Kadanoff-Baym Approach to Entropy Production in $O(N)$ Theory with Next-to-Leading Order Self-Energy

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We investigate entropy production in the $O(N)$ scalar theory based on the Kadanoff-Baym equation. We show that one of the candidate expressions of the kinetic entropy satisfies the H-theorem in the first order of the gradient expansion with the next-to-leading order self-energy of the $1/N$ expansion in the symmetric phase, and that entropy production occurs as the Green's function evolves with nonzero collision term contributions. Entropy production stops at local thermal equilibrium where the collision term contribution vanishes and the maximal entropy state is realized. We examine these features of entropy production in thermalization processes numerically in 1+1 dimensions for a couple of homogeneous cases, where the thermalization can proceed only with the off-shell effects. We find that the entropy production rate γ is larger for smaller N and is found to follow $\gamma \propto (1/N)^\delta$ where $\delta \gtrsim 2$ at strong coupling measured in the unit of bare mass (m), $\lambda = 40m^2$.

§1. Introduction

The 2 Particle Irreducible (2PI) effective functional technique provides a powerful tool to deal with controlled nonequilibrium dynamics with non-secularity and late time universality.[1),2)] It gives the Kadanoff-Baym (KB) equation which describes the dynamics for quantum fluctuations of fields. In 1960s, Luttinger and Ward investigated an expression of thermodynamic potential as a functional of two-point Green's function G.[3)] Based on this functional approach, Baym and Kadanoff studied the analytic continuation of the two-point function $G(x, y)$ in imaginary time formalism to real time analyses .[4),5)] Then Baym reformulated it in terms of variational principle, introducing the so-called Φ-derivable approximation[6)] which is given by a truncated set of closed 2PI diagrams. The main virtue of this approximation is that the resulting equations conserve the charge, energy and momentum of the system. This approach has been reformulated by use of path integral method. First, closed time circuit was proposed by Schwinger[7)] to calculate time-dependent variables for a pure state in a Brownian motion, and by Keldysh,[8)] who did not depend on the path-integral method in this stage. Later on, 2PI action technique was proposed and applied to non-equilibrium physics.[9),10)] Bloch-de Dominicis theorem,[11)] which is finite temperature version of Wick's theorem, is satisfied in nonequilibrium systems.[12),13)] It is applied to a variety of areas in physics, such as cosmology, ultra-relativistic heavy ion collisions, or condensed matter physics.[1),14)] In describing the reheating processes after inflation, it becomes necessary to trace the time evolution of oscillating inflaton field and quantum fluctuations.[15)–18)] In condensed matter physics, it can be applied to Bose-Einstein Condensate (BEC), since 2PI approach is a candidate with properties of gapless excitation and conservation laws.[19),20)] The 2PI approach with the KB equation would be useful also in understanding the early thermalization processes towards quark-gluon plasma formation in high-energy heavy-ion collisions,[21)] while it may be necessary to combine the classical field dynamics leading to the linear rise of entropy from the chaotic nature of the system.[22),23)]

In this paper we focus on the 2PI approach to the scalar $O(N)$ theory, where identical bosons interact one another in the symmetric phase. The $O(N)$ theory has been employed in the vacuum and thermal equilibrium analyses and it is also applied to time dependent phenomena.[1)] It can be used in inflation processes of the early universe, the formation of Bose-Einstein condensate in the laboratory, and the chiral phase transition in heavy ion collisions. The dynamics of the chiral phase transition following the expansion of a quark-gluon plasma produced in relativistic heavy ion collisions has been analyzed by an $O(4)$ σ model in the leading order of the $1/N$ expansion.[24),25)]

In the analysis of the $O(N)$ theory, the large N approximation (or the $1/N$ expansion) has been applied since 1960s or 1970s in both statistical mechanics and quantum field theory.[26)–28)] The $1/N$ expansion has the advantage over the loop expansion that it is not restricted to small couplings. However a naive diagrammatic $1/N$ expansion breaks down when it is applied to the dynamical simulations, because a secular (unbounded) time evolution makes it difficult to describe properly the late-time behavior of quantum field. As an improved treatment, we can take account of the self-consistency in the 1 particle irreducible (1PI) effective action, which corresponds to the first Legendre transformation of the generating functional with respect to the expectation value of the field, but the 1PI effective action is also plagued by secularity and the unitarity violation; for example $\langle \phi(x)^2 \rangle$ can become negative in the late time behavior.[29),30)] In order to extract the stable evolution without the above secularity, we need to go beyond the 1PI technique, and we adopt 2PI technique as a straightforward and powerful prescription. A systematic $1/N$ expansion of the 2PI effective action has been applied to a scalar $O(N)$ theory in the symmetric and broken phases.[31)] By resumming all 2 particle reducible diagrams having higher order secularity with respect to time, 2PI approach with the KB equation realizes the controlled time evolution without secularity and the unitarity violation in far-from-equilibrium dynamics.[32)]

In the literature, we find works on controlled numerical analyses of the next-to-leading order (NLO) skeletons of the $1/N$ expansion in the $O(N)$ theory in $1+1$ dimensions with vanishing classical field,[32)] simulations in the broken phase[16),18)] and estimation of nonthermal fixed point.[33)] It is numerically found that we need to include the NLO contribution in order to describe quantum scattering and thermalization. Recently NNLO of the $1/N$ expansion is derived[34)] and the rapid convergence property for moderate values of $1/N$ is observed.[35)] Numerical analyses show the thermalization of the distribution function derived from the two-point Green's function for the truncated self-energy. However there is no consideration of kinetic entropy and its H-theorem based on the KB equation. Thus we concentrate to examine that the H-theorem is satisfied at the level of the Green's function analytically for the NLO self-energy of the $1/N$ expansion in the symmetric phase. Our aim is to give a criterion whether thermalization occurs or not in off-shell propagation for a given self-energy or collision processes. In ϕ^4 theory with the NLO self-energy of the coupling expansion, numerical simulations without the gradient expansion[36)–40)] shows the resultant thermalization. The proof of the H-theorem[41)–43)] is consistent with these numerical results. The work given in this paper is an extension of our recent work on the H-theorem in ϕ^4 theory to $O(N)$ theory.[43)] We show the H-theorem for local parts of NLO self-energy of the $1/N$ expansion in the $O(N)$ theory. We find that the resultant H-theorem is consistent with numerical analyses performed for the $O(N)$ theory. Judging from this consistency, the proof of the H-theorem might be a criterion to confirm whether thermalization occurs or not without numerical simulation. It is possible to investigate entropy production qualitatively also in non-Abelian gauge theories.[44)]

It should be noted that there is no widely accepted general expression of kinetic entropy, then we need to prove the H-theorem for a given expression in each dynamical system. The present proof shows that the expression in Eq. (2·18) proposed by Kita[42)] is a good candidate of kinetic entropy. The same expression has been also shown to satisfy the H-theorem in non-Abelian gauge theories.[44)]

We also evaluate the evolution of the entropy density in equilibration processes by numerical simulation. Numerical analysis has been done in 1+1 dimension in order to examine the off-shell effects explicitly. It should be noted that the thermalization does not occur when the off-shell effects (spectral width and memory integral) are ignored as in the Boltzmann equation, since two-body collisions only exchange the momenta of two particles in 1+1 dimension. Thus equilibration and entropy production in 1+1 dimension is characteristic in dynamics with off-shell effects such as the Kadanoff-Baym equation. We find that the maximal entropy and the entropy production rate at late times are independent of initial conditions. The NLO effects are found to lead earlier thermalization; the kinetic entropy density derived here shows faster growth and earlier saturation for smaller N. This behavior is consistent with the derivation of the of H-theorem, which implies that a prefactor $\propto 1/N$ appears in the production rate and the spectral width is proportional to $1/\sqrt{N}$ in the first order of the gradient expansion.[43)]

