GAUGE/GRAVITY DUALITY

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References:

Original work:

- Maldacena: hep-th/9711200
- Gubser, Klebanov, Polyakov: hep-th/9802109
- Witten: hep-th/9802150, hep-th/9803131

Early reviews:

- MAGOO: hep-th/9905111
- D'Hoker & Freedman: hep-th/0201253
- Horowitz & Polchinski: gr-qc/0602037

More recent reviews:

- McGreevy: 0909.0518
- VH & Rangamani: 1006.3675
- Polchinski: 1010.6134
- Maldacena: 1106.6073
- Hubeny: 1501.00007
- Harlow: 1802.01040

(Early part of the lectures was based on this review, written for GR audience.)

and many many more....

OUTLINE

- Motivation: path to QG
- Black holes & holographic principle
- Preview of gauge/gravity correspondence
- Historical detour: string theory
- Anti de Sitter geometry
- Elements of gauge/gravity dictionary

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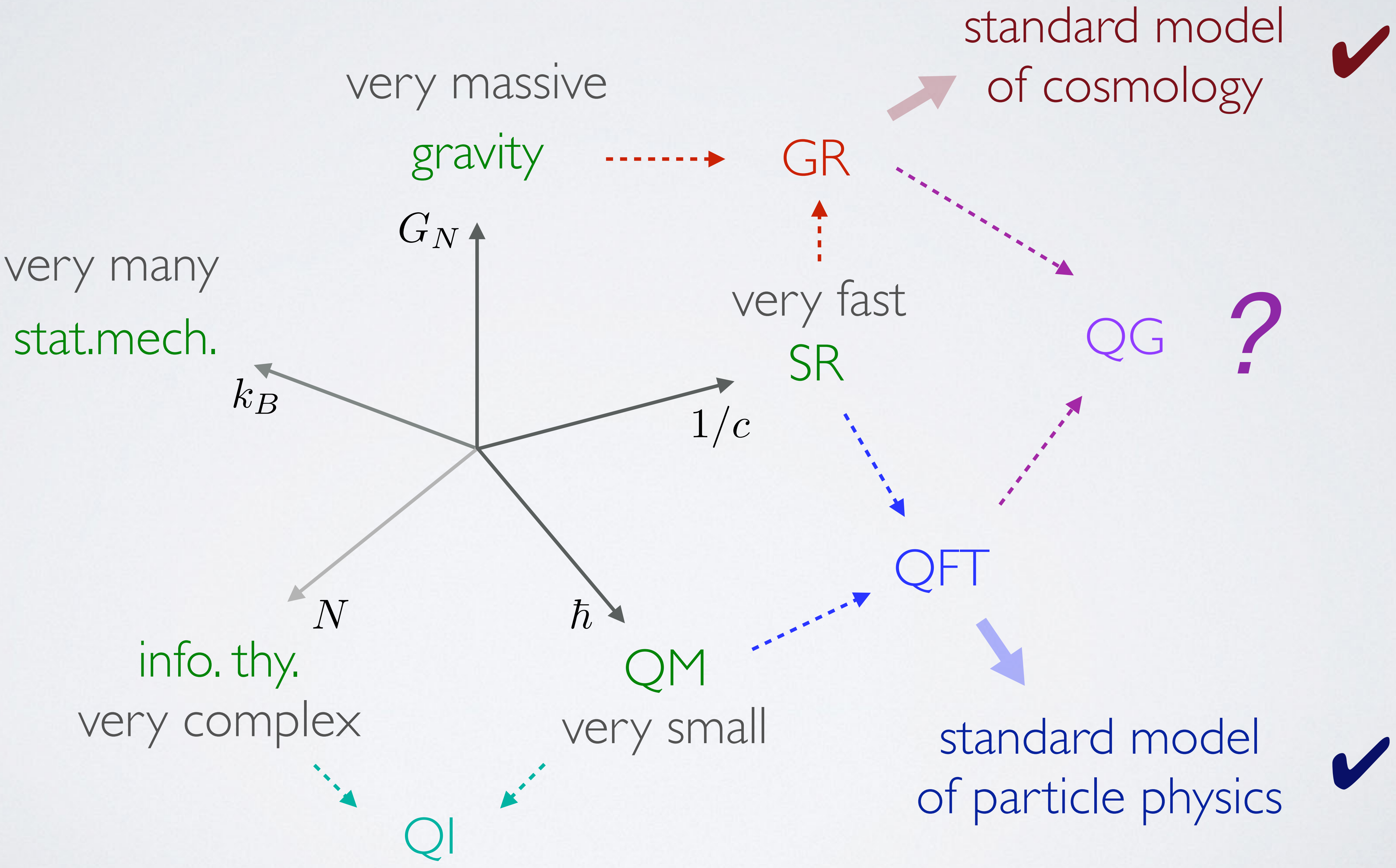
Prologue

Unification in physics: consider extreme regimes…



standard model of cosmology ✓

very massive

gravity

GR

G_N

very many

stat.mech.

very fast

SR

QG ?

k_B

$1/c$

QFT

N

$\hbar$

info. thy.

