The Symmetries of Gravitational Waves and some remarks about the Memory Effect :Part II : Eisenhart-Duval Lifts, Carroll Symmetry, and Gravitational Waves

G. W. Gibbons

DAMTP: University of Cambridge

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What follows is based on work with

- Marco Cariglia
- Christian Duval *
- Peter Horvathy
- Mahmut Elbistan
- Pengming Zhang

*deceased 12th September 2018

The overall message is

There is more to pp-waves than meets the eye

Lecture II consists of

- A brief review of the Eisenhart-Duval Lift and its uses
- Kinematic Algebras and the Galilei, Poincaré and Carroll groups
- Plane gravitational waves and the memory effect

The Eisenhart-Duval Lift

For a Natural Mechanical system * :

$$L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V(x,t) \,, \qquad i = 1, \ldots, n$$

The Eisenhart Duval-Lift is the $n+2$ dimensional metric

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu &= 2L dt^2 + 2dtds \,, \mu = 1, \ldots n, n+1, n+2 \\ &= g_{ij} dx^i dx^j + 2dtds - V(x,t) dt^2 \end{aligned}$$

$K^\mu \partial_\mu = \frac{\partial}{\partial s}$ is a null Killing vector field and since $g_{\mu\nu} K^\nu = \partial_\mu t$ it lies in the null hypersurfaces $t =$ constant.

*We could include $A_i \dot{x}^i$

Roughly speaking, t is Newtonian time and s is Carrollian time. Our Natural Mechanical System has a Hamiltonian

$$H = \frac{1}{2} g^{ij} p_i p_j + V(x,t)\,.$$

Geodesics on the Eisenhart-Duval Lift may be obtained from the super Hamiltonian $\mathcal{H} = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$ subject to the constraint $\mathcal{H} = 0$, i.e.

$$g^{ij} p_i p_j + 2 p_t p_s + 2 V p_s^2 = 0\,.$$

Setting $p_s - = 1$, $p_t = -H$ we see that

The motions of every Natural Mechanical System are the projections on its Newtonian spacetime (whose coordinates are x^i, t) of null geodesics in its Eisenhart-Duval Lift

The correspondence works quantum mechanically:

Suppose ϕ satisfies

$$\nabla_\mu \nabla^\mu \phi(x, t, s) = 0$$

and we assume $\phi = e^{-is}\Psi(x, t)$, then we obtain the Schrödinger equation

$$-\frac{1}{2}\nabla_i \nabla^i \Psi + V(x, t)\Psi = i\frac{\partial \Psi}{\partial t}$$

Symmetry's of null geodsics or wave equations on the Eisenhart-Duval lift descend to symmetries of its Natural Mechanical System provided they normalise the $\mathbb{R}$ *action generated by the null Killing vector field* $\frac{\partial}{\partial s}$.

Conversely constants of the motion $K(x^i, p_i)$ *which Poisson commute with the Hamitonian* H *which are polynomial in* p_i *may be lifted to homogeneous functions* $\widehat{K}(p_i, p_s)$ *which correspond to Killing tensors on the Eisenhart-Duval Lift.*

Even the example $g_{ij} = \delta_{ij}$ and $V(x,t) = 0$ is illuminating.

In the classical case the isometry group of $\mathbb{R}^{n+1,1}$ is the Poincaré group $E(n+1,1)$. It descends to the Galilei group $Gal(n,1)$ which is the symmetry of a free particle moving in Newton-Cartan spacetime.

The quantum mechanical case is more subtle. Elements of the Poincaré group $E(n+1,1)$ involving s linearly induce changes of the phase of the wave function Ψ, This leads to the Bargmann group, the central extension of the Galilei group $Gal(n,1)$.

