The Symmetries of Gravitational Waves and some remarks about the Memory Effect:Part I : Carrollian Structures and the BMS and Conformal BMS Groups

January 28, 2019

This Part is based on three recent papers and one not so recent paper.

C. Duval, G. W. Gibbons and P. A. Horvathy Conformal Carroll groups, Accepted to J. Phys. A [arXiv:1403.4213 [hep-th]]

C. Duval, G. W. Gibbons and P. A. Horvathy, 'Conformal Carroll groups and BMS symmetry,' Class. Quant. Grav. **31** (2014) 092001 [arXiv:1402.5894 [gr-qc]]

C. Duval, G. W. Gibbons, P. A. Horvathy and P. M. Zhang, 'Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time, Class. Quant. Grav. **31** (2014) 085016 [arXiv:1402.0657 [gr-qc]]

C. Duval, G. W. Gibbons and P. Horvathy, Celestial mechanics, conformal structures and gravitational waves, Phys. Rev. D **43** (1991) 3907 [hep-th/0512188].

Contents

- Carrollian and Newton-Cartan Structures obtained from a Bargmann structure in one higher dimension: Duality as projection versus pull back.

- Carrollian Automorphisms and Conformal Automorphisms

- Isomomorphism of Conformal Carroll and Conformal Galilei algebras in $1+1$ spacetime dimensions

- BMS, and Newman-Unti groups as special cases

- Applications

The basic idea of all four papers is to look at non-Einsteinian Relativity Principles from an, albeit anachronistic, Spacetime view point

In our context a Principle of Relativity involves a notion of the invariance of physical laws under passing to a moving frame which we interpret as a symmetry of some sort of spacetime structure.

In effect we follow the path pioneered by Bacry and Levy-Leblond [*] who found all algebras containing rotations, spatial and temporal translations and boosts. All may be regarded as Wigner-Inönü contractions [†] of the two De-Sitter groups.

Note that without boosts we would simply be classifying Aristotelian spacetimes which leads to Helmholtz's classification of congruence geometries [‡] .

[*]H. Bacry and J. Levy-Leblond, Possible kinematics J. Math. Phys. **9** (1968) 1605.

[†]E. Inn, E.P. Wigner (1953). "On the Contraction of Groups and Their Representations". Proc. Nat. Acad. Sci. 39 (6): 51024.

[‡]Über die Thatsachen, welche der Geometrie zu Grunde liegen, in Wissenschaftliche Abhandlungen, Volume II, Leipzig: Johann Ambrosius Barth, 618639. Originally published in the Nachrichten von der Knigl. Gesellschaft der Wissenschaften zu Gttingen, No. 9 (3 June 1868).

The contractions are:

- Newton-Hooke $\Lambda \to O\,, \qquad c \to \infty\,, \qquad c^2\frac{\Lambda}{3}$ finite
- Poincaré $\Lambda \to O\,, \qquad c$ finite.
- Galilei $\Lambda \to O\,, \qquad c \to \infty.$
- Carroll $\Lambda \to O\,, \qquad c \to 0$

There is a certain duality between the Galilei and Carroll groups. In one the future light cone $t > \frac{1}{c}|\mathbf{x}|$ expands to become a future half space $t > 0$. In the other it contracts to become a future half line $t > 0\,, \mathbf{x} = 0$. One allows instantaneous propagation, the other is ultra-local and forbids any propagation.

All kinematic groups have flat invariant model space time which allows a curved generalisation.

For Galilei this is Newton-Cartan spacetime with its degenerate co-metric g^{ij} whose kernel are co-normals of the absolute time slices

Carrollian spacetime. has a degenerate metric g_{ij} whose kernel is tangent to the absolute future *.

*To quote Mrs Thatcher: TINA, i.e. There is no alternative

Well, in our country," said Alice, still panting a little, "you'd generally get to somewhere else if you run very fast for a long time, as we've been doing."

A slow sort of country!" said the Queen. "Now, here, you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!"

For Galilei, boosts act as

$$(t, \mathbf{x}) \to (t, \mathbf{x} - \mathbf{v}t)$$

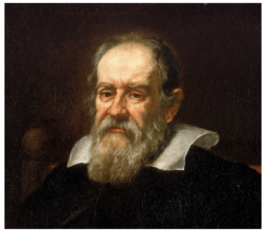
For Carroll, boosts act as

$$(s, \mathbf{x}) \to (s - \mathbf{b} \cdot \mathbf{x}, \mathbf{x})$$

where t is Galilean time and s is Carrollian time.

In 1+1 spacetime dimensions, Galileo and Carroll coincide as groups . since we may interchange Galilean space and with Carrolian time and

vice versa



Taking the limit $c \uparrow \infty$ in the contra-variant Minkowski co- metric

$$-\frac{1}{c^2}\frac{\partial}{\partial t} \otimes \frac{\partial}{\partial t} + \delta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$$

motivates the definition of a Newton-Cartan Spacetime as a quadrupole $\{N, \gamma, \theta, \nabla\}$ where N is a smooth $d+1$ manifold , γ a symmetric semi-positive definite contravariant 2-tensor of rank d with kernel the one-form θ and ∇ a symmetric affine connection w.r.t. which γ and θ are parallel.

