</em>{0,x} \chi_x + 0 \chi_x + 20}, \bar{\chi}<em>x e^{2 \mu U</em>{0,x} \chi_x + 0 \chi_x + 20})$</td>
</tr>
</tbody>
</table>

| $L_{\mu, x}$ | $\text{tr}_c \left[ \prod_{i} U_{0,x_i^c} \right] / N_c$ |

FIG. 17: (Color online) The phase diagram evolution for $m_0 = 0.05$ as a function of $\beta$ in NNLO Haar-measure MFA (left), which is compared with the counterpart in NLO Haar-measure MFA (right) with same parameters. The red-thick-solid line represents the CEP evolution.
of the terms up to \(O(1/g^6)\), and extract from them the leading order terms of the \(1/d\) expansion \(O(1/d^3)\) [50]. For the pure gluonic composites, we take account of the leading order contributions to the Polyakov-loop \(O(1/g^2N_c)\). For the pure gluonic composites, we take account of the leading order contributions to the Polyakov-loop \(O(1/g^2N_c)\). We have introduced a short-hand notation

\[
[AB]_{j,x} = A_x B_{x+j}, \quad (A9)
\]

\[
[ABCD]_{jk,x} = A_x B_{x+j} C_{x+j+k} D_{x+k}, \quad (A10)
\]

and the couplings \(\beta...\) in Eq. (A8) are summarized in Table III.

Second, we introduce the auxiliary fields for the hadronic composites to bosonize the effective action \(S_{\text{eff}}^{\text{NNLO}}\), and perform the static mean-field and saddle-point approximations. The auxiliary fields are summarized in Table II, and the \(S_{\text{eff}}^{\text{NNLO}}\) reduces into

\[
S_{\text{eff}}^{\text{NNLO}} \simeq S_{\text{eff}}^{\text{pol}} + S_{\text{eff}}^{\text{x}}, \quad (A11)
\]

TABLE II: The auxiliary field \(\Phi\) and \((\ell, \bar{\ell})\). See also Table I.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Mean Fields Contents</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\sigma)</td>
<td>(-\langle M \rangle)</td>
</tr>
<tr>
<td>( (\bar{\psi}<em>{\tau \tau}, \psi</em>{\tau \tau}) )</td>
<td>((W^+, \langle W^- \rangle))</td>
</tr>
<tr>
<td>( (\bar{\psi}<em>{\tau \tau}, \psi</em>{\tau \tau}) )</td>
<td>((\langle M \rangle, \langle MMM \rangle))</td>
</tr>
<tr>
<td>( (\bar{\psi}<em>{\tau \tau}, \psi</em>{\tau \tau}) )</td>
<td>(\langle -V^+V^- \rangle, \langle 2MM \rangle)</td>
</tr>
<tr>
<td>( (\bar{\psi}<em>{\tau \tau}, \psi</em>{\tau \tau}) )</td>
<td>(\langle -V^+ \rangle, \langle V^- \rangle)</td>
</tr>
<tr>
<td>( (\bar{\psi}<em>{\tau \tau}, \psi</em>{\tau \tau}) )</td>
<td>((\langle M \rangle, \langle MMM \rangle))</td>
</tr>
<tr>
<td>((\ell, \bar{\ell}))</td>
<td>((\bar{L}<em>{p, x}, L</em>{p, x}))</td>
</tr>
</tbody>
</table>

TABLE III: The coupling coefficients appearing in the effective action/potential. Here, \(g, N_c = 3\), and \(d = 3\) represents the gauge coupling, number of color, and spatial dimension, respectively. See Table II for the auxiliary fields \((\psi..., \bar{\psi}...\)).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b_g)</td>
<td>(d/(2N_c))</td>
</tr>
<tr>
<td>(\beta_2)</td>
<td>((d/(N_c^2g^4)) \cdot (1 + 1/(2g^2)))</td>
</tr>
<tr>
<td>(\beta_s)</td>
<td>((d(d-1)/(8N_c^2g^4)) \cdot (1 + 1/(2g^2)))</td>
</tr>
<tr>
<td>(b_{\sigma}^g)</td>
<td>(b_{\sigma} + 2[\beta_{ss} \psi_{ss} + \beta_{ss} \bar{\psi}<em>{ss} + \beta</em>{ss} (\psi_{ss} + \bar{\psi}_{ss})])</td>
</tr>
<tr>
<td>(\beta_{\sigma}^g)</td>
<td>2 (\beta_{\sigma}^g)</td>
</tr>
<tr>
<td>(\beta_{\sigma})</td>
<td>2 (\beta_{\sigma})</td>
</tr>
<tr>
<td>(\beta_{\tau \tau})</td>
<td>((d(d-1)/(2N_c^2g^4)))</td>
</tr>
<tr>
<td>(\beta_{\tau \tau})</td>
<td>((d(d-1)/(2N_c^2g^4)))</td>
</tr>
</tbody>
</table>

\[S_{\text{eff}} = S_{\text{eff}}^{\text{NNLO}} + S_{\text{eff}}^{\text{pol}}, \quad (A6)\]

with

\[S_{\text{eff}}^{\text{pol}} = -N_c^2 \left( \frac{1}{g^2N_c} \right)^{N_c=1/T} \sum_{x,j,k} \left[ \bar{L}_{p, x} L_{p, x+j} + h.c. \right], \quad (A7)\]

