Phenomenological formulation of relativistic spin hydrodynamics

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Ultra-relativistic heavy ion collisions

**Aim:** study quark-gluon plasma (QGP)

**Found:** QGP behaves like a perfect liquid and hydrodynamics works so well
Another interesting physics: Largest $\omega$ and $B$

**Question:** What happens under huge $\omega$ and/or $B$? Specifically, any changes to QGP properties?
Naïve expectation: QGP is polarized

✓ **Magnetic field** $B$ effect

Zeeman splitting (Landau quantization)

$$E \rightarrow E - s \cdot qB$$

→ charge dependent spin polarization

✓ **Rotation** $\omega$ effect

Bernett effect

$$E \rightarrow E - s \cdot \omega$$

→ charge *independent* spin polarization
Experimental fact

FIG. 5. \( \Lambda (\bar{\Lambda}) \) polarization as a function of the collision centrality in Au+Au collisions at \( \sqrt{s_{NN}} = 200 \text{ GeV} \). Open boxes and vertical lines show systematic and statistical uncertainties. The data points for \( \bar{\Lambda} \) are slightly shifted for visibility.

STAR (2018)
See also talk by T.Niida
How about theory?

Hydrodynamics for spin polarized QGP?

→ Far from complete

See also talk by X.-G. Huang
Hydrodynamics for spin polarized QGP

✓ “Hydro simulations” exist, but...

usual hydro (i.e., hydro w/o spin) is solved

(1) Compute velocity gradient at freezeout and define thermal vorticity \( \tilde{\omega}^{\mu\nu} \equiv \partial^\mu (u^\nu / T) - \partial^\nu (u^\mu / T) \)  

(2) Use Cooper-Frye formula with spin \( f(E) \rightarrow f(E - s \cdot \omega) \), where \( \omega \) is spin chemical potential \((\neq \tilde{\omega} \text{ in general})\)

(3) Assume \( \omega = \tilde{\omega} \) (true only for global equilibrium)

(4) Get spin-dependent hadron yield

✓ Formulation of relativistic spin hydrodynamics is still developing
## Current status of formulation of spin hydro

### Non-relativistic case

- Already established (e.g. micropolar fluid)
  - applied to pheno. and is successful
  - **spin must be dissipative** because of mutual conversion between spin and orbital angular momentum

- **e.g.** Eringen (1998); Lukaszewicz (1999)

- **e.g.** spintronics: 
  - Takahashi et al. (2015)

### Relativistic case

- Some trials exist, but
  - only for “ideal” fluid (no dissipative corrections)
  - some claim **spin should be conserved**
Purpose of this talk

- Formulate relativistic spin hydrodynamics with 1st order dissipative corrections for the first time
- Clarify spin should be dissipative

Outline

1. Introduction
2. Formulation based on an entropy-current analysis
3. Linear mode analysis
4. Summary
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Introduction to hydrodynamics \textit{w/o} spin (1/3)

Hydrodynamics is a low energy effective theory that describes spacetime evolution of IR modes (hydro modes).

- \textbf{Phenomenological formulation} (EFT construction)

\textbf{Step 1}: Write down the \textbf{conservation law}: \quad \begin{align*} 0 &= \partial_{\mu} T^{\mu\nu} \quad \text{4 eqs} \end{align*}

\textbf{Step 2}: Express $T^{\mu\nu}$ i.t.o hydro variables (\textbf{constitutive relation})

- define hydro variables: \{\(\beta, u^\mu\}\) \quad \begin{align*} (u^2 = -1) \end{align*}

- write down all the possible tensor structures of $T^{\mu\nu}$

\[
T^{\mu\nu} = f_1(\beta)g^{\mu\nu} + f_2(\beta)u^\mu u^\nu \\
+ f_3(\beta)\epsilon^{\mu\nu\rho\sigma} \partial_\rho u_\sigma + f_4(\beta)\partial^\mu u^\nu + f_5(\beta)\partial^\nu u^\mu \\
+ f_6(\beta)g^{\mu\nu} \partial^\rho u_\rho + f_7(\beta)u^\mu u^\nu \partial^\rho u_\rho + f_8(\beta)u^\mu \partial_\mu u^\nu + \cdots + O(\partial^2)
\]

