New Exact Spherically Symmetric Solutions in $f(R, \phi, X)$ Gravity by Noether's Symmetry Approach

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Outline

1. Generalised $f(R, \phi, X)$ gravity
2. Noether symmetry and solutions
   - Symmetry reduce Lagrangian
   - Noether symmetry approach
   - Spherically symmetric solutions in $f(R, \phi, X)$
3. Conclusions

Generalised $f(R, \phi, X)$ gravity

- Let us consider the following action

Action considered

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}f(R,\phi,X) + L_{\rm m}\right]. \qquad (1)$$

- Here, $\kappa^2 = 8\pi G$, $L_{\rm m}$ is any matter Lagrangian and f is a function which depends on the scalar curvature R, a scalar field ϕ and a kinetic term being equal to

$$X = -\frac{\epsilon}{2}\,\partial^\mu\phi\partial_\mu\phi\,, \qquad (2)$$

where ϵ is a parameter ($\epsilon = -1$: phantom scalar field and $\epsilon = 1$: canonical scalar field)

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- Some interesting specific cases that can be constructed from this theory:
  - If $f(R,\phi,X) = f(R)$, one recovers $f(R)$ gravity.
  - If $f(R,\phi,X) = \alpha(R) + \gamma(X,\phi)$, one recovers theories minimally coupled with the scalar field.
  - If $f(R,\phi,X) = f(\phi,X)R$, one recovers Brans-Dicke type gravity theories and theories non-minimally couple between the scalar curvature and ϕ or X.
- This theory contains a large number of theories: some of them very interesting for cosmology and also for astrophysics objects.

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Spherical symmetry in Generalised $f(R,\phi,X)$ gravity

- Let us now consider that the space-time is spherically symmetric such as the metric is:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + M(r)(d\theta^2 + \sin\theta^2 d\varphi^2), \qquad (3)$$

where $A(r), B(r)$ and $M(r)$ are functions of the radial coordinate r.

- The scalar curvature becomes

$$R = -\frac{1}{B}\Big[\frac{A''}{A} + \frac{2M''}{M} - \frac{A'B'}{2AB} + \frac{M'A'}{MA} - \frac{M'B'}{MB} - \frac{A'^2}{2A^2} - \frac{M'^2}{2M^2} - \frac{2B}{M}\Big]. \qquad (4)$$

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The field equations for spherically symmetric space-times become

$$f_R\left(2\frac{A''}{A}-\frac{A'B'}{AB}-\frac{A'^2}{A^2}+2\frac{M'A'}{MA}\right)=-2Bf+4\left[f_R''+f_R'\left(\frac{M'}{M}-\frac{B'}{2B}\right)\right],$$
$$f_R\left(2\frac{A''}{A}+4\frac{M''}{M}-\frac{A'B'}{AB}-2\frac{M'B'}{MB}-\frac{A'^2}{A^2}-2\frac{M'^2}{M^2}\right)=-2Bf$$
$$+2f_R'\left(\frac{A'}{A}+2\frac{M'}{M}\right)-2\epsilon f_X\phi'^2,$$
$$f_R\left(2\frac{M''}{M}-\frac{B'M'}{BM}+\frac{M'A'}{MA}-\frac{4B}{M}\right)=-2Bf+4\left[f_R''+\frac{1}{2}f_R'\left(\frac{A'}{A}-\frac{B'}{B}+\frac{M'}{M}\right)\right].\tag{5}$$

where primes denote differentiation with respect to r.
The modified Klein-Gordon equation yields

$$f_X\left[\phi''+\frac{1}{2}\phi'\left(\frac{A'}{A}-\frac{B'}{B}+2\frac{M'}{M}\right)\right]+f_X'\phi'+\epsilon Bf_\phi=0.\tag{6}$$

Spherical symmetry in Generalised $f(R, \phi, X)$ gravity

- Schwarzschild solution is the unique spherically symmetric vacuum solution in GR, but we will see that this no longer holds in $f(R, \phi, X)$ theory of gravity.
- Vacuum solutions do not necessarily imply a null curvature $R = 0$ or $R = const.$, which lead to maximally symmetric solutions, to the contrary in GR.
- The field equations are difficult to treat and it is not so easy to find new spherically symmetric solutions.
- Nowadays, it is vital to explore spherically symmetric solutions in modified gravity to then try to analyse them with observations in order to see any new effects different to GR.

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Spherical symmetry in Generalised $f(R, \phi, X)$ gravity

Important point 1

Many papers in modified gravity have found spherically symmetric solutions considering the case $R = \text{constant}$. Although this case has been widely studied in the literature, this case is more or less trivial: All the higher order terms are zero and therefore, the underlying theory becomes similar to GR+Λ.

Important point 2

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Our approach

In our approach, WE **WILL NOT ASSUME** $R = \text{constant}$, otherwise the theory becomes the same as GR. In this case, finding solutions become a very difficult task.