very complex

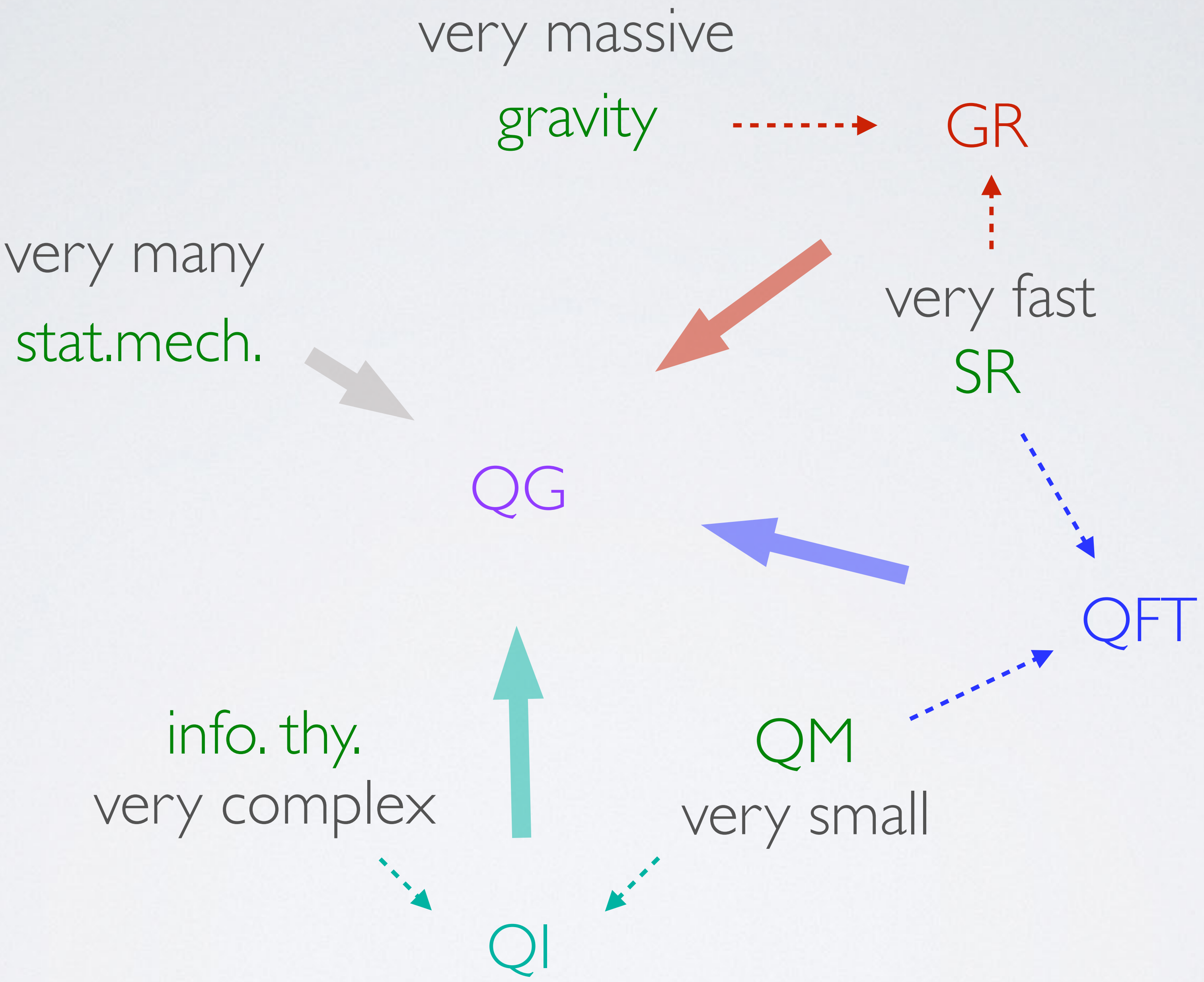
QM

very small

standard model of particle physics ✓

QI

Towards QG…



very massive

gravity

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QFT

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very complex

QM

very small

QI

Gauge/gravity duality substantiates this expectation

Black holes are playing a key role in this quest...

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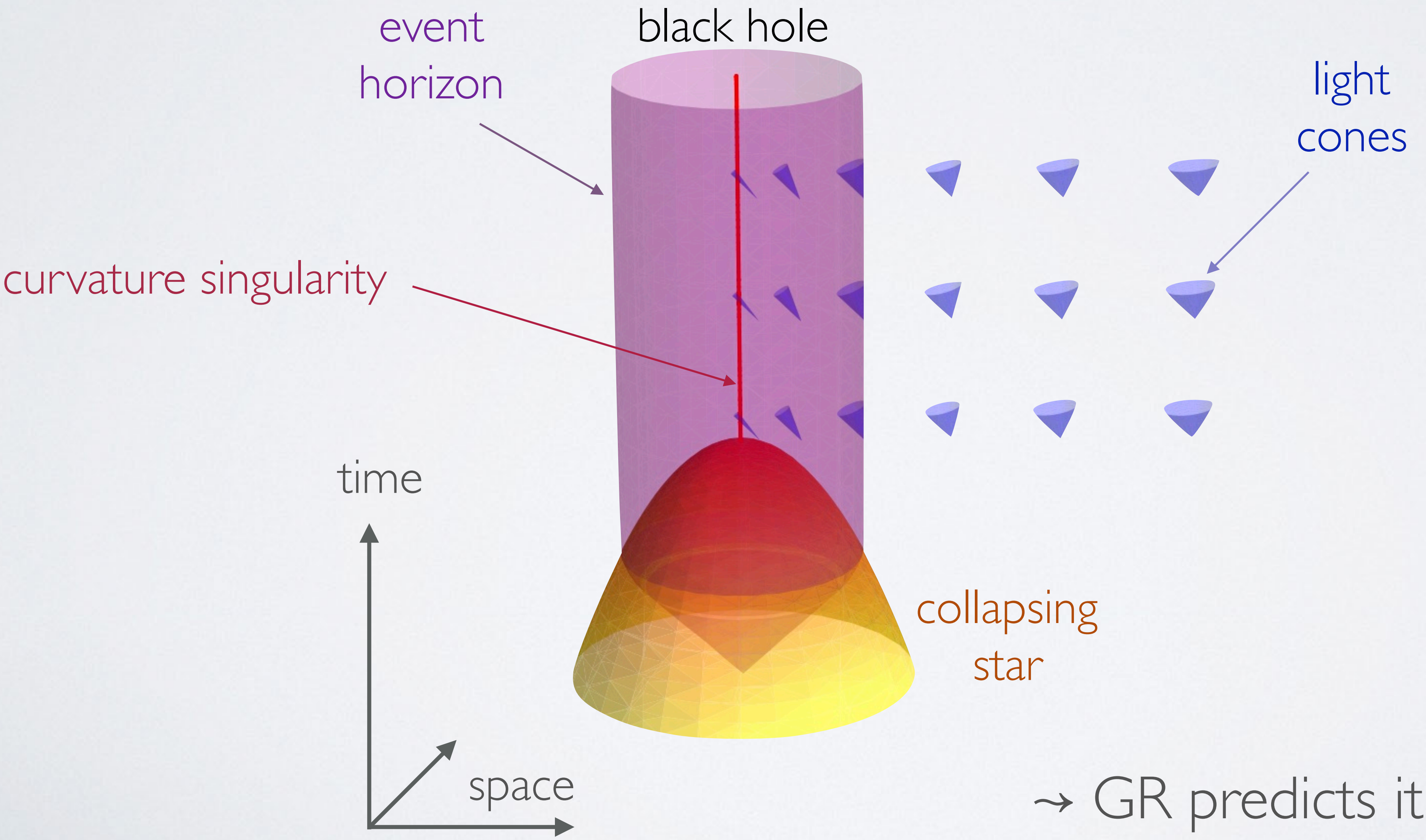
Black holes