This paper is organized as follows. In Sec. 2, we provide a proof of the H-theorem in the $O(N)$ theory with the NLO self-energy of the $1/N$ expansion in the Kadanoff-Baym equation. First we review the Kadanoff-Baym equation with the NLO self-energy in Sec. 2.1. Next in Sec. 2.2 we show that entropy increases in the $O(N)$ theory with the NLO self-energy. In the derivation, we adopt the gradient expansion with respect to the space-time coordinate. Since the KB equation is time-reversal invariant,[45)–47)] we need coarse graining in order to describe thermalization processes. Even if the equation is time-reversal invariant, the increasing complexity of the quantum state leads to the loss of practically obtainable information, since any kind of measurement results in some coarse graining.[22)] When we ignore higher order derivative terms, spatial fluctuations in a small volume are assumed to vanish. Thus the gradient expansion corresponds to introducing coarse graining in dynamics explicitly. In Sec. 3, we show the numerical simulation results on the time evolution of the kinetic entropy derived in this work. The equations of motion for the 2-point Green's functions are solved with the NLO self-energy in 1+1 dimension, and the N dependence of the entropy production rate is discussed. We summarize our work in Sec. 4.

§2. H-theorem in $O(N)$ theory with NLO self-energy in Kadanoff-Baym equation

2.1. *Kadanoff-Baym equation*

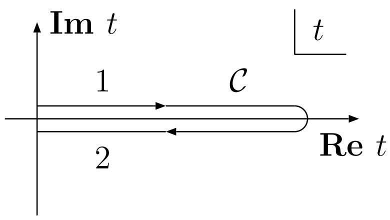

Fig. 1. Integral path in the time coordinate.

In this subsection, we briefly review the Kadanoff-Baym equation with the NLO self-energy of the $1/N$ expansion in the $O(N)$ theory. We follow the notation in Ref. 2). Let us consider a relativistic real scalar field ϕ_a ($a = 1, \cdots N$) with the $O(N)$ symmetric action,

$$S[\phi] = \int_x \left[\frac{1}{2}(\partial \phi_a)^2 - \frac{1}{2} m^2 \phi_a^2 - \frac{\lambda}{4!N} (\phi_a \phi_a)^2 \right] , \tag{2·1}$$

where summation over indices is implied. We adopt a closed time path $\mathcal{C}$ along the real time axis, the path from t_0 to ∞ and from ∞ to t_0 as shown in Fig. 1, in order to trace nonequilibrium dynamics.[7),8)] The 2PI effective action Γ with vanishing mean field $\langle \phi \rangle = 0$ (symmetric phase) is written as

$$\Gamma[G] = \frac{i}{2} \mathrm{Tr} \ln (G)^{-1} + \frac{i}{2} \mathrm{Tr} G_0^{-1} G + \frac{1}{2} \Phi[G] \tag{2·2}$$

with the full two-point Green's function

$$\begin{aligned} G_{ab}(x,y) \equiv& \langle \mathrm{T}_{\mathcal{C}} \phi_a(x) \phi_b(y) \rangle \\ =& \theta_{\mathcal{C}}(x^0 - y^0) \langle \phi_a(x) \phi_b(y) \rangle + \theta_{\mathcal{C}}(y^0 - x^0) \langle \phi_b(y) \phi_a(x) \rangle = \delta_{ab} G(x,y). \end{aligned} \tag{2·3}$$

We have assumed here that the Green's function is diagonal and isotropic in particle components, $G_{ab} = \delta_{ab} G$. Here $G_{ab}(x,y)$ is defined on the closed time path $\mathcal{C}$ as

$$G_{ab}^{21}(x,y) = \langle \phi_a(x) \phi_b(y) \rangle, \tag{2·4a}$$

$$G_{ab}^{12}(x,y) = \langle \phi_b(y) \phi_a(x) \rangle, \tag{2·4b}$$

$$G^{11}_{ab}(x,y) = \theta(x^0 - y^0)G^{21}_{ab}(x,y) + \theta(y^0 - x^0)G^{12}_{ab}(x,y), \tag{2·4c}$$
$$G^{22}_{ab}(x,y) = \theta(y^0 - x^0)G^{21}_{ab}(x,y) + \theta(x^0 - y^0)G^{12}_{ab}(x,y). \tag{2·4d}$$

The upper label "1" represents the path from t_0 to ∞ and "2" represents that from ∞ to t_0 in the closed time path contour $\mathcal{C}$. The term $\Phi[G]/2$ contains infinite series of 2PI diagrams whose lines are given by the full Green's function G. The stationary condition $\delta\Gamma/\delta G = 0$ for the effective action (2·2) gives rise to the Schwinger-Dyson equation for the non-equilibrium Green's function $G(x,y)$, *i.e.* the KB equation,

$$G^{-1}(x,y) = G_0^{-1}(x,y) - \Sigma(x,y) \ . \tag{2·5}$$

The proper self-energy (Σ) and the free Green's function (G_0) are defined as $\Sigma = i\delta\Phi[G]/\delta G$ and $iG_0^{-1}(x,y) = -(\partial_x^2 + m^2)\delta_{\mathcal{C}}(x-y)$, respectively.

The NLO diagrams of the $1/N$ expansion in $\Phi[G]/2$ is given by the series of diagrams shown in Fig.2, which can be written as,[31),32)]

$$\frac{1}{2}\Phi[G] = -\frac{\lambda}{4!N}\int_x G_{aa}(x,x)G_{bb}(x,x) + \frac{i}{2}\mathrm{Tr}\ln B[G] \ , \tag{2·6}$$
$$B(x,y;G) \equiv \delta_{\mathcal{C}}^{(d+1)}(x-y) + \frac{i\lambda}{6N}G_{ab}(x,y)G_{ab}(x,y) \ . \tag{2·7}$$

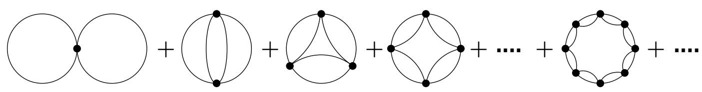

Fig. 2. NLO skeleton diagrams of the $1/N$ expansion in symmetric phase.[31)]

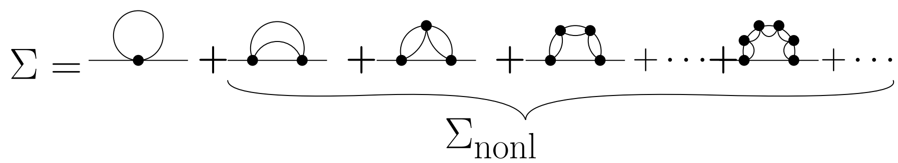

Fig. 3. Local and non-local part of the self-energy.