How can we find solutions?

We adopt the classical Noether approach in order to find the Noether symmetry in $f(R, \phi, X)$ gravity. From the classical Noether theorem, it is shown that the Noether symmetry in $f(R, \phi, X)$ gravity leads to a kind of the first integral of motion, which are able to be solved. We derive exact solutions for the field equations by using the conservation relation coming from the Noether symmetry acquired

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Symmetry reduce Lagrangian for $f(R,\phi,X)$

- One can rewrite the action into its canonical form in such a way that we can reduce the number of degrees of freedom:

$$S_{f(R,\phi,X)} = \int dr \mathcal{L}(A, A', B, B', M, M', R, R', \phi, \phi') \,. \quad (7)$$

- For simplicity let us express the scalar curvature as follows

$$\bar{R} = R^* - \frac{A''}{AB} - \frac{2M''}{BM} \,, \quad (8)$$

where $R^* = \frac{A'B'}{2AB^2} - \frac{A'M'}{ABM} + \frac{A'^2}{2A^2B} + \frac{B'M'}{B^2M} + \frac{M'^2}{2BM^2} + \frac{2}{M}$ contains only first derivatives terms.

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- The symmetry reduced Lagrangian becomes

$$\mathcal{L} = [M(f - Xf_X) + (2 - MR)f_R]\left[f_R\left(M'A' + \frac{AM'^2}{2M}\right) + f_R'\left(MA' + 2AM'\right) - \frac{\epsilon}{2}f_X MA\phi'^2\right]. \quad (9)$$

- Note: B was eliminated using the field equations. Therefore, this point-like Lagrangian contains five degrees of freedom $(A, M, R, \phi$ and $X)$.

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Noether symmetry approach

- The existence of a Noether symmetry implies the existence of a vector field $\mathbf{Y}$ if the Lagrangian $\mathcal{L}(r, q^i, q'^i)$ satisfies

Noether symmetry condition

$$\mathbf{Y}^{[1]}\mathcal{L} + \mathcal{L}(D_r\xi) = D_r K\,, \tag{10}$$

where $\mathbf{Y}^{[1]}$ is the first prolongation of the generator in such a form

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$\mathbf{Y} = \xi \frac{\partial}{\partial r} + \eta^i \frac{\partial}{\partial q^i}$ and $K(r, q^i)$ is a gauge function, D_r is the total derivative operator with respect to r,
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Noether symmetry approach

- The significance of Noether symmetry comes from the following first integral of motion that if $\mathbf{Y}$ is the Noether symmetry generator corresponding to the Lagrangian $\mathcal{L}(r, q^i, q'^i)$, then the Hamiltonian or a conserved quantity associated with the generator $\mathbf{Y}$ is

$$I = -\xi E_{\mathcal{L}} + \eta^i \frac{\partial \mathcal{L}}{\partial q'^i} - K, \tag{12}$$

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Noether symmetry approach for $f(R, \phi, X)$