Conformal transformations of the Eisenhart-Duval lift take null geodsics to null geodesics. The simplest case corresponds to temporal reparametrizations $t = f(\tilde{t})$ and asuume g_{ij} is flat. So that

$$g_{ij}\dot{x}^i\dot{x}^i = \sum_a m_a |d\mathbf{x}_a|^2$$

If $\mathbf{x}_a = \sqrt{f'}\,\mathbf{y}_a$.

$$\tilde{V} = f'V(\sqrt{f'}\,\mathbf{y}_a, f) + \sum m_a |\mathbf{y}_a|^2 \{f, \tilde{t}\}$$

The Einsenhart-Duval metric becomes

$$f'\Big\{g_{ij}dy^i\dot{d}^i + 2d\tilde{t}ds - 2\tilde{V}(d\tilde{t})^2\Big\}$$

$$\{f,\tilde{t}\} = \frac{f'''}{f'} - \frac{3}{2}\Big(\frac{f''}{f}\Big)^2$$

is the Schwarzian derivative which vanishes if $f = \frac{at+b}{ct+d}$ and $f \in SL(2,\mathbb{R})$. In general we induce an extra time dependent oscilator contribution to the potential. If the Schwarzian derivative vanishes the Bargman group is extended to the Schrödinger group. One may apply this to celestial mecanics to map solutions with with a time varying Gravitational constant to those with time independent Gravitational constant. If a Newtonian Cosmological constant is constant one induces a time independent cosmological constant. Perhaps this has relevance to dark matter.

If $g_{ij} = \delta_{ij}$ one may isometrically embed the Eisenhart-Duval Lift into $\mathbb{R}^{n+2,2}$ as an $(n+1)$-brane $Z^A = Z^A(x^i, s, t)$.

$$
\begin{aligned}
Z^i &= x^i \\
Z^{n+1} &= \frac{1}{\sqrt{2}}(tV + s + t) \\
Z^{n+2} &= \frac{1}{\sqrt{2}}(tV + s - t) \\
Z^{n+3} &= \frac{1}{\sqrt{2}}(V + \frac{1}{2}t^2) \\
Z^{n+4} &= \frac{1}{\sqrt{2}}(V - \frac{1}{2}t^2)
\end{aligned}
$$

If $\nabla_i \nabla^i V = 0$ then $\Box Z^A = 0$ and therefore satisfies the brane equations of motion. This holds for the example of celestial mechanics and Newtonian Cosmology.

* †

$$m_a\ddot{\mathbf{x}}_a = \sum_{b\neq a} -\frac{Gm_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|^3}(\mathbf{x}_a - \mathbf{x}_b)\,.$$

$$L = \sum_a \frac{1}{2} m_a \dot{\mathbf{x}}_a^2 + \frac{1}{2}\sum_{a\neq b}\sum_b G\frac{m_a m_b}{|\mathbf{x}_a - \mathbf{x}_b|}$$

$$\mathbf{x}_a = a(t)\mathbf{r}_a\,, \qquad \dot{\mathbf{r}}_a = 0, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G\rho_0}{a^3}\,.$$

$$4\pi G\rho_0 m_a \mathbf{r}_a = -\sum_{b\neq a} Gm_a m_b \frac{(\mathbf{r}_a - \mathbf{r}_b)}{|\mathbf{r}_a - \mathbf{r}_b|^3}.$$

[*] G. F. R. Ellis and G. W. Gibbons,Discrete Newtonian Cosmology, Class. Quant. Grav. **31** (2014) 025003 [arXiv:1308.1852 [astro-ph.CO]]

[†] G. F. R. Ellis and G. W. Gibbons, Discrete Newtonian Cosmology: Perturbations, Class. Quant. Grav. **32** (2015) no.5, 055001 [arXiv:1409.0395 [gr-qc]].