The two-point correlation function $G(x,y)$ and nonlocal part of self-energy $\Sigma(x,y)$ (Fig. 3) in the Schwinger-Keldysh formalism are expressed by the spectral and the statistical part,

$$\rho = i\left(G^{21} - G^{12}\right), \ \ F = \frac{1}{2}\left(G^{21} + G^{12}\right), \tag{2·8}$$
$$\Sigma_\rho = i\left(\Sigma^{21}_{\mathrm{nonl}} - \Sigma^{12}_{\mathrm{nonl}}\right), \ \ \Sigma_F = \frac{1}{2}\left(\Sigma^{21}_{\mathrm{nonl}} + \Sigma^{12}_{\mathrm{nonl}}\right). \tag{2·9}$$

Notice that $F(x,y) = F(y,x)$, $\rho(x,y) = -\rho(y,x)$, $\Sigma_F(x,y) = \Sigma_F(y,x)$ and $\Sigma_\rho(x,y) = -\Sigma\rho(y,x)$. The spectral function $\rho(x,y)$ contains the information on how many states are available, while the statistical function $F(x,y)$ represents how many particles exist. Then KB equation (2·5) can be reexpressed by using $F(x,y)$ and $\rho(x,y)$ as

$$\left[\partial_x^2 + M^2(x;F)\right] F(x,y) = -\left[\Sigma_\rho, F\right]_{t_0}^{x^0} + \left[\Sigma_F, \rho\right]_{t_0}^{y^0} \ , \tag{2·10}$$

$$\left[\partial_x^2 + M^2(x;F)\right] \rho(x,y) = -\left[\Sigma_\rho, \rho\right]_{y^0}^{x^0} \ , \tag{2·11}$$

where $M^2(x;F) = m^2 + \lambda \frac{N+2}{6N} F(x,x)$. We have introduced a short-hand notation, $[P,Q]_{t_1}^{t_2} \equiv \int_{t_1}^{t_2} dz^0 d^d z P(x,z) Q(z,y)$. Here self-energies are given by

$$\hat{\Sigma}_F(x,y) = F(x,y)\hat{D}_F(x,y) - \frac{1}{4}\rho(x,y)\hat{D}_\rho(x,y) \ , \tag{2·12}$$

$$\hat{\Sigma}_\rho(x,y) = \rho(x,y)\hat{D}_F(x,y) + F(x,y)\hat{D}_\rho(x,y) \ . \tag{2·13}$$

The functions $\hat{D}_F$ and $\hat{D}_\rho$ are given by resummation:

$$\hat{D}_F(x,y) = -\frac{\lambda}{3N}\hat{\Pi}_F(x,y) + \left[D_F, \hat{\Pi}_\rho\right]_{t_0}^{y^0} - \left[D_\rho, \hat{\Pi}_F\right]_{t_0}^{x^0} , \tag{2·14}$$

$$\hat{D}_\rho(x,y) = -\frac{\lambda}{3N}\hat{\Pi}_\rho(x,y) - \left[\hat{\Pi}_\rho, D_\rho\right]_{y^0}^{x^0} , \tag{2·15}$$

with the functions $\hat{\Pi}_F$ and $\hat{\Pi}_\rho$ given as,

$$\hat{\Pi}_F(x,y) = \frac{\lambda}{6}\left[F^2(x,y) - \frac{1}{4}\rho^2(x,y)\right] \ , \quad \hat{\Pi}_\rho(x,y) = \frac{\lambda}{3}F(x,y)\rho(x,y) \ . \tag{2·16}$$

Here notice that $\hat{\Pi}_F$ and $\hat{\Pi}_\rho$ represent the chain part of the diagrams in Fig. 2. In the large N limit, $N \to \infty$, the R.H.S. of Eqs. (2·10) and (2·11) vanish and neither thermalization nor damping of unequal-time Green's functions occurs.

2.2. *A proof of the H-theorem*

In this section, we provide a proof of the H-theorem for KB equation with NLO self-energy of the $1/N$ expansion. First we give the kinetic entropy from the Kadanoff-Baym (or equivalently Schwinger-Dyson) equation (2·5). We adopt G^{12} and G^{21} as independent functions in the Green's function. We start from the Kadanoff-Baym equation (2·5), set $t_0 \to -\infty$, Fourier transform for the relative coordinate $x-y$ to momentum p, take the 1st order of the gradient expansion,[48),49)] and use the Kadanoff-Baym Ansatz $G^{12} = -i\rho f$ and $G^{21} = -i\rho(1+f)$ with a real function f. Then we arrive at

$$\partial_\mu s^\mu(X) = \frac{1}{2}\int \frac{d^{d+1}p}{(2\pi)^{d+1}} \, C(X,p) \ln \frac{G^{12}(X,p)}{G^{21}(X,p)} \ , \tag{2·17}$$

where we have defined the entropy current $s^\mu(X)$ as

$$s^\mu(X) = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \left[\frac{\rho}{i}\left(p^\mu - \frac{1}{2}\frac{\partial \mathrm{Re}\Sigma_R(X,p)}{\partial p_\mu}\right) + \frac{\Sigma_\rho}{i}\frac{1}{2}\frac{\partial \mathrm{Re} G_R(X,p)}{\partial p_\mu}\right] \sigma(X,p) \ , \tag{2·18}$$

with $i(\Sigma^{11} - \Sigma^{22}) = 2\mathrm{Re}\Sigma_R$, $i(G^{11} - G^{22}) = 2\mathrm{Re}G_R$ and $\sigma(X,p) = -f \ln f + (1 + f)\ln(1+f)$.[43),50)–53)] The form of entropy density is equivalent to that in thermal equilibrium defined in Ref. 51) in the leading order of the gradient expansion. The memory correction term[41)] may cause deviation from Eq. (2·18), but it is in the higher order $O(\partial^2/\partial X^2)$ of the gradient expansion. The term C,

$$C(X,p) = i\left[\Sigma_\rho(X,p)\, F(X,p) - \Sigma_F(X,p)\, \rho(X,p)\right] \tag{2·19}$$

may be identified as the collision term in the Boltzmann limit, as derived in the leading order of the gradient expansion in Ref. 54). The procedure to derive (2·17) is the same as that in the previous work.[43)] Here we notice that the leading order of the gradient expansion of $C(X,p)$ is already the 1st order $\sim O(\partial/\partial X)$.[41),42)] We assume here that the interval of $x^0 - y^0$ can take sufficiently large values for a fixed X^0.

The remaining task is to prove that the R.H.S. of Eq. (2·17) is positive definite. Since Eq. (2·17) is local in space-time, we omit the common space-time variable X in the Fourier transformed quantity in the later discussion of this section.

Next we shall Fourier transform $\hat{\Sigma}_F$ and $\hat{\Sigma}_\rho$ in addition to $\hat{D}_F$ and $\hat{D}_\rho$ with $\hat{\Pi}_F$ and $\hat{\Pi}_\rho$.[54)] We start from $\hat{D}_\rho(p)$ and $\hat{D}_F(p)$ in Eqs. (2·14) and (2·15), which are necessary to calculate self-energies. We calculate $\hat{D}_R(p)$ and $\hat{D}_A(p)$ by multiplying $\theta(x^0 - y^0)$ and $-\theta(y^0 - x^0)$ in (2·15) and Fourier transform. $\hat{D}_R(p)$ is obtained as,

$$\begin{aligned}
\hat{D}_R(x,y) \equiv& \theta(x^0 - y^0)\hat{D}_\rho(x,y) = -\frac{\lambda}{3N}\theta(x^0 - y^0)\hat{\Pi}_\rho(x,y) - \theta(x^0 - y^0)\left[\hat{\Pi}_\rho, \hat{D}_\rho\right]_{x^0}^{y^0} \\
=& -\frac{\lambda}{3N}\theta(x^0 - y^0)\hat{\Pi}_\rho(x,y) - \int_{t_0}^{\infty} dz\ \hat{\Pi}_\rho(x,z)\hat{D}_\rho(z,y)\theta(z^0 - y^0)\theta(x^0 - z^0) \\
=& -\frac{\lambda}{3N}\hat{\Pi}_R(x,y) - \left[\hat{\Pi}_R, \hat{D}_R\right]_{t_0}^{\infty} ,
\end{aligned} \tag{2·20}$$

where we have used $\theta(x^0-y^0)\theta(x^0-z^0) = \theta(y^0-z^0)\theta(x^0-y^0) + \theta(z^0-y^0)\theta(x^0-z^0)$. By taking $t_0 \to -\infty$, carrying out Fourier transformation, and using the gradient expansion, $\hat{D}_R(p)$ is written in the leading order of the gradient expansion as

$$\hat{D}_R(p) = -\frac{\lambda}{3N}\hat{\Pi}_R(p) - \hat{\Pi}_R(p)\hat{D}_R(p)\ . \tag{2·21}$$