Taking the limit $c \uparrow \infty$ in the co-variant Minkowski metric

$$-c^2 dt^2 + \delta_{ij} dx^i dx^j$$

motivates the definition of a Carrollian Spacetime as a quadrupole $\{C, g, \xi, \nabla\}$ where N is a smooth $d+1$ manifold , g a symmetric semi-positive definite co-variant 2-tensor of rank d with kernel the vector field ξ and ∇ a symmetric affine connection w.r.t. which ξ and ∇ are parallel.

The standard flat case is $C = \mathbb{R} \times \mathbb{R}^d$, $g_{ij} = \delta_{ij}$, $\xi = \frac{\partial}{\partial s}$, $\Gamma_\mu{}^\nu{}_\lambda = 0$ where s is Carrollian time. The isometry group of the Carrollian metric contains

$$x^i \to x^i \,, \qquad s \to s + f(x^i)$$

and so is infinite dimensional but if we require that the Carrolian automorphisms preserve the connection ∇ we obtain the standard finite dimensional Carroll group.

All the kinematic groups have a description in terms of Lorentzian geometry in 4+1 spacetime dimensions.

• Minkowski spacetime arises from a Kaluza-Klein reduction on a spacelike translation as shown by Kaluza and Klein.

• Newton-Cartan spacetime from a reduction on a null translation as shown by Duval and Kunzle.

• Carrollian spacetime arises as the pull-back to a null hyperplane.

Indeed given any null surface (like future null infinity) $\mathcal{I}^+$ Carrollian structures come into play.

We dfine a Bargmann Manifold as a triple $\{B, G, \xi\}$ where B is a (d+2) manifold, G a Lorentzian metric (i.e non-degenerate and signature $(d+1, 1)$ and a null vector field ξ which is parallel w.r.t. the Levi-Civita connection of G. The standard flat Bargmann structure is given by $B =$, $\xi = \frac{\partial}{\partial s}$ with

$$G = \delta_{ij} dx^i dx^j + dt \otimes ds + ds \otimes dt$$

Note that *both* s and t are null coordinates.

The standard flat Newton-Cartan structure is obtained by pushing forward the flat Bargmann structure to the quotient or lightlike shadow or null reduction $N = B/(\mathbb{R}\xi)$ The Bargmann group consists of those isometries of B which preserve ξ. This is a central extension of the Galilei group, the centre being generated by ξ.

One may also obtain the central extension of the conformal Schroedinger group, the symmetry of the free Schroedinger equation as the those conformal transformations of $d+2$-dimensional Minkowski spacetime which commute with the action of $\mathbb{R}\xi$.

A massles scalar field in $\mathbb{E}^{d+1,1}$ is invariant under conformal transformations

$$2\frac{\partial^2\phi}{\partial t\partial s}\phi(s,t,x^i)+\nabla^2\phi=0\,.$$

set

$$\xi\phi=-im\phi\,,\qquad\qquad \phi=e^{-ims}\Psi(t,x^i)$$

then

$$i\frac{\partial\Psi}{\partial t}=-\frac{1}{2m}\nabla^2\Psi\,.$$

The standard flat Carroll structure is obtained by pulling back the flat Bargmann structure to a null hypersurface $t =$ constant. The Carroll group consists of this isometries of B which commute with the pull back.

Note, by a Lie-algebra co-homology argument it has been shown that that the Carroll group admits no central extension.

A non-standard Carroll structure may be obtained by taking the product $C = \mathbb{R} \times \Sigma_d$ where Σ_d with Riemmannian metric $\widehat{g}$ and $g = \widehat{g} \oplus 0 \times du^2$ and $\xi = \frac{\partial}{\partial u}$, where u is a coordinate on $\mathbb{R}$. For ∇ we *could* take the Levi-civita connection of $\{\Sigma, \widehat{g}\}$.

For a general Carroll structure $\{C, g, \xi\nabla\}$ we define the Conformal Carroll group of level N as consisting of diffomeorphisms a such that

$$a^{\star}\widehat{g} = \Omega^{2}\widehat{g}\,, \qquad a_{\star} = \Omega^{-\frac{2}{N}}\xi$$

For the flat Carroll structure this has Killing vactor fields

$$X = (\omega_{ij}x_j + \gamma_i(\chi - 2\kappa_i x_i) + \kappa_i x_j x_j)\frac{\partial}{\partial x_i} + \Big(\frac{2}{N}(\chi - 2\kappa_j x_j)u + T(x_k)\Big)\frac{\partial}{\partial u}$$

This is *infinite dimensional* because of the super-translations $T(x_i)$ which have conformal weight $= -\frac{2}{N}$, i.e. are densities of weight $\nu = -\frac{2}{Nd}$. The quantity $z = \frac{2}{N}$ is known as a dynamical exponent.