Kinematic Algebras and the Carroll Group

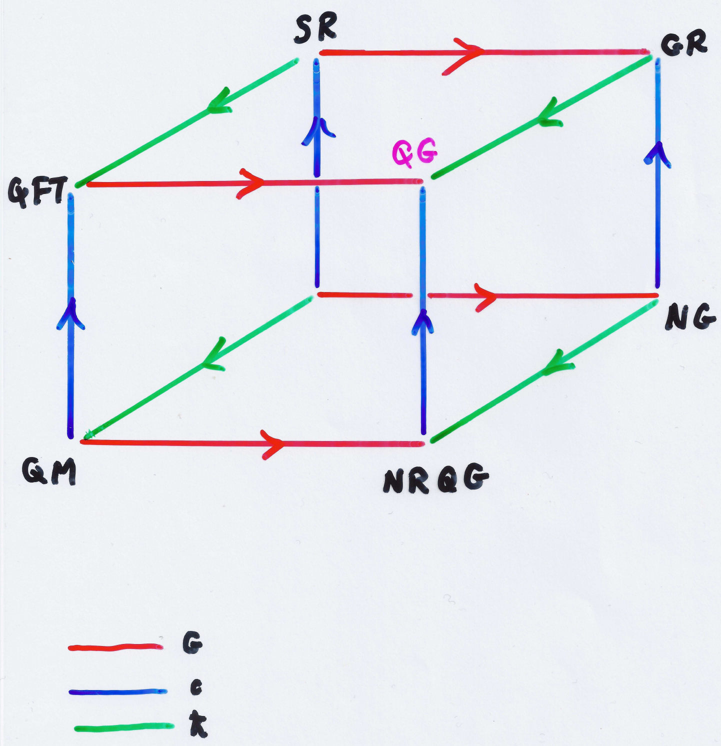
In 1967 a paper entitled "Possible Kinematics"* Levy-Leblond and Bacry classified all Lie algebras up to isomorphism spanned by $\{\mathbf{J}, \mathbf{P}, \mathbf{K}, H\}$.

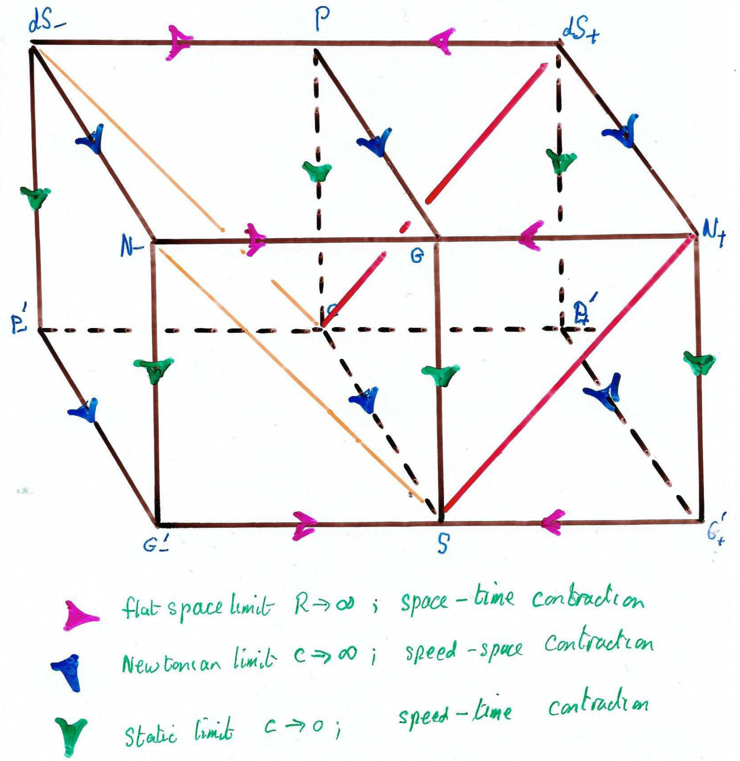
$$\Big[\mathbf{J}, \mathbf{J}\Big] = \mathbf{J}, \quad \Big[\mathbf{J}, \mathbf{P}\Big] = \mathbf{P}, \quad \Big[\mathbf{J}, \mathbf{K}\Big] = \mathbf{K}, \quad \Big[\mathbf{J}, H\Big] = 0.$$