By solving this equation with respect to $\hat{D}_R(p)$, we obtain,

$$\hat{D}_R(p) = -\frac{\lambda}{3N}\frac{\hat{\Pi}_R(p)}{1 + \hat{\Pi}_R(p)}\ , \tag{2·22}$$

where we have taken only the leading order (local parts) of the gradient expansion. *) In a similar way $D_A(X,p) = -\mathrm{F.T.}\theta(y^0 - x^0)\hat{D}_\rho(x,y)$ can be written as

$$\hat{D}_A(p) = -\frac{\lambda}{3N}\frac{\hat{\Pi}_A(p)}{1 + \hat{\Pi}_A(p)}. \tag{2·23}$$

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*) Here it might be necessary to extract the nonlocal parts in estimating memory correction terms[55)] although it is not assured that the nonlocal parts is within the 1st order gradient expansion.

By using Eqs. (2·22) and (2·23), we obtain

$$\hat{D}_\rho(p) = \hat{D}_R(p) - \hat{D}_A(p) = -\frac{\lambda}{3N}\frac{\hat{\Pi}_\rho(p)}{\left[1+\hat{\Pi}_R(p)\right]\left[1+\hat{\Pi}_A(p)\right]} = -\frac{\lambda_{\rm eff}(p)}{3N}\hat{\Pi}_\rho(p). \tag{2·24}$$

We have defined the effective coupling as $\lambda_{\rm eff}(p) = \lambda/[1+\hat{\Pi}_R][1+\hat{\Pi}_A]$.

We obtain $\hat{D}_F(p)$ in a similar way. $\hat{D}_F(x,y)$ is rewritten as

$$\begin{aligned}\hat{D}_F(x,y) =& -\frac{\lambda}{3N}\hat{\Pi}_F(x,y) + \left(\left[\hat{D}_F,\hat{\Pi}_\rho\right]_{t_0}^{y^0} - \left[\hat{D}_\rho,\hat{\Pi}_F\right]_{t_0}^{x^0}\right)\\ =& -\frac{\lambda}{3N}\hat{\Pi}_F(x,y) - \left(\left[\hat{D}_F,\hat{\Pi}_A\right]_{t_0}^{\infty} + \left[\hat{D}_R,\hat{\Pi}_F\right]_{t_0}^{\infty}\right) .\end{aligned} \tag{2·25}$$

The Fourier transform $\hat{D}_F(p)$ is obtained by taking $t_0 \to -\infty$ and using Eq. (2·22) as

$$\hat{D}_F(p) = -\frac{\lambda}{3N}\hat{\Pi}_F - \left(\hat{D}_F\hat{\Pi}_A + \hat{D}_R\hat{\Pi}_F\right) = -\frac{\lambda_{\rm eff}(p)}{3N}\hat{\Pi}_F(p) . \tag{2·26}$$

We substitute $\hat{D}_\rho$ and $\hat{D}_F$ in Eqs. (2·24) and (2·26) into Eqs. (2·12) and (2·13), and get the Fourier transformed self-energies $\hat{\Sigma}_F(p)$ and $\hat{\Sigma}_\rho(p)$ as,

$$\begin{aligned}\hat{\Sigma}_F(p) =& \int \frac{d^{d+1}q}{(2\pi)^{d+1}}\left[F(q)\hat{D}_F(p-q) - \frac{1}{4}\rho(q)\hat{D}_\rho(p-q)\right]\\ =& -\int \frac{d^{d+1}q}{(2\pi)^{d+1}}\frac{\lambda_{\rm eff}(p-q)}{3N}\left[F(q)\hat{\Pi}_F(p-q) - \frac{1}{4}\rho(q)\hat{\Pi}_\rho(p-q)\right] ,\end{aligned} \tag{2·27}$$

$$\begin{aligned}\hat{\Sigma}_\rho(p) =& \int \frac{d^{d+1}q}{(2\pi)^{d+1}}\left[\rho(q)\hat{D}_F(p-q) + F(q)\hat{D}_\rho(p-q)\right]\\ =& -\int \frac{d^{d+1}q}{(2\pi)^{d+1}}\frac{\lambda_{\rm eff}(p-q)}{3N}\left[\rho(q)\hat{\Pi}_F(p-q) + F(q)\hat{\Pi}_\rho(p-q)\right],\end{aligned} \tag{2·28}$$

$\hat{\Pi}_F(p)$ and $\hat{\Pi}_\rho(p)$ can be written from (2·16) as

$$\hat{\Pi}_F(p) =\frac{\lambda}{6}\int \frac{d^{d+1}q}{(2\pi)^{d+1}}\left[F(p-q)F(q) - \frac{1}{4}\rho(p-q)\rho(q)\right] , \tag{2·29}$$

$$\hat{\Pi}_\rho(p) =\frac{\lambda}{3}\int \frac{d^{d+1}q}{(2\pi)^{d+1}}\; F(p-q)\rho(q) . \tag{2·30}$$

Now we can express the collision term $C(p)$ in Eq. (2·19) in terms of $G^{12,21}$ and $\lambda_{\rm eff}$,

$$C(p_1) =\frac{\lambda}{18N}\int d\Gamma_2\, K(p_1,p_2,p_3,p_4) , \tag{2·31}$$

$$K(p_1,p_2,p_3,p_4) \equiv G^{12}(p_1)G^{12}(p_2)G^{21}(p_3)G^{21}(p_4)$$

$$-G^{21}(p_1)G^{21}(p_2)G^{12}(p_3)G^{12}(p_4) \ , \tag{2·32}$$

$$d\mathit{\Gamma}_j \equiv \prod_{i=j}^{4} \frac{d^{d+1}p_i}{(2\pi)^{d+1}} \delta^{d+1}(p_1+p_2-p_3-p_4)\lambda_{\rm eff}(p_1+p_2) \ . \tag{2·33}$$

Here we have used Eqs. (2·27), (2·28), (2·29), (2·30), and (2·8), and the symmetry relations, $F(-q) = F(q)$, $\rho(-q) = -\rho(q)$.