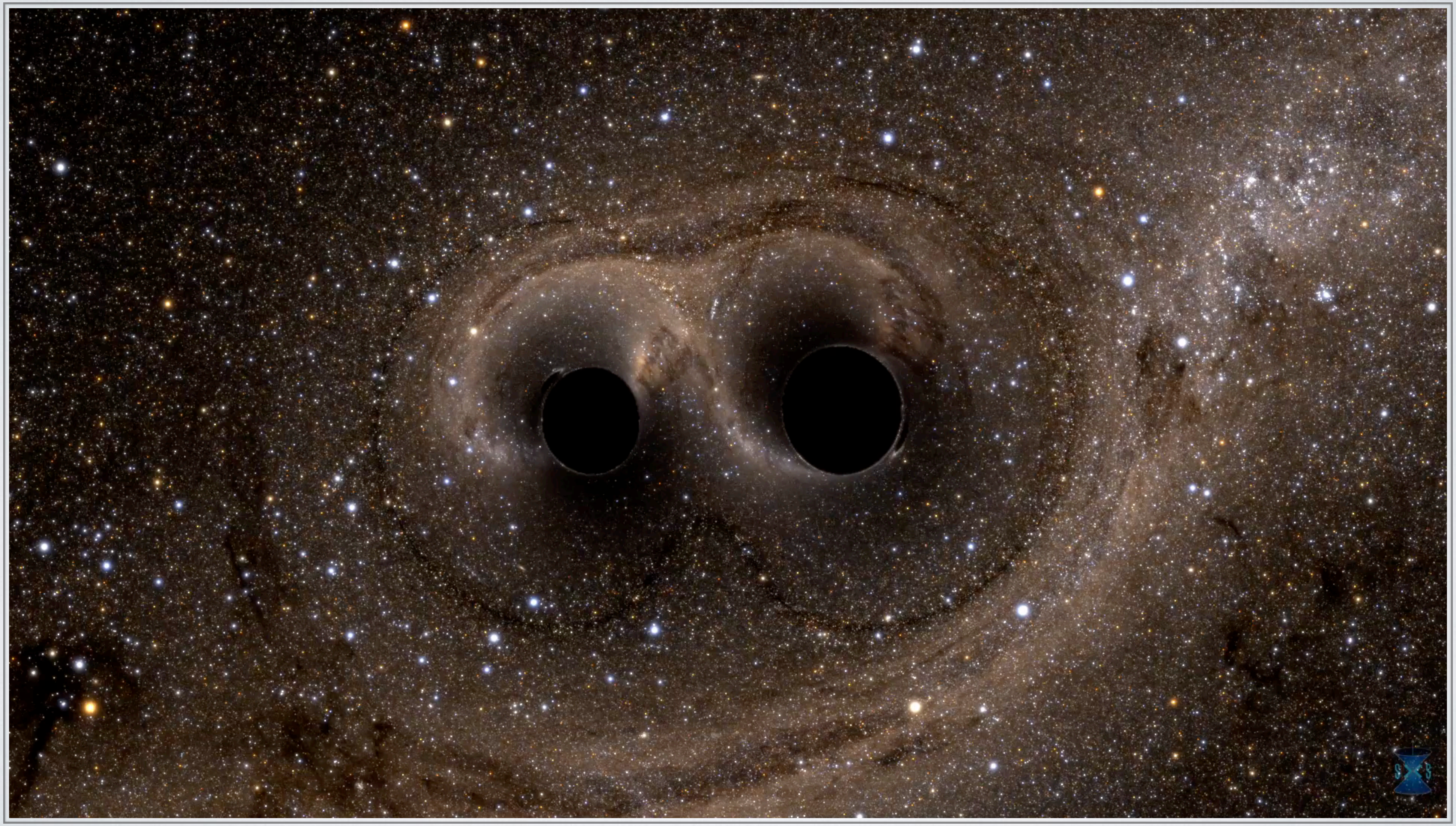
- Black hole = region of spacetime which cannot communicate with the external universe
- In nature, typically occurs as endpoint of gravitational collapse



→ GR predicts its own downfall!