- By using the reduced Lagrangian for $f(R, \phi, X)$ in spherically symmetric space-time into the Noether's condition one gets a system of 26 partial differential equations.
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\begin{aligned}
&\xi_{,A}=0,\quad \xi_{,M}=0,\quad \xi_{,R}=0,\quad \xi_{,\phi}=0,\quad \xi_{,X}=0,\quad K_{,r}=0,\\
&f_R\eta^2_{,r}+M\left(f_{RR}\eta^3_{,r}+f_{R\phi}\eta^4_{,r}+f_{RX}\eta^5_{,r}\right)-\frac{1}{F}K_{,A}=0,\quad f_{RX}\left(M\eta^1_{,r}+A\eta^2_{,r}\right)-\frac{1}{F}K_{,X}=0,\\
&f_R\left(\eta^1_{,r}+\frac{A}{M}\eta^2_{,r}\right)+2A\left(f_{RR}\eta^3_{,r}+f_{R\phi}\eta^4_{,r}+f_{RX}\eta^5_{,r}\right)-\frac{1}{F}K_{,M}=0,\\
&f_{RR}\left(M\eta^1_{,r}+A\eta^2_{,r}\right)-\frac{1}{F}K_{,R}=0,\quad f_{RR}\left(M\eta^1_{,r}+A\eta^2_{,r}\right)-\epsilon MAf_X\eta^4_{,r}-\frac{1}{F}K_{,\phi}=0,\\
&f_{RR}\left(M\eta^1_{,R}+2A\eta^2_{,R}\right)=0,\quad f_{RR}\left(M\eta^1_{,\phi}+2A\eta^2_{,\phi}\right)+f_{R\phi}\left(M\eta^1_{,R}+2A\eta^2_{,R}\right)=0,\\
&f_{RR}\left(M\eta^1_{,X}+2A\eta^2_{,X}\right)+f_{RX}\left(M\eta^1_{,R}+2A\eta^2_{,R}\right)=0,\quad f_{RX}\left(M\eta^1_{,X}+2A\eta^2_{,X}\right)=0,\\
&f_{R\phi}\left(M\eta^1_{,X}+2A\eta^2_{,X}\right)+f_{RX}\left(M\eta^1_{,\phi}+2A\eta^2_{,\phi}\right)-\epsilon MAf_X\eta^4_{,X}=0,\quad f_R\eta^2_{,A}+M\left(f_{RR}\eta^3_{,A}+f_{R\phi}\eta^4_{,A}+f_{RX}\eta^5_{,A}\right)=0,\\
&\quad f_R\left(\eta^1_{,A}+\frac{A}{M}\eta^2_{,A}+\eta^2_{,M}-\xi_{,r}\right)+M\left(f_{RR}\eta^3_{,M}+f_{R\phi}\eta^4_{,M}+f_{RX}\eta^5_{,M}\right)+2A\left(f_{RR}\eta^3_{,A}+f_{R\phi}\eta^4_{,A}+f_{RX}\eta^5_{,A}\right)\\
&\qquad+\frac{1}{F}\left[f_RF_{,M}\eta^2+(Ff_R)_{,R}\eta^3+(Ff_R)_{,\phi}\eta^4+(Ff_R)_{,X}\eta^5\right]=0,\\
&\qquad\quad \frac{f_R}{M}\eta^2_{,R}+f_{RR}\left(\eta^1_{,A}+\frac{2A}{M}\eta^2_{,A}+\eta^3_{,R}-\xi_{,r}\right)+f_{R\phi}\eta^4_{,R}+f_{RX}\eta^5_{,R}\\
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&\qquad\qquad+\frac{1}{F}\left[f_{R\phi}(FM)_{,M}\frac{\eta^2}{M}+(Ff_{R\phi})_{,R}\eta^3+(Ff_{R\phi})_{,\phi}\eta^4+(Ff_{R\phi})_{,X}\eta^5\right]=0,\qquad (1)\\
&\qquad\quad \frac{f_R}{M}\eta^2_{,X}+f_{RX}\left(\eta^1_{,A}+\frac{2A}{M}\eta^2_{,A}+\eta^5_{,X}-\xi_{,r}\right)+f_{RR}\eta^3_{,X}+f_{R\phi}\eta^4_{,X}\\
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&\qquad f_R\left[\frac{\eta^1}{A}+\frac{M}{F}\left(\frac{F}{M}\right)_{,M}\eta^2+\frac{2M}{A}\eta^1_{,M}+2\eta^2_{,M}-\xi_{,r}\right]+4M\left(f_{RR}\eta^3_{,M}+f_{R\phi}\eta^4_{,M}+f_{RX}\eta^5_{,M}\right)\\
&\qquad\quad+\frac{1}{F}\left[(Ff_R)_{,R}\eta^3+(Ff_R)_{,\phi}\eta^4+(Ff_R)_{,X}\eta^5\right]=0,\\
&\qquad \frac{f_R}{2A}\left(\eta^1_{,R}+\frac{A}{M}\eta^2_{,R}\right)+f_{RR}\left[\frac{\eta^1}{A}+\frac{F_{,M}}{F}\eta^2+\frac{M}{2A}\eta^1_{,M}+\eta^2_{,M}+\eta^3_{,R}-\xi_{,r}\right]+f_{R\phi}\eta^4_{,R}+f_{RX}\eta^5_{,R}\\
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&\qquad\quad+\frac{1}{F}\left[(Ff_{RX})_{,R}\eta^3+(Ff_{RX})_{,\phi}\eta^4+(Ff_{RX})_{,X}\eta^5\right]=0,\\
&\qquad\qquad f_X\left[\frac{\eta^1}{A}+\frac{(MF)_{,M}}{MF}\eta^2+2\eta^4_{\phi}-\xi_{,r}\right]-\frac{2\epsilon}{MA}f_{R\phi}\left(M\eta^1_{,\phi}+2A\eta^2_{,\phi}\right)+\\
&\qquad\qquad\quad+\frac{1}{F}\left[(Ff_X)_{,R}\eta^3+(Ff_R)_{,\phi}\eta^4+(Ff_X)_{,X}\eta^5\right]=0,
\end{aligned}
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Noether symmetry approach for $f(R,\phi,X)$: cases

We split the study in 4 different types of $f(R,\phi,X)$:

1. $f(R,\phi,X) = f_0 R^n$, **Power-law** $f(R)$
2. $f(R,\phi,X) = f_0 R + f_1 X^q - V(\phi)$, **Minimally coupled theory.**
3. $f(R,\phi,X) = f_0 \phi^m R^n + f_1 X^q - V(\phi)$, **non-minimally coupled theory between the scalar field and** R.
4. $f(R,\phi,X) = U(\phi,X)\,R$, **Brans-Dicke type.**