The simplest case is to assume that all other brackets vanish, this is known as *Static*. If we demand that the boosts $\mathbf{K}$ are non-compact there are 11 posibilities, among them the the Galilei group $G(3,1)$ and the Carroll group $C(3,1)$. All such deformations of the Static algebra are Wigner-Inönü contractions of the de-Sitter groups. For Galilei the additonal non-vanishing Lie bracket Lie is $\Big[H, \mathbf{K}\Big] = H$. For Carroll the additonal non-vanishing Lie bracket Lie is $\Big[\mathbf{P}, \mathbf{K}\Big] = H$.

*see J. Figueroa-O'Farrill, Classification of kinematical Lie algebras, arXiv:1711.05676 [hep-th]. for a recent extension to all dimensions

The Cube of Theories





Levy-Leblond had previously defined the Carroll group as the limit of the Poincaré group as $c \downarrow 0$ just as one may define the Galilei group as the limit of the Poincaré group as $c \uparrow 0$.

$$\text{Carroll :} \qquad \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + dx^i dx^j \longrightarrow dx^i dx^i$$

$$\text{Galilei :} \qquad \eta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} = -c^{-2} \frac{\partial}{\partial t} \frac{\partial}{\partial t} + \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^i} \longrightarrow \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^i}$$

From The Eisenhart-Duval lift point view $C(3,1)$ is the subgroup of $SO(4,1)$ stabilizing the null hyperplane $t = constant$ in $R^{4,1}$ with metric

$$g_{\mu\nu} dx^\mu dx^\nu = d\mathrm{x}^2 + 2dtds$$

$C(2,1)$ arises in Dirac's Light Front quantization which uses as initial surface a null hyperplane replacing the Euclidean group $E(3)$ of the more usual spacelike hyperplane used in canonical quantization. The dynamics is then implemented by the Galileo group $Gal(2,1)$.

As such $C(2,1)$ plays a central rôle in high energy scattering in the Infinite Momentum Frame. It arises as a symmetries of supergravity solutions used in the AdS/CFT approach to CFT's in the infinite momentum frame. *

*M. Cvetic, H. Lu and C. N. Pope, Space-times of boosted p-branes and CFT in infinite momentum frame, Nucl. Phys. B **545** (1999) 309 [hep-th/9810123]. D. Brecher, A. Chamblin and H. S. Reall, AdS / CFT in the infinite momentum frame,' Nucl. Phys. B **607** (2001) 155 doi:10.1016/S0550-3213(01)00170-5 [hep-th/0012076]. H. Kim, 'Supergravity duals to the noncommutative N=4 SYM theory in the infinite momentum frame, Phys. Rev. D **68** (2003) 066007 [hep-th/0306124].

Plane Gravitational Waves

If $g_{ij} = \delta_{ij}$, the Eisenhart-Duval metric is what is known as a pp-wave or Brinkmann wave. If the potential $V(x,t)$ is harmonic in x but with *arbitrary* dependence on t it is Ricci flat. All local invariants vanish and thus it is a solution of almost any theory of gravity without a cosmological term constructed from just a metric tensor $g_{\mu\nu}$. The linearity of the harmonic condition reflects the Kerr-Schild structure. Plane gravitational waves are a special case

$$g_{\mu\nu} dX^\mu dX^\nu = dX^i dX^i + 2 dU dV + K(X^i, U) dU^2$$

In the physical dimension,

$$K = K_{ij}(U) X^i X^j = \mathcal{A}_+(U)\Big((X^1)^2 - (X^2)^2\Big) + 2\mathcal{A}_\times(U) X^1 X^2$$