Light bending

Background: ordinary starry sky; Foreground: merging black holes

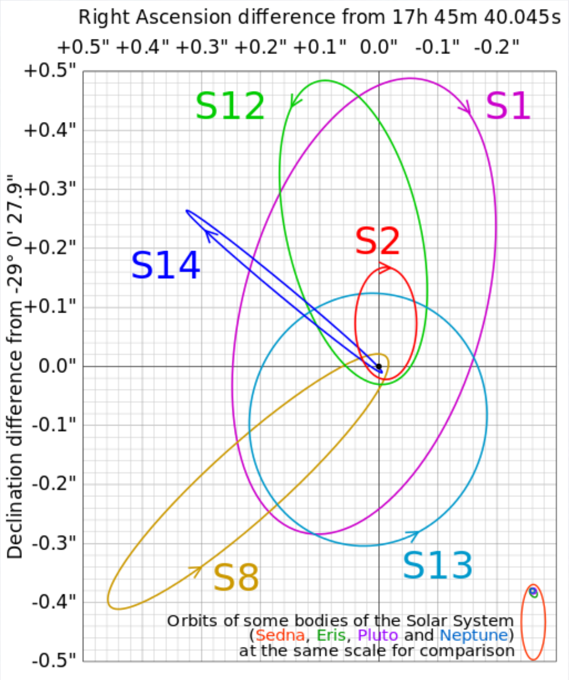


LIGO (SXS) simulation of the event 1.3 billion years ago (two black holes merged into a single one) which led to the first direct detection of gravitational waves in 2015

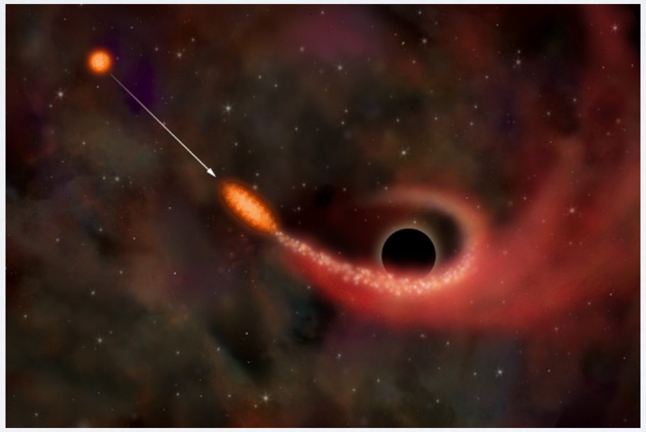
Observing black holes

- No light escapes from black hole, so we cannot see it directly
- But we see the effects of the spacetime on surrounding matter:

stellar orbits



stars get ripped apart



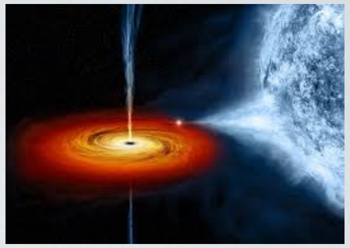
accreting matter gets heated & radiates

- And now we have entered the exciting era of gravitational wave detection

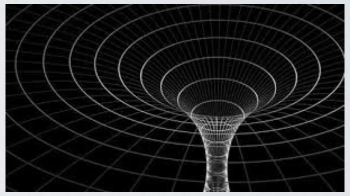
⇒ We can learn a lot from far away…

Many guises of black holes

- Astronomical objects
  powering the most energetic processes in the universe

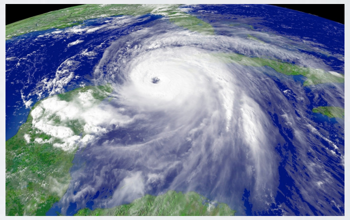


- Mathematically elegant

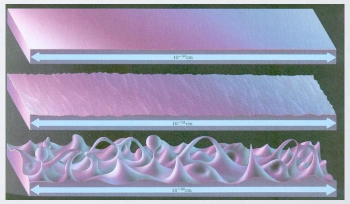


- Lie at heart of profound dualities

- Remarkably related to ‘ordinary’ systems



- May hold a key to quantum gravity



Black hole characteristics

- Important characteristics of a black hole:
  - mass M
  - horizon area A
  - surface gravity κ
    (= force exerted at infinity to keep a unit mass suspended at horizon)

- Cf. essential characteristics of ordinary matter (e.g. fluid)
  - temperature T
  - energy E
  - entropy S

Black hole thermodynamics

- Important characteristics of a black hole:
  - mass M
  - horizon area A
  - surface gravity κ
    (= force exerted at infinity to keep a unit mass suspended at horizon)

Laws of BH mechanics mimic laws of thermodynamics:

0. κ is constant over horizon for stationary BH	0. T is constant over system in thermal equilibrium
1. $dM = (1/8\pi)\,\kappa\,dA + \Omega_H\,dJ$	1. $dE = T\,dS +$ work terms
2. $\delta A \geq 0$ in any process	2. $\delta S \geq 0$ in any process
3. Impossible to achieve $\kappa = 0$ by a physical process	3. Impossible to achieve $T = 0$ by a physical process