- simplify the tensor structures by (assumptions in hydro)

\begin{enumerate}
  \item symmetry
  \item power counting \rightarrow \textbf{gradient expansion}
  \item other physical requirements \rightarrow \textbf{thermodynamics} (see next slide)
\end{enumerate}

\textbf{Hydrodynamic eq.} = conservation law + constitutive relation
Introduction to hydrodynamics w/o spin (2/3)

✓ Constraints by thermodynamics

Expand $T^{\mu \nu}$ i.t.o derivatives

$$T^{\mu \nu} = T^{\mu \nu}_{(0)} + T^{\mu \nu}_{(1)} + O(\partial^2)$$

In static equilibrium $T^{\mu \nu} \rightarrow T^{\mu \nu}_{(0)} = (e, p, p, p)$, so that

$$T^{\mu \nu}_{(0)} = e u^\mu u^\nu + p(g^{\mu \nu} + u^\mu u^\nu)$$

1st law of thermodynamics says

$$ds = \beta de, \ s = \beta(e + p)$$

By using EoM $0 = \partial_\mu T^{\mu \nu}$, div. of entropy current $S^\mu = s u^\mu + O(\partial)$ can be evaluated as

$$\partial_\mu S^\mu = -T^{\mu \nu}_{(1)} \partial_\mu (\beta u_\nu) + O(\partial^3)$$

2nd law of thermodynamics requires $\partial_\mu S^\mu \geq 0$, which is guaranteed if RHS is expressed as a semi-positive bilinear as

$$-T^{\mu \nu}_{(1)} \partial_\mu (\beta u_\nu) = \sum_{X_i \in T_{(1)}} \lambda_i X_i^{\mu \nu} X_i^{\nu \mu} \geq 0 \text{ with } \lambda_i \geq 0$$

ex) heat current: $2h^{(\mu \nu)} \equiv h^\mu u^\nu + h^\nu u^\mu \in T^{\mu \nu}_{(1)}$ ($u_\mu h^\mu = 0$)

$$\Rightarrow T^{\mu \nu}_{(1)} \partial_\mu (\beta u_\nu) = -\beta h^\mu (\beta \partial_{\perp \mu} \beta^{-1} + u^\nu \partial_\nu u^\mu) \geq 0$$

$$\Rightarrow h^\mu = -\kappa (\beta \partial_{\perp \mu} \beta^{-1} + u^\nu \partial_\nu u^\mu) \text{ with } \kappa \geq 0$$
Introduction to hydrodynamics w/o spin (3/3)

✓ Constitutive relation up to 1\textsuperscript{st} order w/o spin

\[
T_{(0)}^{\mu\nu} = eu^{\mu}u^{\nu} + p(g^{\mu\nu} + u^{\mu}u^{\nu})
\]

\[
T_{(1)}^{\mu\nu} = -2k \left( Du^{(\mu} + \beta \partial_{\perp}^{(\mu} \beta^{-1}) u^{\nu)} \right) - 2\eta \partial_{\perp} <u^{\nu}> - \zeta (\partial_{\mu} u^{\mu}) \Delta^{\mu\nu}
\]

- heat current
- shear viscous effect
- bulk viscous effect

✓ Hydrodynamic equation w/o spin

\textbf{Hydrodynamic eq.} = conservation law + constitutive relation

\begin{align*}
\text{Euler eq.} & \quad 0 = \partial_{\mu} T^{\mu\nu} \\
\text{Navier-Stokes eq.} & \quad 0 = \partial_{\mu} T^{\mu\nu}
\end{align*}

\[
T^{\mu\nu} = T_{(0)}^{\mu\nu} \\
T^{\mu\nu} = T_{(0)}^{\mu\nu} + T_{(1)}^{\mu\nu}
\]