The coordinates (X^i, U, V) are global. The null Killing vector $\frac{\partial}{\partial V}$ is covariant constant, lies in the wave fronts $U =$ constant and generates Carrollian time translations. There are in general four additional symmeries. To reveal them we pass to Baldwin-Jeffery-Rosen coordinates (x^i, u, v) which are always somewhere singular.

$$X^i = P^i_j x^j \,, \quad U = u \,, \quad V = v - \frac{1}{4} x^i \dot a(u)_{ij} x^j \,, \quad a = P^t P \,.$$

$$g_{\mu\nu} dx^\mu dx^\nu = a(u)_{ij} dx^i dx^j + 2 du dv$$

$$\dot P = K P \,, \qquad P^t \dot P = \dot P^t P \,, \qquad L = \dot a^{-1} a \,.$$

$$K = P(\dot L + \frac{1}{2} \dot L^2) P^{-1} \,, \qquad \mathrm{Tr}(\dot L + \frac{1}{2} \dot L^2) = 0 \,.$$

* Transverse translations and null translations mutually commute. The full set of isometries act multiply transitively on the wave fronts and are

$$x^i \to x^i + H(u)c^i\,, \quad v \to v - b_i x^i - \frac{1}{2} b_i H_{ij} b_j + f\,, \quad H(u) = \int^u a(t)^{-1} dt$$

which make up the Carroll group in 2+1 dimensions with the rotations discarded $C(2,1) \to G \to O(2)$, $C(2,1)/G = O(2)$.

If $a = 1$ we recover the Carrol group For circularly polarized gravitational waves the is an additional screw symmetry.

*C. Duval, G. W. Gibbons, P. A. Horvathy and P.-M. Zhang, Carroll symmetry of plane gravitational waves, Class. Quant. Grav. **34** (2017) no.17, 175003 doi:10.1088/1361-6382/aa7f62 [arXiv:1702.08284 [gr-qc]].

The timelike geodesics of plane gravitational waves are most readiily solved, using the conserved quantities arising from the isometry group in BJR coordinates. They may then be translated back to the more physical B cordinates. In practice this means solving the Sturm-Liouvile equations for $P^i_j(U)$ which typically can only be done numerically.

The main cases are

- Monochromatic Gravitational Waves including Plane Polarised Gravitational Waves and Circularly Polarised Gravitational Waves.

- Sandwich Waves for which $K_{ij}(U) = 0$ outside a finite interval of time U.

Sandwich Waves represent a pulse of gravitational radiation as might arise from black hole mergers. Either side of the pulse is flat Minkowki spacetime. This is manifest in B coordinates but not in BJR coordinates. This fact gives rise to the Memory Effect

[*] .

[*]P. M. Zhang, M. Elbistan, G. W. Gibbons and P. A. Horvathy, SturmLiouville and Carroll: at the heart of the memory effect," Gen. Rel. Grav. **50** (2018) no.9, 107 doi:10.1007/s10714-018-2430-0 [arXiv:1803.09640 [gr-qc]].

P. M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, Velocity Memory Effect for Polarized Gravitational Waves," JCAP **1805** (2018) no.05, 030 doi:10.1088/1475-7516/2018/05/030 [arXiv:1802.09061 [gr-qc]].

P.-M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, Soft gravitons and the memory effect for plane gravitational waves, Phys. Rev. D **96** (2017) no.6, 064013 doi:10.1103/PhysRevD.96.064013 [arXiv:1705.01378 [gr-qc]].

P.-M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, The Memory Effect for Plane Gravitational Waves, Phys. Lett. B **772** (2017) 743 doi:10.1016/j.physletb.2017.07.050 [arXiv:1704.05997 [gr-qc]].