Black holes as thermodynamic objects

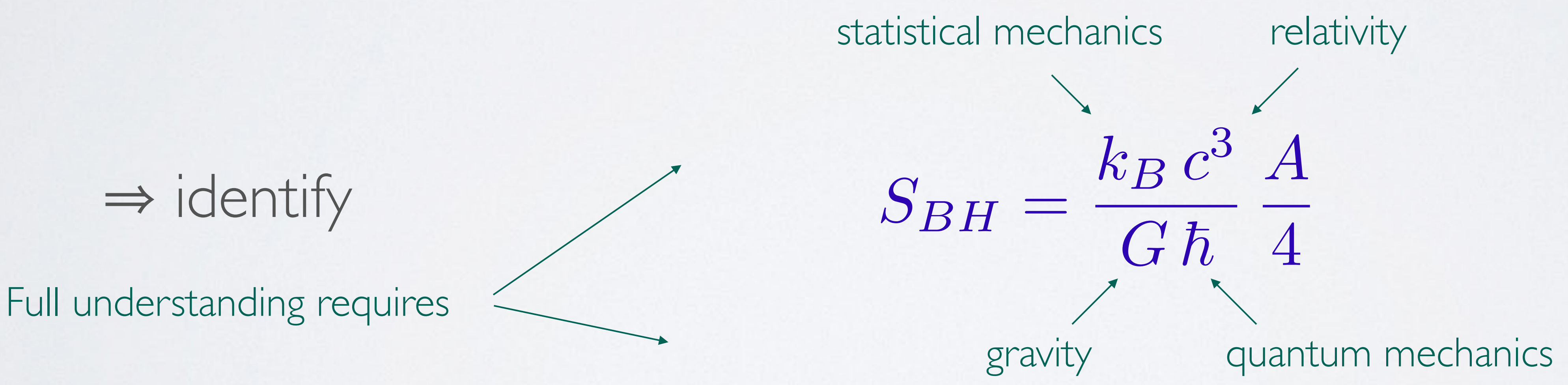
- $S \sim A$ motivated by gedanken-experiments of matter falling into BH

[Bekenstein]

- $T \sim \kappa$ substantiated by semi-classical calculations [Hawking]:

black holes radiate at temperature $T_{BH} = \frac{\hbar\, c}{k_B}\, \frac{\kappa}{2\,\pi}$

⇒ identify



statistical mechanics

relativity

$$S_{BH} = \frac{k_B\, c^3}{G\, \hbar}\, \frac{A}{4}$$

Full understanding requires

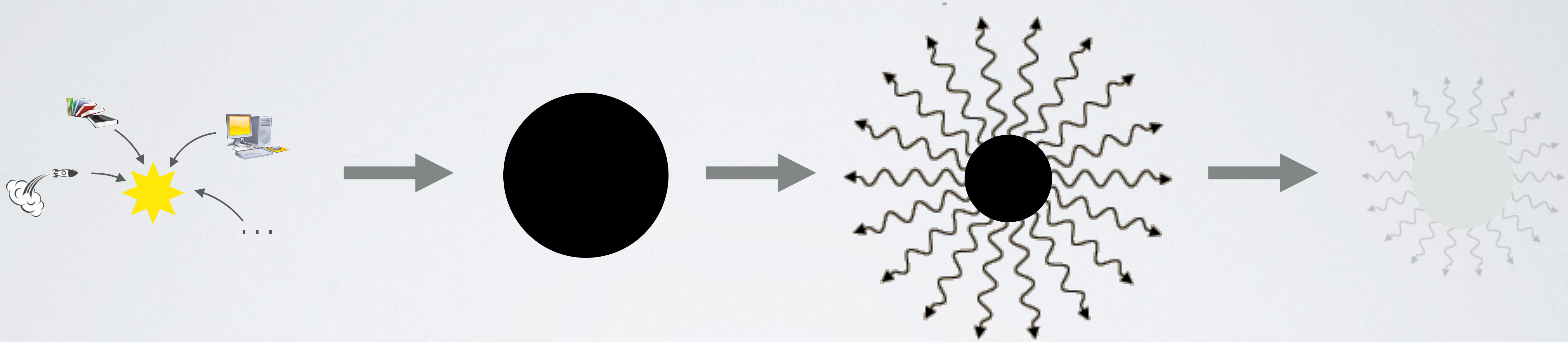
gravity

quantum mechanics

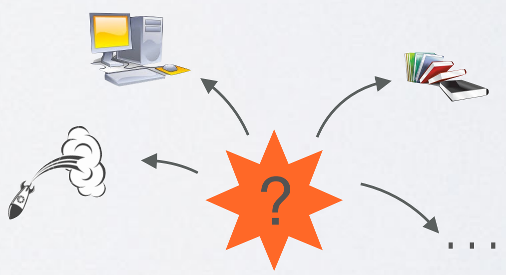
- Natural question: statistical mechanics origin of BH entropy?

Black hole Information Paradox

- According to general relativity, information which falls into a black hole is lost forever.



- According to quantum mechanics, information can never be lost.



- This clash provides an important clue toward formulating a unified theory.

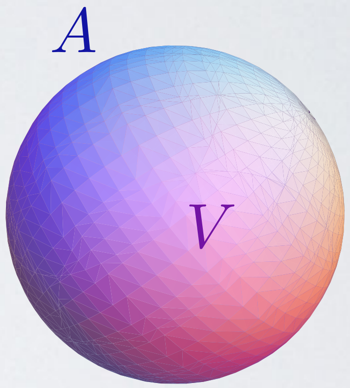
Entropy bound

- Generalized Second Law: combined matter + BH entropy increases

  → Spherical entropy bound (slowly evolving systems): ['t Hooft, Susskind]

$$S_{\text{matter}} \le \frac{A}{4} \qquad \Rightarrow \text{ entropy } S \text{ is not extensive:}$$

$$S \not\propto V$$



Intuitive expectation:

- Amount of information we can store in a region should scale with the volume…
- … in absence of gravity.