Finally, the divergence of the entropy current per particle component in Eq. (2·17) can be written as

$$\begin{aligned}
\partial_\mu s^\mu(X) = & \frac{\lambda}{36N} \int d\mathit{\Gamma}_1 \, K(p_1,p_2,p_3,p_4) \ln \left[\frac{G^{12}(p_1)}{G^{21}(p_1)} \right] \\
= & \frac{\lambda}{36N} \cdot \frac{1}{2} \int d\mathit{\Gamma}_1 \, K(p_1,p_2,p_3,p_4) \ln \left[\frac{G^{12}(p_1)G^{12}(p_2)}{G^{21}(p_1)G^{21}(p_2)} \right] \\
= & \frac{\lambda}{36N} \cdot \frac{1}{4} \int d\mathit{\Gamma}_1 \, K(p_1,p_2,p_3,p_4) \ln \left[\frac{G^{12}(p_1)G^{12}(p_2)G^{21}(p_3)G^{21}(p_4)}{G^{21}(p_1)G^{21}(p_2)G^{12}(p_3)G^{12}(p_4)} \right] \\
= & \frac{\lambda}{36N} \cdot \frac{1}{4} \int d\mathit{\Gamma}_1 \, \left[G^{12}(p_1)G^{12}(p_2)G^{21}(p_3)G^{21}(p_4) - G^{21}(p_1)G^{21}(p_2)G^{12}(p_3)G^{12}(p_4) \right] \\
& \times \ln \left(\frac{G^{12}(p_1)G^{12}(p_2)G^{21}(p_3)G^{21}(p_4)}{G^{21}(p_1)G^{21}(p_2)G^{12}(p_3)G^{12}(p_4)} \right) \geq 0 \ .
\end{aligned} \tag{2·34}$$

In this derivation, we have utilized the inequality $(x-y)\ln(x/y) \geq 0$, and the symmetry of K under the exchange of momenta, $p_1 \leftrightarrow p_2$ and $(p_1,p_2) \leftrightarrow (p_3,p_4)$,

$$K(p_1,p_2,p_3,p_4) = K(p_2,p_1,p_3,p_4) = -K(p_3,p_4,p_1,p_2) \ . \tag{2·35}$$

Hence we have proved the H-theorem for the NLO self-energy of the $1/N$ expansion in the 1st order gradient expansion. We can see that any change of Green's functions with nonzero collision term $C = i\left[\hat{\mathit{\Sigma}}_\rho F - \hat{\mathit{\Sigma}}_F \rho\right] \neq 0$ contributes to entropy production.

The equality in Eq. (2·34) holds when $K = 0$ is satisfied for all combinations of momenta which satisfy the energy-momentum conservation. This condition is realized when G^{21}/G^{12} is a linear function of p,

$$\ln \frac{G^{21}(X,p)}{G^{12}(X,p)} = \ln \frac{1+f(X,p)}{f(X,p)} = \alpha(X) + \beta^\mu(X)p_\mu \tag{2·36}$$

where $\alpha(X)$ and $\beta(X)$ are arbitrary functions for the center coordinate X. Since particle number is not conserved without for any additional symmetry of $O(N)$ theory, $\alpha(X)$ becomes zero. It is shown by use of the relation $G^{12}(-p_i) = G^{21}(p_i)$ $(i = 1,2,3,4)$ in Eq. (2·34) Thus we notice that the local Bose distribution function is realized in equilibrium,

$$f(X,p) = \frac{1}{e^{\beta^\mu(X)p_\mu} - 1}. \tag{2·37}$$

Equation (2·31) shows that C actually has the form of the collision term of bosons. The first term in K represents the loss term, which is proportional to the product of probability $G^{12}(p_1)G^{12}(p_2) \propto f(p_1)f(p_2)$ for particles having p_1 and p_2, and proportional to the bosonic enhancement factor in the final state, $G^{21}(p_3)G^{21}(p_4) \propto (1+f(p_3))(1+f(p_4))$, as schematically shown in Fig. 4.

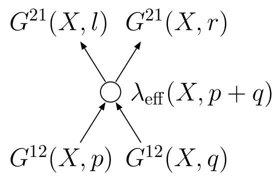

Fig. 4. Collision term C

§3. Numerical analysis of the kinetic entropy evolution in 1+1 dimensions

3.1. *Entropy production*

We shall now examine the time evolution of the kinetic entropy (2·18) derived in the previous section. We simulate the time evolution of the 2-point Green's functions in the NLO of the $1/N$ expansion in 1+1 dimension, and we discuss the N dependence of the entropy production rate. We assume the system is spatially homogeneous and the Green's functions are diagonal in particle components $G_{ab} = \delta_{ab}G$, then it is enough to consider the Fourier transformed Green's functions with respect to $\mathbf{x}-\mathbf{y}$. It should be noted that there is no thermalization in the Boltzmann equation in 1+1 dimension, since the two-body collision only results in the exchange of two momenta. Thus entropy production described here purely comes from the off-shell effects.

The initial condition of the spectral function ρ is constrained by the equal-time commutation relations. Fourier transformed spectral functions at $x^0 = y^0 = t^0$ are given as

$$\rho_{ab}(x^0,y^0;\mathbf{p})\Big|_{x^0=y^0} = 0 \ , \quad \partial_{x^0}\rho_{ab}(x^0,y^0;\mathbf{p})\Big|_{x^0=y^0} = \delta_{ab}. \tag{3·1}$$

We give initial conditions of the statistical function F in terms of the initial number distribution $n^0_{\mathbf{p}}$ as

$$\begin{aligned} F(x^0,y^0;\mathbf{p})\Big|_{x^0=y^0} &= \frac{1}{\omega^0_{\mathbf{p}}}\left[\frac{1}{2}+n^0_{\mathbf{p}}\right] \\ \partial_{x^0}F(x^0,y^0;\mathbf{p})\Big|_{x^0=y^0} &= 0 \\ \partial_{x^0}\partial_{y^0}F(x^0,y^0;\mathbf{p})\Big|_{x^0=y^0} &= \omega^0_{\mathbf{p}}\left[\frac{1}{2}+n^0_{\mathbf{p}}\right] \end{aligned} \tag{3·2}$$

where the matrix δ_{ab} is factorized, $\omega^0_{\mathbf{p}} = \sqrt{\mathbf{p}^2+M_0^2}$, and M_0^2 including the leading order shift in the $1/N$ expansion is self-consistently determined by

$$M_0^2 = m^2 + \frac{\lambda}{6}\int\frac{d^dp}{(2\pi)}F(t_0,t_0,\mathbf{p}). \tag{3·3}$$

Here we must subtract $\frac{\lambda}{6}\int\frac{d^dp}{(2\pi)}\frac{1}{2\sqrt{\mathbf{p}^2+m^2}}$ in the R.H.S. to renormalize the tadpole mass shift. Starting from this common initial condition, dynamical NLO effects are investigated in the later discussions. As we explain later, the mass shift due to the interaction is significant, and it is numerically important to include the large mass shift effects for large λ/m^2 in the initial condition to obtain stable results.

The Fourier transformed KB equation for the statistical $F(x^0, y^0; \mathbf{p})$ and spectral $\rho(x^0, y^0; \mathbf{p})$ functions in Eqs. (2·10) and (2·11) read,

$$\left(\partial_{x^0}^2 + \mathbf{p}^2 + M^2(x^0; F)\right) F(x^0, y^0; \mathbf{p}) = -\,[\hat{\Sigma}_\rho, F]_{t_0}^{x_0} + [\hat{\Sigma}_F, \rho]_{t_0}^{y_0}\ , \tag{3·4}$$

$$\left(\partial_{x^0}^2 + \mathbf{p}^2 + M^2(x^0; F)\right) \rho(x^0, y^0; \mathbf{p}) = -\,[\hat{\Sigma}_\rho, \rho]_{y^0}^{x_0} \tag{3·5}$$

where $[P, Q]_{t_1}^{t_2}(x_0, y_0; \mathbf{p}) \equiv \int_{t_1}^{t_2} dz^0 P(x_0, z_0; \mathbf{p}) Q(z_0, y_0; \mathbf{p})$. The squared mass M^2 contains subleading terms in the $1/N$ expansion,

$$M^2(x^0; F) = m^2 + \lambda\frac{N+2}{6N}\int\frac{d^d\mathbf{k}}{(2\pi)^d} F(x^0, x^0; \mathbf{k}). \tag{3·6}$$

We can obtain Fourier transformed self-energies Σ_F and Σ_ρ in Eqs. (2·12) and (2·13) and functions D_F and D_ρ in Eqs. (2·14) and (2·15) in a similar way.