If $N = 2$, $z = 1$ and we have symmetry between the scaling of space and time.

If $d = 1$, using the isomorphism between the Carroll and Galilei algebras described above we obtain the Conformal Galilei algebra. CGA introduced by many people in a variety of contexts.

The isometry group of the flat Carroll structure is obtained by setting $\Omega = 1$. Its Lie algebra is also infinite dimensional, because of the supertanslations. Requiring that the connection is preserved reduces the Carroll Lie algebra to the standard finite dimensonal case obtained by Levy-Lebond and Bacry.

For a general Carollian structure , the Conformal Carroll group is generated by

$$X = Y + \Big(\frac{\lambda}{N} + T(x)\Big)\frac{\partial}{\partial u}$$

where Y is a confomal vector field of $\{\Sigma, \hat{g}\}$

$$\mathcal{L}_Y g = \lambda g$$

generating

$$x \to \phi(x)\,, \qquad u \to \Omega^{\frac{2}{N}}(u + \alpha(x))$$

Example

If $\{\Sigma_d, \hat{g}\} = \{S^1, d\theta^2\}$ we get Diff(S^1) semi-direct product super translations of weight $\nu = -\frac{2}{N}$ generated by the vector field

$$X = Y(\theta)\frac{\partial}{\partial\theta} + \Big(\frac{2}{N}Y'(\theta) + T(\theta)\Big)\frac{\partial}{\partial u}\,.$$

whose algebra is an extension of the Witt or Virasoro algebra.

Example

If $\{\Sigma_d, g\} = \{S^2, d\theta^2 \, + \sin^2\theta d\phi^2\}$ and $N = 2$ we get

$$PSL(2,\mathbb{C}) \ltimes \mathcal{T}$$

where $\mathcal{T}$ are half densities on S^2 which is the Bondi-Metzner-Sachs Group

Which was originally discovered as the asymptotic symmetry group of an asymptotically flat four-dimensional spacetime. The BMS Group has an obvious generalisation to S^d for all $d > 2$. However this generalistion does not appear to coincide with the asymptotic symmetry group of an asymptotically flat spacetime of dimension greater than four.

We may weaken our requirement on the conformal Carroll group so that $\alpha(x)$ only preserves the conformal class of the Carroll metric g. Since the vector field ξ spans the kenel of g, α is only requred to preserve the *direction* of ξ. Thus its generating vector field X need only satisfy

$$\mathcal{L}_X \xi = \mu(x,u)\xi \,, \qquad \Rightarrow X = Y + F(u,x)\frac{\partial}{\partial u} \tag{1}$$

where $F(u,x)$ is an arbitrary function of *both* x *and* x, and Y is a conformal Killing field of the Riemannian manifold $\{\Sigma_d, \widehat{g}\}$

Example

If $\{\Sigma_d, \widehat{g}\} = \{\Sigma_d, g\} = \{S^2, d\theta^2 \ + \sin^2\theta d\phi^2\}$ we obtain the Newman-Unti Group which was introduced by these authors in the study of asymptotically flat four-dimensional spacetimes.

• Carrollian and BMS symmetries have a number of applications to various topics of current interest to string theorists and holography which was the original motivation for the work reported in this lecture.

• Using our enhanced understanding of the Carroll group We were able construct Carrollian-invariant theories of electromagnetism.

• Using a method of Souriau we constructed theories of Carrollian massive and massless particles. One finds the former do not move, consitent with other view points.

• Perhaps the most intriguing is to Schild or Null Strings, that is strings whose two-dimensional world sheet carries a Carrollian metric, i.e is a two-dimensional null surface. It turns out that Souriau's procedure for obtaining dynamical systems invariant under a group G applied to massless "particles " leads to Schild Strings.

Souriau's procedure starts with an Evolution space $\{V, \sigma\}$ where σ is a closed 2-form, $d\sigma = 0$. By virtue of the closure of σ ist kernel $\ker(\sigma)$ defines a (i.e continuous asigment of a vector sub-space $\ker(\sigma) \subset TM$) which is *integrable*, i.e for which $A, B \in \ker(\sigma) \to [A, B] \in \ker(\sigma)$. The space of leaves $\{U, \sigma\}$ is thus a sympletic manifold called the Space of Motions, i.e the space of histories of the system defined by $\{V, \sigma\}$.

For a Lie group G one may choose for $V = G/H$ a co-adjoint orbit of G in $\mathfrak{g}$ eqipped with its so called Kostant-Kirilov-Souriau 2-form. For a spacetime group, such as the Carroll group we may also project the leaves defined by $\ker(\mathrm{Kostant} - \mathrm{Kirilov} - \mathrm{Souriau}\, 2 - \mathrm{form})$ in the spacetime,(another coset).

If the projected orbits have dimension $p+1$ we have a p-brane. The case $p=0$ is a particle and if $p=1$ we have a string.

Cranking through this machinery we found that if G = Carroll we obtain a Schild String.