Noether's theorem implies that in BJR coordinates

$$a_{ij}\frac{dx^j}{du} = p_i\,, \tag{1}$$

where p_i is an constant momentum . if $a_{ij} = \delta_{ij}$ befoe the pulse it will be *time dependent* after the pulse. In fact two particles at relative rest before the pulse will move with non-zero relative velocity after the pulse This is clear in B coordinates. $P^i_j(U)$ determines a local diffeomorphism of two coordinate patches of Minkowski spacetime. The memory effect was first discovered in the days of Weber-type Bar Detectors for Gravitational Radiation

From: * we take the following equations :

$$\frac{d^2x}{dt^2} + \frac{\omega_0}{Q}\frac{dx}{dt} + \omega_0^2 x = -c^2 l R_{1010}$$

$$R_{i0j0} = \frac{G}{3r}\frac{d^4 D_{ij}}{dt^4}(t-r)\,,$$

*G. W. Gibbons and S. W. Hawking, Theory of the detection of short bursts of gravitational radiation, Phys. Rev. D **4** (1971) 219

We deduced that if the quadrupole moment is initially and finally time independent, as might be expected for the gravitational collapse of a massive star, then three integrals of the signal must vanish

$$\int_{t_i}^{t_f} dt \int_{t_i}^{t} dt' \int_{t_i}^{t'} dt'' R_{0i0j}(t'') = 0,$$

In which case the signal must change sign at least three times. We provided a sketch of a signal which changed sign exactly three times which appears to have mislead some people to think that we had claimed that it must always change sign three times. By contrast for what is now called a *flyby* we pointed out that only

$$\int_{t_i}^{t_f} dt R_{0i0j}(t)$$

need vanish. We did not labour the point of how this might affect the displacement $x(t)$ of the detector after a pulse like signal has passed.

Later Zeldovich and Polnarev * were considering likeley signals from dense clusters of massive stars or collapsed objects noted that that after a pulse has passed,according to linear theory the metric perturbation h_{ij} satisfies

$$\frac{d^2 h_{ij}}{dt^2} = 0\,.$$

whose solution is

$$h_{ij} = h^1_{ij} t + h^0_{ij}\,, \qquad h^1_{ij}, h^0_{ij} \quad \text{constant}$$

*Ya. B. Zel'dovich and A. G. Polnarev, "Radiation of gravitational waves by a cluster of superdense stars," Astron. Zh. **51**, 30 (1974) [Sov. Astron. **18** 17 (1974)].

and stated that:

> *. . . another, nonresonance, type of detector is possible, consisting of two noninteracting bodies (such as satellites). the values of h_{ij} after the encounter of two objects differs fromthe value before the encounter. As a result the distance between a pair of free bodies should change, and in principle this effect might possibly serve as a nonresonance detector. [. . .] One should note that although the distance between the free bodies will change, their relative velocity will actually become vanishingly small as the flyby event concludes.*


Я.Б. ЗЕЛЬДОВИЧ
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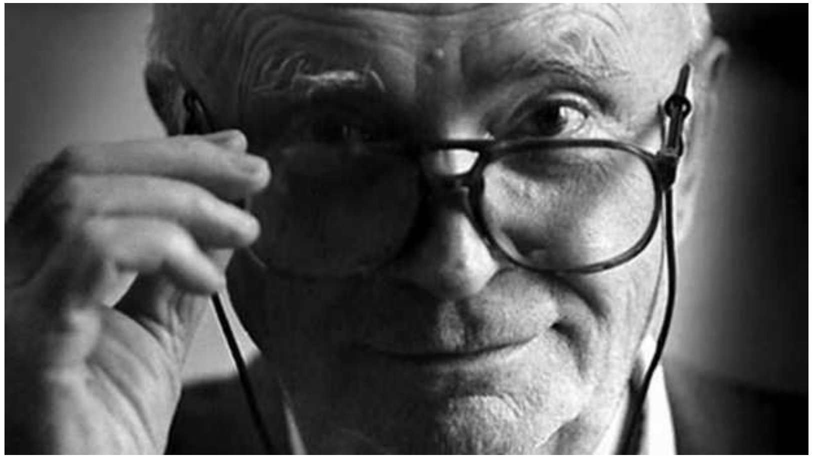
Subsequently * Braginsky and Grischuk dubbed this the Memory effect

*V B Braginsky and L P Grishchuk, Kinematic resonance and the memory effect in free mass gravitational antennas, Zh. Eksp. Teor. Fiz. **89** 744-750 (1985) [Sov. Phys. JETP 62, 427 (1985)].