We solve the coupled equation (3·4) and (3·5) on the lattice. We discretize the space $L = 2N_s a_s$ into $2N_s$ grid points $x_n = na_s$ $(n = -N_s, -N_s+1, \cdots, N_s-1, N_s)$ with a_s the lattice spacing and apply the periodic boundary condition. Then the momentum is discretized as $p_n = \frac{2\pi n}{L}$, and $-\partial_x^2$ is replaced with $\mathbf{p}^2 \equiv \frac{4}{a_s^2}\sin^2\left(\frac{a_s p_n}{2}\right)$, which removes much of the lattice artifacts.[56] We use $N_s = 40$ which is sufficient to study the momentum dependence. We have confirmed that the simulation with $N_s = 80$ has no appreciable differences numerically. We adopt the parameters of bare mass $ma_s = 0.3$ and coupling $\lambda/m^2 = 40$. We solve the evolution with the time step $a_t/a_s = 0.25$, and we discuss the results with $N = 10, 5, 4, 3$ and 2.

In the same way as in Ref. 43), we prepare the two different types of the initial conditions, "tsunami" distribution and the Woods-Saxon (WS) distribution for the number distribution function $n_{\mathbf{p}}^0$ in (3·2),

$$n_{\mathbf{p}}^T = \frac{1}{\mathcal{N}_T}\exp\left[-\frac{(|p_x| - p_T)^2}{2\sigma^2}\right]\ , \tag{3·7}$$

$$n_{\mathbf{p}}^{WS} = \frac{1}{\mathcal{N}_{WS}}\frac{1}{e^{(\sqrt{\mathbf{p}^2+m^2}-p_{WS})/\kappa}+1}\ , \tag{3·8}$$

with $\sigma^2/m^2 = 6.0\times(2\pi/mL)^2$, $p_T = 9\cdot 2\pi/L$, $\mathcal{N}_T = 0.25$, $p_{WS}/m = 3.4$, $\kappa/m = 0.45$ and $\mathcal{N}_{WS} = 0.5$. The "tsunami" initial condition with two peaks at $\pm p_T$ may be regarded as a toy model of the nuclear collisions. The WS initial condition is prepared to investigate the sensitivity of the evolution to the initial condition. The above parameters in the WS case are tuned so that the two initial conditions give the same energy for $\lambda/m^2 = 40$ and $N = 10$.

In Fig. 5, we show the time evolution of the number distribution $n_{\mathbf{p}}$ for $N = 10$

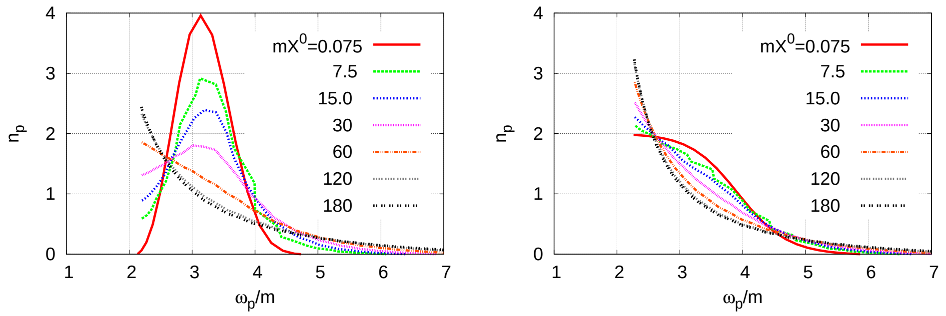

Fig. 5. Evolution of the distribution function $n_{\mathbf{p}}(p_x)$ from the tsunami (left) and WS (right) initial conditions ($\lambda/m^2 = 40,\ N = 10$).

as a function of $\tilde{\omega}_{\mathbf{p}}$ defined as,

$$n_{\mathbf{p}}(X^0) + \frac{1}{2} = \sqrt{F\frac{\partial^2 F}{\partial x^0 \partial y^0} - \left\{\frac{\partial F}{\partial x^0}\right\}^2}\Bigg|_{x^0=y^0=X^0}, \tag{3·9}$$

$$\tilde{\omega}_{\mathbf{p}}(X^0) = \sqrt{\frac{1}{F}\frac{\partial^2 F}{\partial x^0 \partial y^0}}\Bigg|_{x^0=y^0=X^0}. \tag{3·10}$$

With the tsunami initial condition shown in the left panel of Fig. 5, the Gaussian peak dumps rapidly and the high and low momentum components grow up gradually with time. The number distribution function is found to approach the Bose distribution function $n_{\mathbf{p}}^{\rm eq} = 1/(e^{(\tilde{\omega}_{\mathbf{p}}-\mu)/T} - 1)$ with $T/m = 2.06$ (2.12) and $\mu/m = 1.65$ (1.57) at the time $mX^0 = 180$ (300). At very late time $mX^0 \geq 300$, this chemical potential tends to vanish moderately. The time evolution from the WS initial condition shown in the right panel of Fig. 5 also converges to the Bose distribution with time. In these simulations, the errors in energy are within 1 %.

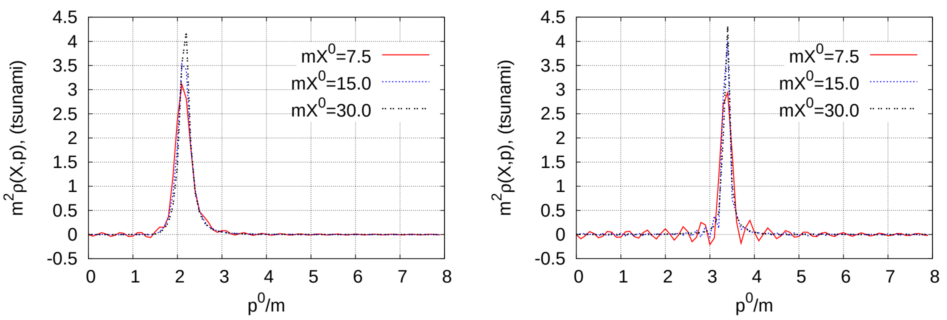

Fig. 6. Evolution of the spectral function $\rho(X^0, p_0, p_x)$ at $p_x = 0$ (left panel) and $p_x = 2\pi \cdot 10/L$ (right panel) for the tsunami initial condition ($\lambda/m^2 = 40,\ N = 10$).

Next we show the time evolution of the spectral function $\rho(X, p)$ in Fig. 6 for $p_x = 2\pi n/L$ with $n = 0$ and 10 for the tsunami initial condition. We can see the clear peak structure near $p^0/m \sim \sqrt{M^2 + p_x^2}/m$ at later times. The shift of the squared mass $M^2 - m^2 \sim 3m^2$ is comparable to the squared bare mass due to the large coupling $\lambda/m^2 = 40$. At early times, we find oscillatory behaviors of $\rho(p)$ as functions of p^0 in the spectral function as explained in Ref. 43). Similar early time oscillation is observed in the occupation number function $f(X,p)$, whose logarithm is necessary to obtain the kinetic entropy. At later times ($mX^0 > 20$), such oscillatory behavior disappears and it becomes possible to define the kinetic entropy in Eq. (2·18).