TABLE IV: Quantities which govern the property of the effective potential. See Table III for the couplings \((b_{\sigma}, \beta_{\sigma}, \beta_{\tau \tau})\) and Table II for the auxiliary fields \((\sigma, \bar{\psi}_{\tau \tau}, \psi_{\tau \tau}, \bar{\psi}_{\tau \tau})\).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Meanings</th>
</tr>
</thead>
<tbody>
<tr>
<td>(m_q)</td>
<td>(m_q'/\sqrt{Z_+ Z_-})</td>
<td>dynamical quark mass</td>
</tr>
<tr>
<td>(m_q')</td>
<td>(\hat{m}_q = b_q \sigma + m_0)</td>
<td></td>
</tr>
<tr>
<td>(\sqrt{Z_+ Z_-})</td>
<td>(Z_+ = 1 + \beta_{\sigma} \bar{\psi}_{\tau \tau})</td>
<td>wave function</td>
</tr>
<tr>
<td>(E_q)</td>
<td>(\sinh^{-1} \hat{m}_q)</td>
<td>quark excitation energy</td>
</tr>
<tr>
<td>(\hat{\mu})</td>
<td>(\mu - \log \sqrt{Z_+ / Z_-})</td>
<td>shifted chemical potential</td>
</tr>
</tbody>
</table>
TABLE V: The thermal excitation effects $\mathcal{P}^I_n$ and $Q^I$ in the quark determinant of the Weiss MFA, Eq. (A21). The left column represents the excitation channel with the label $I$ in the text: (M, B, Q, D) stands for (mesonic,baryonic,quark,diquark) excitation. The quark excitation energy $E_q$ and modified chemical potential $\tilde{\mu}$ appearing in the third column are explained in Table IV. In the right column, $I_n$ represents a modified Bessel function with the argument $x = 4dN_c(\beta/(2N_c^2))^{1/T}\ell$.

<table>
<thead>
<tr>
<th>Excitation ($I$)</th>
<th>$N^I_n$</th>
<th>$Q^I$ ($\Phi$)</th>
<th>$P^I_n(1/\sqrt{T})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MMM</td>
<td>0</td>
<td>$(2\cosh(E_q/T))^{N_c}$</td>
<td>$P^\text{MMM}_n$</td>
</tr>
<tr>
<td>MQQ</td>
<td>0</td>
<td>$2\cosh(E_q/T)$</td>
<td>$P^\text{MQQ}_n$</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>$e^{N_c\tilde{\mu}/T}$</td>
<td>$P^\text{B}_n = P^\text{MMM}_n^{-1}$</td>
</tr>
<tr>
<td>B</td>
<td>$-3$</td>
<td>$e^{-N_c\tilde{\mu}/T}$</td>
<td>$P^\text{B}_n = P^\text{MMQ}_n^{-1}$</td>
</tr>
<tr>
<td>MMQ</td>
<td>1</td>
<td>$e^{\mu/T}(2\cosh(E_q/T))^2$</td>
<td>$P^\text{MMQ}_n$</td>
</tr>
<tr>
<td>MMQ</td>
<td>$-1$</td>
<td>$e^{-\mu/T}(2\cosh(E_q/T))^2$</td>
<td>$P^\text{MMQ}_n$</td>
</tr>
<tr>
<td>MD</td>
<td>2</td>
<td>$e^{2\mu/T}2\cosh(E_q/T)$</td>
<td>$P^\text{MD}_n = P^\text{MMQ}_n$</td>
</tr>
<tr>
<td>MD</td>
<td>$-2$</td>
<td>$e^{-2\mu/T}2\cosh(E_q/T)$</td>
<td>$P^\text{MD}_n = P^\text{MMQ}_n$</td>
</tr>
<tr>
<td>DQ</td>
<td>1</td>
<td>$e^{\mu/T}$</td>
<td>$P^\text{DQ}<em>n = (2I</em>{n+1}^3-I_{n-1}+I_{n+1}^3-I_{n-1}^3)$</td>
</tr>
<tr>
<td>QD</td>
<td>$-1$</td>
<td>$e^{-\mu/T}$</td>
<td>$P^\text{QD}<em>n = (2I</em>{n+1}^3-I_{n-1}+I_{n+1}^3-I_{n-1}^3)$</td>
</tr>
</tbody>
</table>