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4. $f(R,\phi,X) = U(\phi,X)\,R$, **Brans-Dicke type.**

Noether symmetry approach for $f(R, \phi, X)$: cases

We split the study in 4 different types of $f(R, \phi, X)$:

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Case I: $f(R,\phi,X) = f_0 R^n$

- For $M(r) = r^2, R = R_0 r^p$ and $p = (n-2)/(4n^2-10n+7)$

New analytical solution 1

$$A(r) = r^{\frac{(n-1)(2n-1)}{4n^2-10n+7}}\Big[A_0 R_0^{\frac{2n^2-3n+1}{n-2}} + \frac{I_3(n-1)\left(4n^2-10n+7\right)R_0^{3-2n}}{4f_0^2 n^3(n-2)\left(8n^2-19n+12\right)} \times \log\left(1-\frac{2nr^{\frac{-8n^2+19n-12}{4n^2-10n+7}}}{(n-1)R_0}\right)\Big] + \frac{I_3\left(4n^2-10n+7\right)R_0^{2-2n}}{2f_0^2 n^2(n-2)\left(8n^2-19n+12\right)} r^{\frac{-6n^2+16n-11}{4n^2-10n+7}}.$$

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Case I: $f(R,\phi,X) = f_0 R^2$ (specific subcase $n = 2$)

- For $M(r) = r^2$ and $f(R,\phi,X) = f_0 R^2$ (quadratic gravity):

New analytical solution 2

$$A = A_0\left(1+\frac{q}{r}\right)^{1-\frac{4I_3}{\alpha}}\left[R_0 r^2\left(1+\frac{q}{r}\right)^{\frac{2I_3}{\alpha}}-4\right]^{-1}, \tag{13}$$

$$B = \frac{4}{\left(1+\frac{q}{r}\right)}\left[4-R_0 r^2\left(1+\frac{q}{r}\right)^{\frac{2I_3}{\alpha}}\right]^{-1}. \tag{14}$$

- This solution is asymptotically flat if $0 < I_3/\alpha < 1/4$ and can describe a black hole since its horizons are at $r = -q$ and when $r^2 R_0(q/r+1)^{2I_3/\alpha} = 4$.

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Case II: $f(R, \phi, X) = f_0 R + f_1 X^q - V(\phi)$

- We found many solutions under this theory. One interesting one is

Analytical solution 3 ($q = 2, I_1 = -2I_3$ and $V = V_1 e^{-V_0 \phi}$)

$$A(r) = A_0 + \frac{(A_0 I_2 - 4A_1 I_3^2)}{2I_3 r} - \frac{A_1 I_2}{r^2}, \quad B(r) = 4f_0^2 \left(\frac{A_0 r - 2A_1 I_3}{I_2 + 2I_3 r} \right)$$

- This metric is asymptotically flat and has two horizons at the surfaces $r_1 = 2A_1 I_3 / A_0$ and $r_2 = -I_2/(2I_3)$, i.e $A(r) = 0$ at these surfaces.
- This metric behaves as Schwarzschild and additional contribution ($A_0 = 1, A_1 = 0, I_2 = -4mI_3$ and $I_3 = 2f_0^2$ one gets Schwarzschild).

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- For this case, taking $n = q = 1$:

Analytical solutions 4

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- These are also new spherically symmetric solutions in this non-minimally couple theory between the scalar field and the Ricci scalar.
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Analytical solution 5 ($M(r) = r^2$ and $V = V_0 r - V_1$)

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$$r_\pm = \frac{1}{12 V_0 B_1 I_1}\left[-(I_1 + 3V_0 I_3) \pm \sqrt{(24 B_1 V_1 + 1) I_1^2 + 6V_0 I_1 (I_3 - 8B_1 K_0) + 9V_0^2 I_3^2}\right], \quad (19)$$

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Conclusions

- **Without assuming $R = \textbf{constant}$**, we have found new spherically symmetric solutions in different theories such as: power-law $f(R) = f_0 R^n$ gravity, non-minimally coupling models between the scalar field and the Ricci scalar $f(R,\phi,X) = f_0 R^n \phi^m + f_1 X^q - V(\phi)$, non-minimally couplings between the scalar field and a kinetic term $f(R,\phi,X) = f_0 R^n + f_1 \phi^m X^q$, and also in extended Brans-Dicke gravity $f(R,\phi,X) = U(\phi,X)R$.
- Some of these solutions can represent new black holes solutions in this extended theory of gravity
- The Noether symmetry in $f(R,\phi,X)$ gravity yields the solvable first integral of motion. With the conservation relation obtained from the Noether symmetry, the exact solutions for the field equations can be found.

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