\vdots

\vdots

\vdots
Formulation of hydrodynamics with spin (1/4)

✓ Strategy is the same

Phenomenological formulation

Step 1: Write down the conservation law

Step 2: Construct a constitutive relation
   - define hydro variables
   - write down all the possible tensor structures
   - simplify the tensor structures
     (1) symmetry
     (2) gradient expansion
     (3) thermodynamics
Formulation of hydrodynamics with spin (2/4)

Step 1: Write down the conservation law

(1) energy conservation

\[ 0 = \partial_\mu T^{\mu\nu} \]
(canonical)

(2) total angular momentum conservation

\[ 0 = \partial_\mu M^{\mu,\alpha\beta} \]
\[ = \partial_\mu (L^{\mu,\alpha\beta} + \Sigma^{\mu,\alpha\beta}) \]
\[ = \partial_\mu (\chi^{\alpha} T^{\mu\beta} - \chi^{\beta} T^{\mu\alpha} + \Sigma^{\mu,\alpha\beta}) \]
\[ \therefore \partial_\mu \Sigma^{\mu,\alpha\beta} = T^{\alpha\beta} - T^{\beta\alpha} \]

✓ Spin is **not** conserved if (canonical) \( T^{\mu\nu} \) has anti-symmetric part \( T_{(a)}^{\mu\nu} \)

✓ There’s **no** a priori reason (canonical) \( T^{\mu\nu} \) must be symmetric

Consequence

(1) **Spin must not be a hydro mode** in a strict sense
(2) Nevertheless, it behaves *like* a hydro mode if \( T_{(a)}^{\mu\nu} \ll 1 \)

→ inclusion of dissipative nature is important
Formulation of hydrodynamics with spin (3/4)

Step 2: Construct a constitutive relation for $T^{\mu \nu}, \Sigma^{\mu, \alpha \beta}$

(1) define hydro variables

Introduce **spin chemical potential** $\{\beta, u^\mu, \omega^{\mu \nu}\}$ with $\omega^{\mu \nu} = -\omega^{\nu \mu}$

✓ $\{\beta, u^\mu, \omega^{\mu \nu}\}$ are independent w/ each other at this stage ($\omega^{\mu \nu}$ ≠ thermal vorticity)

(2) simplify the tensor structure by thermodynamics

Expand $T^{\mu \nu}, \Sigma^{\mu, \alpha \beta}$, i.t.o derivatives

$$T^{\mu \nu} = eu^\mu u^\nu + p(g^{\mu \nu} + u^\mu u^\nu) + T^{\mu \nu}_{(1)} + O(\partial^2), \quad \Sigma^{\mu, \alpha \beta} = u^\mu \sigma^{\alpha \beta} + O(\partial^1)$$

where I defined **spin density** $\sigma^{\alpha \beta}$

Generalizing **1st law of thermodynamics with spin** as

$$ds = \beta(de - \omega_{\mu \nu} d\sigma^{\mu \nu}), \quad s = \beta(e + p - \omega_{\mu \nu} \sigma^{\mu \nu})$$

With EoMs, div. of entropy current $S^\mu = su^\mu + O(\partial)$ can be evaluated as

$$\partial_\mu S^\mu = -T^{\mu \nu}_{(1s)} \frac{\partial_\mu (\beta u_\nu)}{2} + \partial_\nu (\beta u_\mu) - T^{\mu \nu}_{(1a)} \left\{ \frac{\partial_\mu (\beta u_\nu) - \partial_\nu (\beta u_\mu)}{2} - 2 \beta \omega_{\mu \nu} \right\} + O(\partial^3)$$

✓ In global equilibrium $\partial_\mu S^\mu = 0$, so that $\omega = $ thermal vorticity.