where

$$S^\text{P}_{\text{eff}} = \sqrt{Z^*Z} \sum_{xy} \chi_x G^{-1}_{xy}(\tilde{m}_q, \tilde{\mu}) \chi_y,$$

$$G^{-1}_{xy}(\tilde{m}_q, \tilde{\mu}) = \tilde{m}_q \delta_{xy} + \frac{\delta_{xy}}{2} \left[ e^{\tilde{\mu}U_{0,x}\delta_{x+\bar{0},y}} - e^{-\tilde{\mu}U_{0,x}^\dagger\delta_{x-\bar{0},y}} \right],$$

$$S^\text{X}_{\text{eff}} = N_T N_c \left[ \beta_s \sigma^2 + \frac{1}{2} \beta_s \tilde{\psi}_s^\dagger \tilde{\psi}_s^\dagger \right],$$

Here, the dynamical quark mass $\tilde{m}_q$, the shifted quark chemical potential $\tilde{\mu}$, and the wave function renormalization factor $\sqrt{Z^*Z}$ are summarized in Table IV, and the $N_{\ell(s)}$ represents the temporal (spatial) lattice extension.

Third, we carry out the Gaussian integral over the staggered quarks ($\chi, \bar{\chi}$) in Eq. (A12) in the anti-periodic boundary condition. The resultant quark determinant at finite $T$ is then calculated by using the Matsubara method in the Polyakov gauge for temporal link variables [42].

$$\int \mathcal{D}[\chi, \bar{\chi}] e^{-S^\text{eff}} = \prod_{x} \left[ e^{N_c(\log \sqrt{Z^*Z}+E_q)/T} \right.$$

$$\times \det_c \left[ \begin{array}{ll} 1 + N_c L_{p,x} e^{-(E_q-\tilde{\mu})/T} \\ 1 + N_c \tilde{L}_{p,x} e^{-(E_q+\tilde{\mu})/T} \end{array} \right].$$

with $E_q = \sinh^{-1} \tilde{m}_q$. Temperature $T$ is now considered as a continuous valued number (see the appendix in Ref. [24] for details). The Polyakov loop $L_{p,x}$ has appeared in the determinant via the quark hopping wrapping around the temporal direction in addition to the Plaquette effects Eq. (A7).

Finally, we evaluate the $L_{p,x}$ effects in the path integral over the temporal link variable $U_0$ in two approximation schemes: Haar measure and Weiss MFA. In the former, we replace the Polyakov loop $L_{p,x}$ contained in Eq. (A7) and (A15) as well as the Haar measure of the $U_0$ path integral with a constant mean-field $(\ell, \bar{\ell})$ instead of performing the $U_0$ path integral. In the latter, we introduce a mean-field $(\ell, \bar{\ell})$ via the extended Hubbard-Stratonovich transformation [25] in Eq. (A7), and exactly carry out the $U_0$ path integral to include the fluctuation effects from $(\ell, \bar{\ell})$ [28].

As a result, we obtain the effective potential

$$\mathcal{F}^{\text{MFA/IV}}(\Phi, \ell, \bar{\ell}; \beta, m_0, T, \mu) = \mathcal{F}_X(\beta) + \mathcal{F}^{\text{MFA}}(\Phi, \beta, m_0, T, \mu) + \mathcal{F}^{\text{IV}}(\ell, \bar{\ell}, T) + \mathcal{O}(1/g^0, 1/\sqrt{d}).$$

The auxiliary field term is given by Eq. (A14) and common in both Haar measure MFA and Weiss MFA,\n
$$\mathcal{F}_X(\beta) = S^\text{eff}(N_c N_c^2).$$

The quark determinant and the Polyakov loop effects are given
as

\begin{equation}
F^{\text{det}}_v = -N_c E_q - N_c \log \sqrt{Z_+ Z_-} - T \left( \log \mathcal{R}_q(E_q - \mu, \bar{\ell}, \bar{\ell}) + \log \mathcal{R}_q(E_q + \mu, \bar{\ell}, \bar{\ell}) \right),
\end{equation}

\begin{equation}
\mathcal{R}_q(x, y, \bar{y}) \equiv 1 + N_c(e^{-x/T} + e^{-2x/T}) + e^{-3x/T}
\end{equation}

\begin{equation}
F_{v}^{W} = -2T dN_c^2 \left( \frac{1}{g^2 N_c} \right)^{1/T} \bar{\ell} \ell - T \log \mathcal{R}_{\text{Haar}}(\ell, \bar{\ell}),
\end{equation}