Consideration is given to two effects in the motion of free masses subjected to gravitational waves, kinematic resonance and the memory effect. In kinematic resonance, a systematic variation in the distance between the free masses occurs, provided the masses are free in a suitable phase of the gravitational wave. In the memory effect, the distance between a pair of bodies is different from the initial distance in the presence of a gravitational radiation pulse. Some possible applications [. . .] to detect gravitational radiation . . .

Actually, as we have seen the distance can be expected to be time dependent in general.




Gravitational waves as optical media In a curved spacetime Maxwell's equations are

$$dF = 0\,, \qquad dG = 0$$

where

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu\,, \qquad G = \frac{1}{2} G_{\mu\nu} dx^\mu \wedge dx^\nu\,,$$

and if

$$\begin{aligned} F &= -E_i dt \wedge dx^i + \frac{1}{2}\epsilon_{ijk} B_k dx^i \wedge dx^j \\ G &= H_i dt \wedge dx^i + \frac{1}{2}\epsilon_{ijk} D_k dx^i \wedge dx^j \end{aligned}$$

they then become

$$\begin{aligned} \nabla \cdot \mathbf{D} &= 0\,, \qquad \partial_t \mathbf{B} = -\nabla \times \mathbf{E}\,, \\ \nabla \cdot \mathbf{B} &= 0\,, \qquad \partial_t \mathbf{D} = \nabla \times \mathbf{H}\,. \end{aligned}$$

The constitutive relation is $G = \star F$, i.e.

$$D_i = \epsilon_{ij} E_j + \alpha_{ij} H_j \,, \qquad B_i = \mu_{ij} H_j + \alpha_{ji} E_j$$

where ϵ_{ij}, the permitivity tensor, and μ_{ij} the permeablity tensor are

$$\mu_{ij} = \epsilon_{ij} = -\sqrt{-g} \frac{g^{ij}}{g_{00}} \,, \qquad \alpha_{ij} = -\beta_{ij} = \epsilon_{ijk} \frac{g_{0k}}{g_{00}} \,.$$

In BJR coordinates we find $\alpha_{ij} = 0$ the permeability depends only on u.

$$\epsilon_{ij} = \sqrt{\det a}\, a^{-1}_{ij} \,, i = 1, 2, \qquad \epsilon_{33} = \sqrt{\det a} \,.$$

Plane gravitational wave are the Eisenhart-Duval lift of two oscillators, one with positive time dependent freqency squared and the other with negative time dependent freqency squared. There is an enormous literature on such systems. In the present setting this makes use of coordinate transformations, especially conformal transformations to reduce the problem to simpler, sometimes time independent, problems

Monochromatic plane polarised waves : $X^i(U)$ behave like charged particles (ions) in a Paul Trap.

Nonochromatics Circularly polarised waves : $X^i(U)$ behave like ions in a modified version of a Penning Trap. *.

*P.-M. Zhang, M. Cariglia, C. Duval, M. Elbistan, G. W. Gibbons and P. A. Horvathy, Ion Traps and the Memory Effect for Periodic Gravitational Waves, Phys. Rev. D **98** (2018) no.4, 044037 doi:10.1103/PhysRevD.98.044037 [arXiv:1807.00765 [gr-qc]].

Conclusion

William Blake once wrote

> To see a World in a Grain of Sand And a Heaven in a Wild Flower Hold Infinity in the palm of your hand And Eternity in an hour

Auguries of Innocence

I hope I have convinced you that the similar sentiments may be applied to pp-waves.

Thank you for your attention