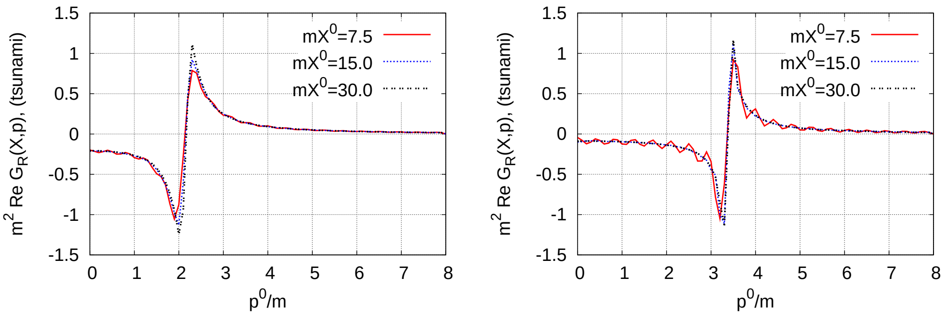

Fig. 7. Evolution of the $\mathrm{Re}G_R(X^0, p^0, p_x)$ at $p_x = 0$ (left panel) and $p_x = 2\pi \cdot 10/L$ (right panel) for the tsunami initial condition ($\lambda/m^2 = 40$, $N = 10$).

In Fig. 7, we show the evolution of the real part of retarded Green's function, $\mathrm{Re}\, G_R$ for $p_x = 2\pi n/L$ with $n = 0$ and 10 for tsunami initial condition. This quantity and the retarded self-energy Σ_R are necessary to calculate the entropy density (2·18). These functions are given by the principal values of the integrals,

$$G_R = P\int \frac{dp'^0}{2\pi i} \frac{\rho(X^0, p'^0, p_x)}{p^0 - p'^0} \ , \quad \Sigma_R = P\int \frac{dp'^0}{2\pi i} \frac{\Sigma_\rho(X^0, p'^0, p_x)}{p^0 - p'^0} \ . \qquad (3\cdot 11)$$

These relations are derived by using the well known relations $G_R(x,y) = i(G^{11} - G^{12}) = \theta(x^0 - y^0)\rho(x,y)$ and $\Sigma_R(x,y) = i(\Sigma^{11} - \Sigma^{12}) = \theta(x^0 - y^0)\Sigma_\rho(x,y)$ and taking the real part. Then we need the integrand at the point $p^0 = p'^0$. Since we need principal values, we subtract $\rho(X^0, p^0, p_x)$ and $\Sigma_\rho(X^0, p^0, p_x)$ in the numerator of Eq. (3·11), then we can estimate the integrand by the derivatives $-\frac{\partial\rho(p'^0)}{\partial p'^0} = \frac{\rho(p'^0)-\rho(p^0)}{p^0-p'^0}\Big|_{p'^0\to p^0}$ and $-\frac{\partial\Sigma_\rho(p'^0)}{\partial p'^0} = \frac{\Sigma_\rho(p'^0)-\Sigma_\rho(p^0)}{p^0-p'^0}\Big|_{p'^0\to p^0}$. In this figure we find decreasing, increasing and decreasing behavior around the peak of spectral functions in Fig. 6. This behavior is originated from the convolution of the function $1/p^0$ and two-peak structure spectral function. We find oscillating behaviors at early times, but at $mX^0 \gtrsim 10$ the retarded functions are nearly static since the spectral function does not change so drastically.

We show the kinetic entropy $s(KB)$ in Eq. (2·18) as a function of time X^0 for the tsunami and WS initial conditions in Fig. 8. As an exploratory estimate, we simply neglect the contributions from the region where $\rho(X,p)$ is negative. As expected from the H-theorem, the kinetic entropy increases monotonically in the later stage

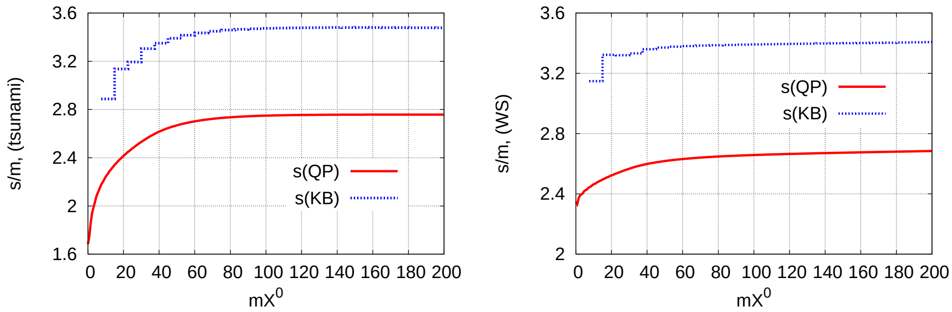

Fig. 8. Evolution of entropy density s(KB) (2·18) denoted by the histogram and its quasiparticle approximation (the limit of $\Sigma_{\rm nonl} \to 0$) of the kinetic entropy s(QP) (3·12) shown in a curve for the tsunami (left) and WS (right) initial conditions ($\lambda/m^2 = 40$, $N = 10$).

$mX^0 > 20$ in both cases. Asymptotic value of entropy has little dependence on the initial conditions. The deviation is about 1 % ($s^{\rm tsunami}(KB)/s^{\rm WS}(KB) = 1.017$) at $mX^0 = 260$.

Here we also show the quasiparticle (QP) approximation of the entropy density, $s(QP)$. In the QP approximation, we take the limit of $\Sigma_{\rm nonl} \to 0$, and the entropy current is obtained by using the number distribution $n_{\mathbf{p}}(X^0)$ defined in Eq. (3·9) as

$$s^\mu \to \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu}{\epsilon_p} \left[-n_{\mathbf{p}} \ln n_{\mathbf{p}} + (1+n_{\mathbf{p}}) \ln(1+n_{\mathbf{p}}) \right] \ . \qquad (3\cdot 12)$$

The time component of this entropy current is $s(QP)$. Because of the effects of finite width of spectral function, $s(KB)$ is larger than $s(QP)$. Since the spectral function is nicely peaked near $p^0 \sim \sqrt{M^2 + \mathbf{p}^2}$, $s(QP)$ gives a reasonable estimate of entropy production, while it is not based on the H-theorem of the KB equation.

3.2. *N dependence*

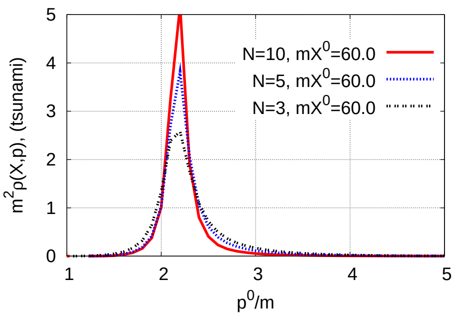

Fig. 9. Spectral function $\rho(p^0, p_x = 0)$ at $mX^0 = 60.0$ for $N = 10$, 5 and 3 for tsunami initial condition with $\lambda/m^2 = 40$.

We now discuss the N dependence of the kinetic entropy $s(KB)$ and the equilibration time. The kinetic entropy is sensitive to the spectral width, which is larger for smaller N as shown in Fig. 9. Thus we expect shorter equilibration time for smaller N.

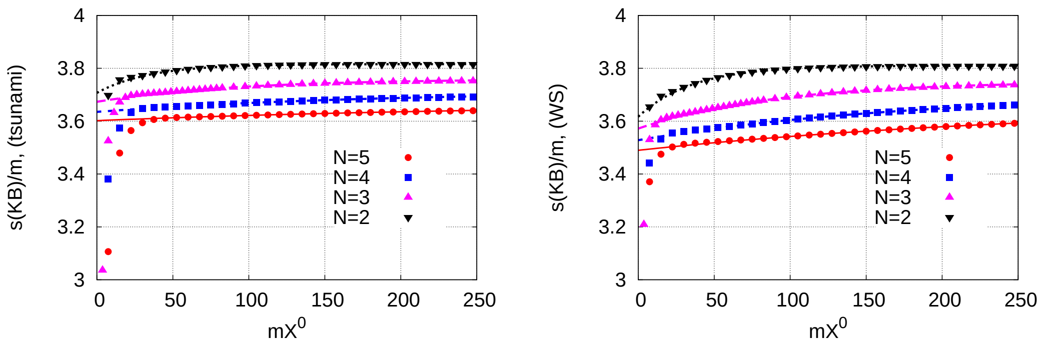

Fig. 10. Entropy density s^0(KB)$/m$ (2·18) for the tsunami (left) and WS (right) initial conditions with N =5,4,3 and 2 for $\lambda/m^2 = 40$. Curves show the exponential fit with (3·13).