✓ 2nd law of thermodynamics $\partial_\mu S^\mu \geq 0$ gives strong constraint on $T^{\mu \nu}_{(1)}$
Formulation of hydrodynamics with spin (4/4)

✓ **Constitutive relation for** $T^{\mu\nu}$ **up to 1st order with spin**

\[ T^{\mu\nu}_{(0)} = e u^\mu u^\nu + p (g^{\mu\nu} + u^\mu u^\nu) \]

- Heat current
- Shear viscous effect
- Bulk viscous effect

\[ T^{\mu\nu}_{(1)} = -2\kappa \left( Du^{(\mu + \beta \partial_\perp (\mu \beta^{-1}) u^\nu)} - 2\eta \partial_\perp <u^\mu u^\nu> - \zeta (\partial_\mu u^\mu) \Delta^{\mu\nu} \right) \]

\[ -2\lambda \left( -Du^{[\mu + \beta \partial_\perp [\mu \beta^{-1} + 4u_{\rho} \omega_{\rho[\mu]}) u^\nu]} - 2\gamma \left( \partial_\perp u^\mu u^\nu) - 2\Delta^\mu_\rho \Delta^\nu_\lambda \omega_{\rho\lambda} \right) \]

“boost heat current”  “rotational (spinning) viscous effect”

**NEW!**

- Relativistic generalization of a non-relativistic micropolar fluid
- “boost heat current” is a relativistic effect

✓ **Hydrodynamics equation up to 1st order with spin**

\[ 0 = \partial_\mu (T^{\mu\nu}_{(0)} + T^{\mu\nu}_{(1)} + O(\partial^2)) \]

\[ \partial_\mu (u^\mu \sigma^{\alpha\beta}) = T^{\alpha\beta}_{(1)} - T^{\beta\alpha}_{(1)} + O(\partial^2) \]
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Linear mode analysis (1/2)

**Setup:** small perturbations on top of static equilibrium

\[
i \partial_t \left( \frac{\delta \beta}{\delta u^\mu} \right) = M \left( \frac{\delta \beta}{\delta \omega^{\mu \nu}} \right) + O(\delta^2)
\]

\[
\beta = \beta_0
\]

\[
u^\mu = (1, 0)
\]

\[
\omega^{\mu \nu} = 0
\]

propagate if Im \( \omega = 0 \)

dissipate if Im \( \omega < 0 \)

unstable if Im \( \omega > 0 \)
Linear mode analysis (2/2)

✓ Hydro w/o spin \( \{\beta, u^\mu\} \)

4 modes

2 sound modes \( \omega = \pm c_s k + O(k^2) \)

2 shear modes \( \omega = -i \frac{\eta k^2}{e + p} + O(k^4) \)

where \( c_s^2 \equiv \partial p/\partial e \)

✓ Hydro with spin \( \{\beta, u^\mu, \omega^{\mu\nu}\} \)

4 modes

2 sound modes \( \omega = \pm c_s k + O(k^2) \)

2 shear modes \( \omega = -i \frac{\eta k^2}{e + p} + O(k^4) \)

+ 6 dissipative modes

3 “boost” modes \( \omega = -2i \tau_b^{-1} + O(k^2) \)

3 “spin” modes \( \omega = -2i \tau_s^{-1} + O(k^2) \)

where \( \tau_s \equiv \frac{\partial \sigma^{ij}/\partial \omega^{ij}}{4\gamma}, \tau_b \equiv \frac{\partial \sigma^{i0}/\partial \omega^{i0}}{4\lambda} \)

✓ We explicitly confirmed that spin is dissipative

✓ The time-scale of the dissipation is controlled by the new viscous constants \( \gamma, \lambda \)
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Summary

✓ Spin polarization of QGP is one of the hottest topics in HIC. But, its theory, in particular hydrodynamic framework, has not been developed well

✓ **Relativistic spin hydrodynamics with 1st order dissipative corrections is formulated** based on the phenomenological entropy-current analysis

✓ **Spin must be dissipative** because of the mutual conversion between the orbital angular momentum and spin

✓ Linear mode analysis of the spin hydrodynamic equation also suggests that spin must be dissipative

**Outlook:** extension to 2nd order, Kubo formula, application to cond-mat, numerical simulations, and **start something new with you!**