\begin{equation}
\mathcal{R}_{\text{Haar}}(\ell, \bar{\ell}) \equiv 1 - 6\bar{\ell}\ell - 3(\bar{\ell}\ell)^2 + 4(\ell N_c + \bar{\ell} N_c),
\end{equation}

in Haar measure MFA case, and

\begin{equation}
F_{v}^{W} + F_{\text{det}}^{W} = 2T dN_c^2 \left( \frac{1}{g^2 N_c} \right)^{1/T} \bar{\ell} \ell - T \log \left[ \sum_{I} Q^I(\Phi) P^I(\ell, \bar{\ell}) \right],
\end{equation}

\begin{equation}
P^I(\ell, \bar{\ell}) = \sum_{n=\infty}^{\infty} \left( \frac{\sqrt{\bar{\ell}/\ell}}{N_{c}^{n} + N_{c}^{n}} \right) \mathcal{P}^I_n \left( \sqrt{\bar{\ell}/\ell} \right),
\end{equation}

in Weiss MFA case. In Eqs. (A21) and (A22), the index $I$ labels a pattern of thermal excitations of the quark composites, and the fermionic thermal excitation effects $Q^I$, the Polyakov loop thermal excitation effects $P^I_n$, and the baryon number index $N_{b}^{I}$ are summarized in Table V.

As indicated in Eq. (A19) and (A21), the $Z_3$ symmetric term remains in the Haar measure MFA, but not in the Weiss MFA up to the first $\bar{\ell}\ell$ term. In the latter, the path integral over the temporal link variable $U_0$ which accounts for the summation over the Polyakov loop fluctuations spoils the $Z_3$ symmetry in the presence of the dynamical quarks. In heavy quark mass limit $m_0 \to \infty$, the $Z_3$ symmetry recovers in the Weiss MFA as follows: In the effective potential of Weiss MFA, the factor $\left( \sqrt{\bar{\ell}/\ell} \right)^{-N_{c}^{n} + N_{b}^{I}}$ in Eq. (A22) gives a unique source of the explicit $Z_3$ symmetry breaking ($\ell, \bar{\ell} \to (\Omega \ell, \Omega^{-1} \bar{\ell})$, $\Omega \in Z_3$). For $m_0 \to \infty$ or equivalently $E_q \gg T, \mu$, the three mesonic thermal excitation $Q^{I=\text{MM}M}$ in Table V becomes dominant, and it does not carry the baryon number $N_{b}^{I=\text{MM}M} = 0$. Therefore, the Eq. (11) reduces to

\begin{equation}
F_{v}^{W} + F_{\text{det}}^{W} = 2T dN_c^2 \left( \frac{1}{g^2 N_c} \right)^{1/T} \bar{\ell} \ell - T \log \left[ \mathcal{P}_{\text{MM}M}^{I=\text{MM}M} \left( \sqrt{\bar{\ell}/\ell} \right) \right].
\end{equation}

This expression is invariant under the $Z_3$ transformation, $(\ell, \bar{\ell} \to (\Omega \ell, \Omega^{-1} \bar{\ell})$ with the property $\Omega^{N_{c}^{n} + N_{b}^{I}} = 1$ for $N_{c} = 3$.

Finally, we consider the confinement limit $(\ell, \bar{\ell} \to 0)$ in the Weiss MFA. The quark determinant effect (A21) includes the Polyakov loop thermal excitation $P^I_n$, which are solely characterized by the nth-order modified Bessel functions as shown in Table V. In the limit $(\ell, \bar{\ell} \to 0)$, the 0th-order modified Bessel function remains finite $(I_0(x \to 0) = 1)$ while the others vanishes $(I_n(x \to 0) = 0)$. Consequently, the only thermal excitations which carry the baryon number 0 and $\pm 3$ survives in Table V, and the effective potential reduces into the one which we have derived in our previous work [25],

\begin{equation}
F_{\text{eff}}^{W}(\Phi; \ell, \bar{\ell}; \beta, m_0, T, \mu)|_{\ell, \bar{\ell} = 0} \to F_{\text{eff}}^{\text{NLO}}(\Phi; \beta, m_0, T, \mu) = F_X(\Phi, \beta) - T \log \left[ (2 \cosh \frac{E_q}{T})^{N_{c}} - 4 \cosh \frac{E_q}{T} + 2 \cosh \frac{N_{c} \mu}{T} \right].
\end{equation}

\[\text{A24}\]


[38] P. de Forcrand, private communication: For the temporal lattice extension $N_t = 2$, the lattice bare coupling associated with the chiral phase transition ($\beta_c$) is found to be 3.67 at the bare quark mass $m_0 = 0.025$ and 3.81 at $m_0 = 0.05$ for one species of staggered fermion.


[51] In fact, the PNJL model is invented from the SC-LQCD [37].