In Fig. 10, we show the N dependence of the kinetic entropy evolution with the tsunami (left panel) and WS (right panel) initial conditions with $N = 5, 4, 3$ and 2. In order to evaluate the entropy production rate (inverse of the equilibration time) qualitatively, we fit the kinetic entropy at late times with the following fit function,

$$s^0 = s_{\max} - A \exp\left(-\gamma X^0\right) \ . \tag{3·13}$$

In Table I, we show the maximal entropy density $s_{\max}$, produced entropy A and the entropy production rate γ, which are obtained by fitting the kinetic entropy in the time range $100 \le mX^0 \le 180$, $80 \le mX^0 \le 150$, $70 \le mX^0 \le 120$ and $60 \le mX^0 \le 100$ for $N = 5, 4, 3$ and 2, respectively.

Table I. Thermalization constants of kinetic entropy and temperature at late time. Parameters in Eq. (3·13) and temperature at $mX^0 = 180$ for "tsunami" (left) and WS (right) initial conditions for $N = 2, 3, 4$ and 5 with $\lambda/m^2 = 40$. Each parameter is scaled by the bare mass.

N	λ	$s_{\max}$	A	γ	T	$s_{\max}$	A	γ	T
2(OS)	40	3.82	0.11	0.029	3.3	3.81	0.19	0.028	3.2
3(OS)	40	3.76	0.087	0.011	2.8	3.75	0.18	0.011	2.6
4(OS)	40	3.71	0.078	0.0057 ± 0.0001	2.5	3.71	0.18	0.0056 ± 0.0002	2.3
5(OS)	40	3.67	0.067	0.0035 ± 0.0002	2.4	3.67	0.17	0.0035 ± 0.0001	2.1

We find that $s_{\max}$ and γ are almost independent of the initial conditions. Their N dependence may be qualitatively understood from the spectral width; since the width is proportional to $1/\sqrt{N}$, we expect larger asymptotic entropy and faster entropy production for smaller N. In the proof of the H-theorem, we find that the prefactor is proportional to $1/N$ in Eq. (2·34), but the N dependence of the integrand is non-trivial. In Fig. 11, we show the N dependence of the entropy production rate γ/m. It seems that the entropy production rate would have the N dependence of $\gamma/m \propto 1/N^\delta$, where $\delta \sim 2.3$. The distribution functions converge to Bose distribution with temperature $T/m = 2.1 - 3.3$ at $mX^0 = 180$. Their values of

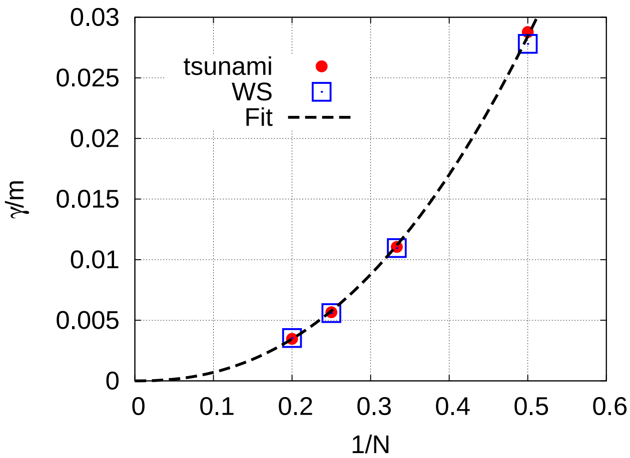

Fig. 11. Entropy production rate for the tsunami (filled circle) and the WS (square) initial conditions with $N = 5, 4, 3$ and 2 ($\lambda/m^2 = 40$).

temperature seem to be a little dependent on initial conditions, but they converge to the same value for each N at later times.

The present analyses show that $\delta \sim 2.3 > 2$ and the damping rate may be in the higher order of the $1/N$ expansion. With Next-to-Next-to-Leading Order (NNLO) terms of the $1/N$ expansion, it is difficult to apply the same procedure in the proof of the H-theorem, and is out of the scope of this article. It should be noted that as is shown in Ref. 35), damping rate of unequal Green's function or spectral width shows rapid convergent behavior for small N, while the change of effective mass in NLO and NNLO is significant at small N. Having this observation in mind, we expect that thermalization time in NNLO may be similar to that in NLO.

§4. Summary

In this article, we have discussed the entropy density and the H-theorem in the Kadanoff-Baym equation for the $O(N)$ theory with the next-to-leading order (NLO) self-energy of the $1/N$ expansion. While many numerical analyses show thermalization takes place in the Kadanoff-Baym equation, it is desired to confirm thermalization analytically for a given form of the self-energy. We have confirmed that the H-theorem is satisfied for the relativistic kinetic entropy. We have adopted the gradient expansion and taken account of the 1st order terms of the Kadanoff-Baym equation. The gradient expansion is an appropriate approximation in treating moderately changing systems in space-time. Since we ignore higher order derivative terms, fluctuations in a small space-time volume are assumed to vanish. Thus the gradient expansion corresponds to a coarse graining procedure. Time-reversal invariance is violated by the gradient expansion of the (time reversible) Kadanoff-Baym equation. Then it is suggested that the gradient expanded KB dynamics obtains the time arrow at the level of the Green's function, and that the observed system at late time will reach the maximum entropy state where local equilibrium is realized. The reliability of the gradient expansion and the form of the self-energy are important in the proof of the H-theorem. From the proof of the H-theorem we find that any change of the Green's function with nonzero collision term contributes to entropy production. Entropy production is ensured at the level of the Green's function, and we do not have to take quasiparticle approximation. When local thermal equilibrium is realized, the collision term contribution vanishes and entropy ceases to increase. The proof of H-theorem is consistent with the numerical analyses of the $O(N)$ theory. Thus the proof might provide a criterion whether thermalization occurs even in the cases when numerical analyses are difficult to perform.

We have also shown entropy production in numerical simulation with two types of initial conditions, tsunami and Woods-Saxon structure in momentum space, for the NLO self-energy of the $1/N$ expansion, and evaluated the time evolution of kinetic entropy density for various N in 1+1 dimensions. In order to estimate our entropy density we have to resolve sharp structure of Fourier transformed spectral and statistical functions. We have demonstrated that we can estimate the kinetic entropy density $s(KB)$ in the later stage. For smaller N, the spectral width becomes wider and entropy production is promoted more. This behavior is expected from the proof of H-theorem and consistent with it. We also estimated asymptotic behavior of entropy production with exponential fit. The maximal entropy density and the entropy production rate are independent of initial conditions. The larger entropy production rate γ is realized for smaller N. While the prefactor of the entropy production rate is proportional to $1/N$ in the proof of the H-theorem in Eq. (2·34), the N dependence of the integrand is not trivial. Numerically, we find that the entropy production rate has stronger $1/N$ dependence as $\gamma/m \propto (1/N)^\delta$, where $\delta \sim 2.3 > 1$ in the strong coupling region of $\lambda/m^2 = 40$ with $T/m = 2.1 - 3.3$. Numerical and analytical analyses of the entropy production rate for various coupling and N would be an interesting direction for further investigation.

Acknowledgment

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