  With gravity, sufficiently large info storage will collapse to a black hole

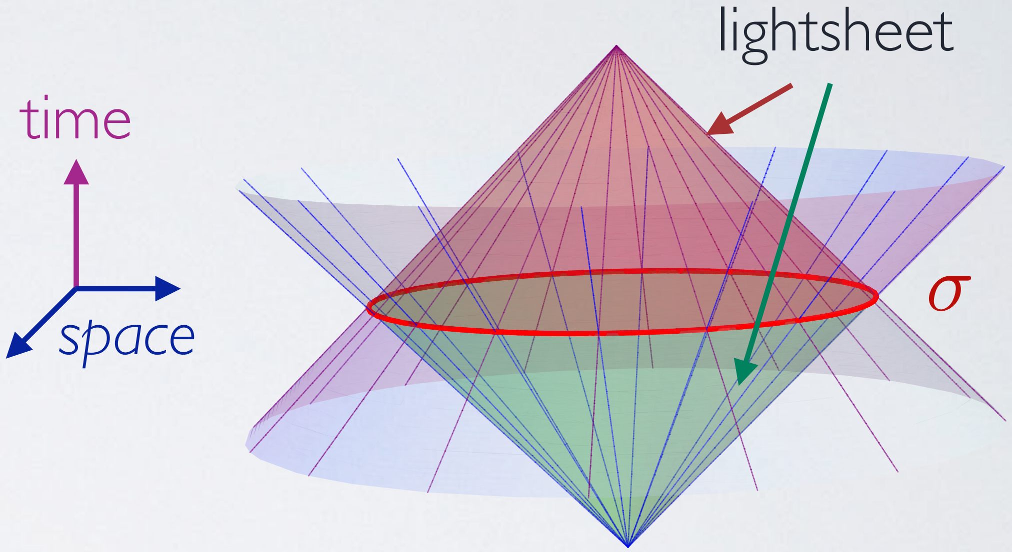
  ⇒ can't exceed S = A/4

Holographic Principle

- Covariant entropy bound: full spacetime construct [Bousso]

  Entropy on any lightsheet L of a surface σ cannot exceed the area of σ:

$$S(L) \leq \frac{A(\sigma)}{4}$$



- Holographic Principle: in a theory of gravity, the number of degrees of freedom describing the physics on lightsheet $L(\sigma)$ cannot exceed $A(\sigma)/4$

→ physical equivalence between 2 theories living in different # of dimensions!

- Concrete realization: gauge/gravity duality

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Gauge/gravity duality

also known as the AdS/CFT correspondence

first derived within string theory [Maldacena '97]
(but applies more generally)

Gauge/gravity duality

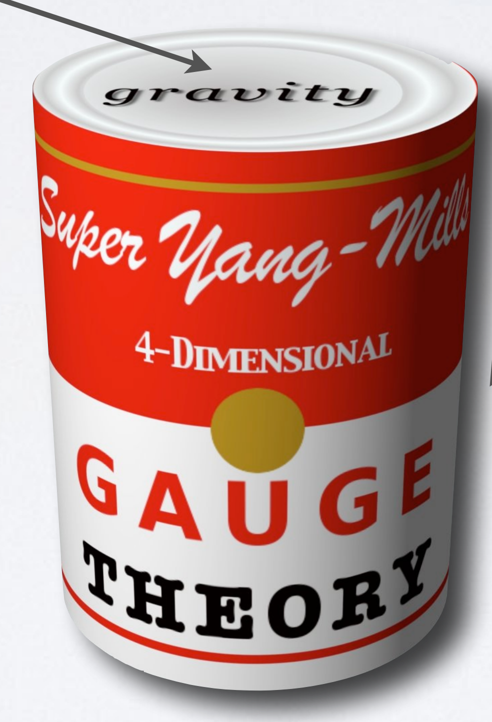
also known as the AdS/CFT correspondence

String theory (gravity) $\Longleftrightarrow$ gauge theory (CFT)

"in bulk" asymp. AdS

"on boundary"

'soup can' diagram of AdS:

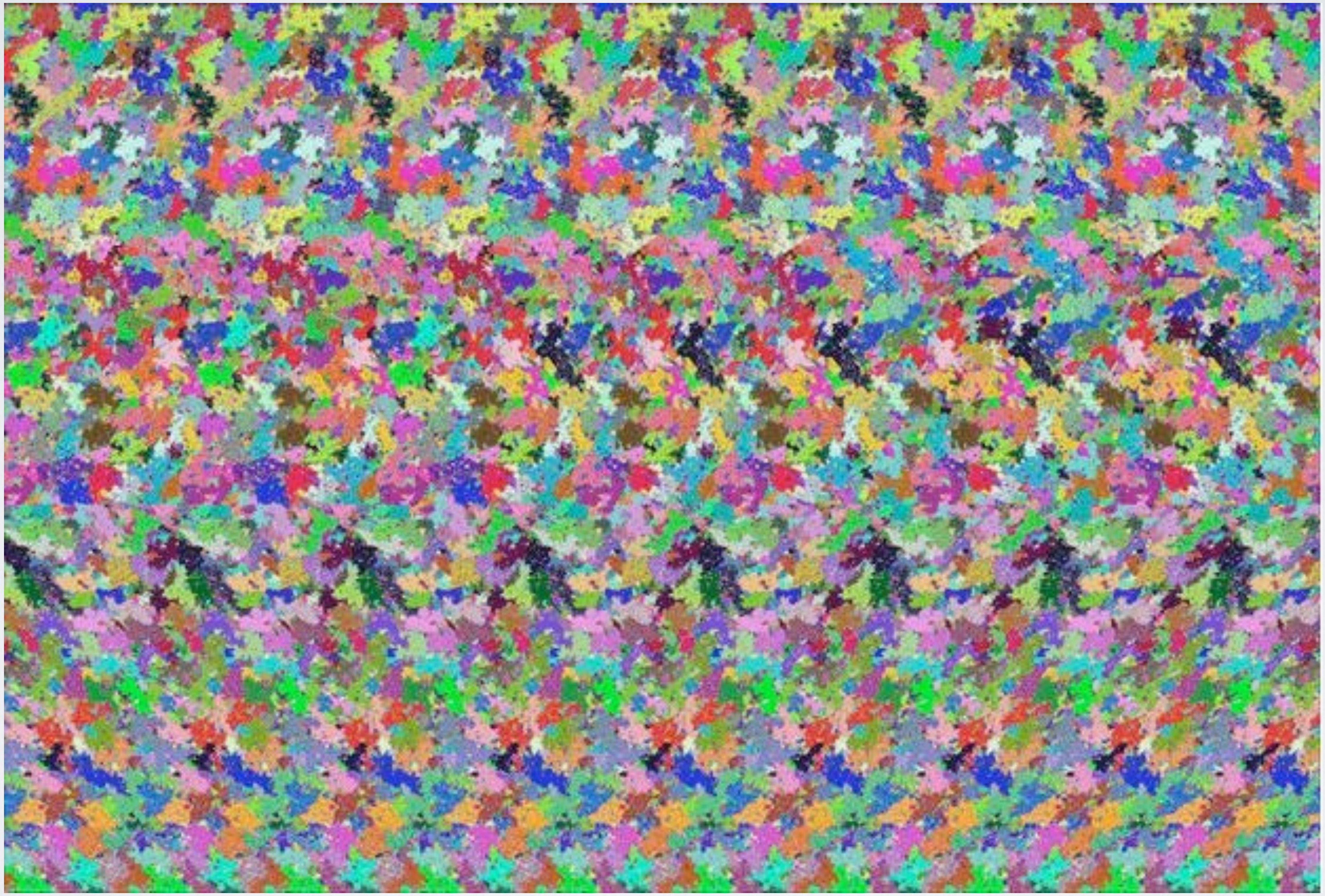


here label is everything...

- Since the two descriptions live in different number of dimensions, we call such a correspondence holographic.

Gauge/gravity duality

- better analogy: stereogram...



...but infinitely more complicated

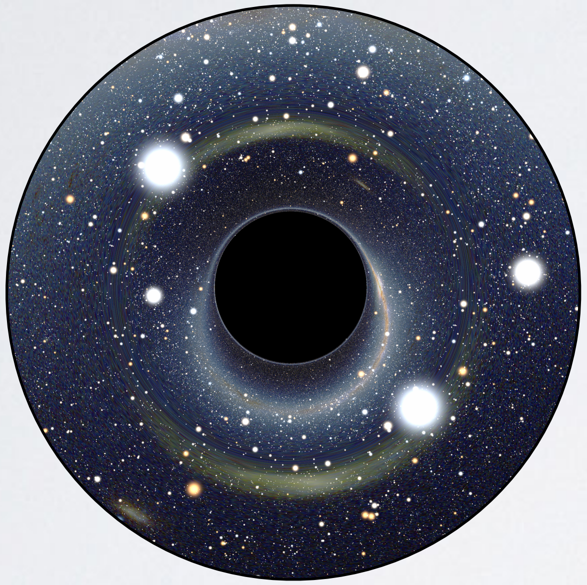
Applications of gauge/gravity duality

String theory (gravity) $\Longleftrightarrow$ field theory (no gravity)

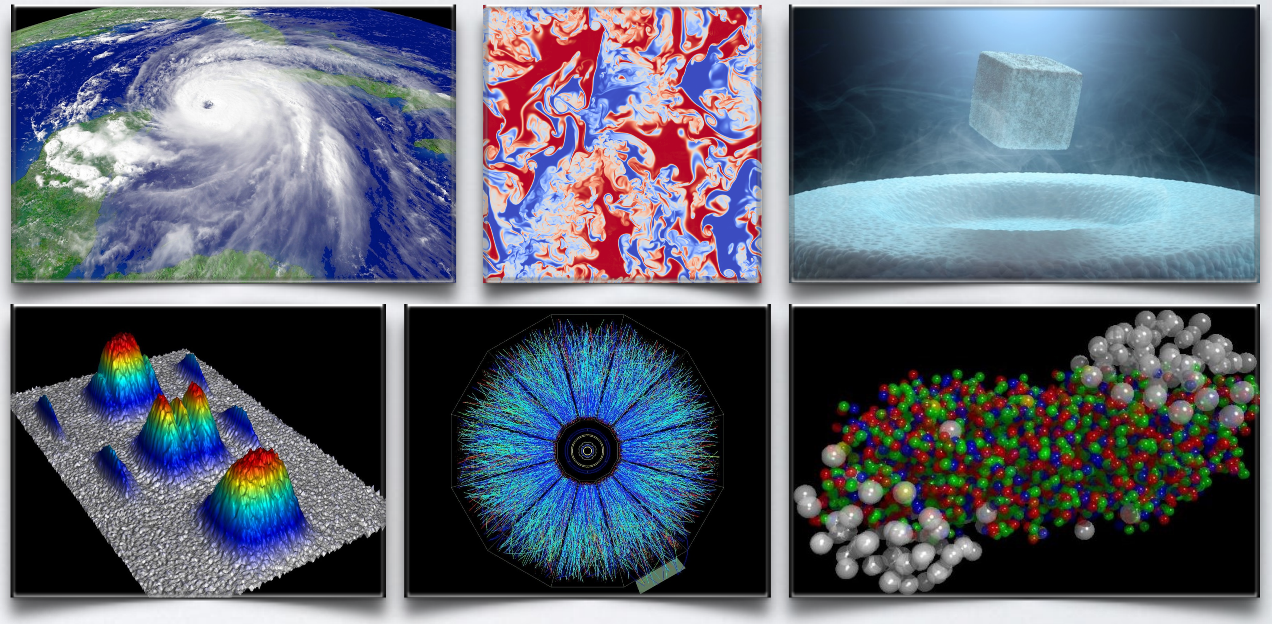
"in bulk" = higher dimensions

"on boundary" = lower dimensions

describes gravitating systems, e.g. black holes



describes experimentally accessible systems



Invaluable tool to:

- Study strongly interacting field theory (hard, but describes many systems) by working with higher-dimensional gravity on AdS (easy).
- Study quantum gravity in AdS (hard, but needed to understand spacetime) by using the field theory (easy for certain things)

Pre-requisite:

We need to understand the AdS/CFT dictionary…

- How does bulk spacetime emerge from the CFT?
  - Which CFT quantities give the bulk metric?
  - What determines bulk dynamics (Einstein's eq.)?
  - How does one recover a local bulk operator from CFT quantities?
- What part of bulk can we recover from a restricted CFT info?
  - What bulk region does a CFT state (at a given instant in time) encode?
  - What bulk region does a spatial subregion of CFT state encode?
- (How) does the CFT "see" inside a black hole?
  - Does it unitarily describe black hole formation & evaporation process?
  - How does it resolve curvature singularities?

Fortuitously, using GR technology goes a long way…

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String Theory

- Compelling candidate for QG
  - it is a self-consistent quantum theory which contains gravity (GR)
- Unifies all fundamental forces and particles
  - these are manifested as excitations of a fundamental string
    (cf. different notes produced by a violin string…)


elementary particle

fundamental string

$10^{-33} cm$

- This "tames" the infinities in computations, simplifies interactions:

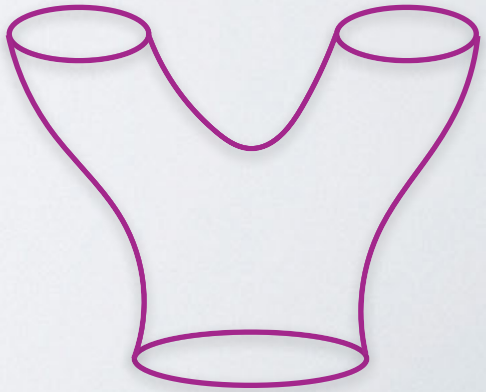

time

space

probability (strength of interaction) = g_s

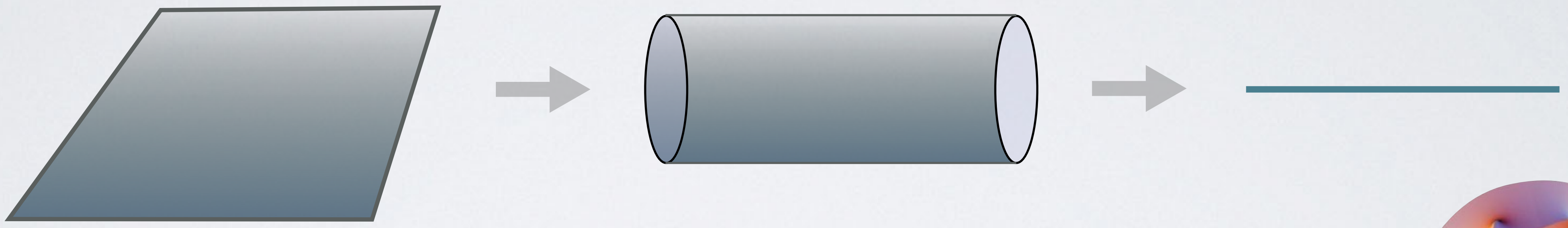
Spacetime in string theory

- string quantization → graviton (as one of the modes of the string)
  - = quantum of gravity (~ ripple in spacetime)
  - hence unlike ordinary QFT where gravitons are impossible, string theory actually *requires* gravity
- new type of fuzziness of spacetime
  - in addition to QM uncertainty principle, blurriness on string scale
  - this bounds how precisely we can describe spacetime
- 2-dimensional worldsheet description:
  - string position described by fields on the string
  - spacetime geometry described by interaction strength
  - Einstein's equation for gravity comes out naturally ✔

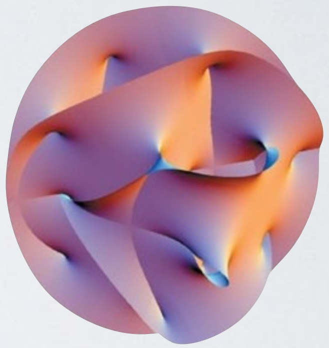


Spacetime in string theory

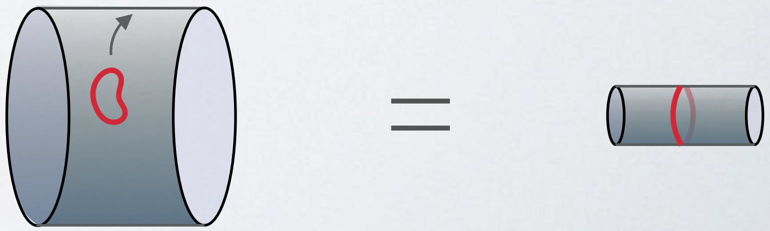
- Consistency requires extra dimensions (9 space + 1 time)
  - these can however be small (“compactified”)



  - in fact the internal space gives rise to matter we see.

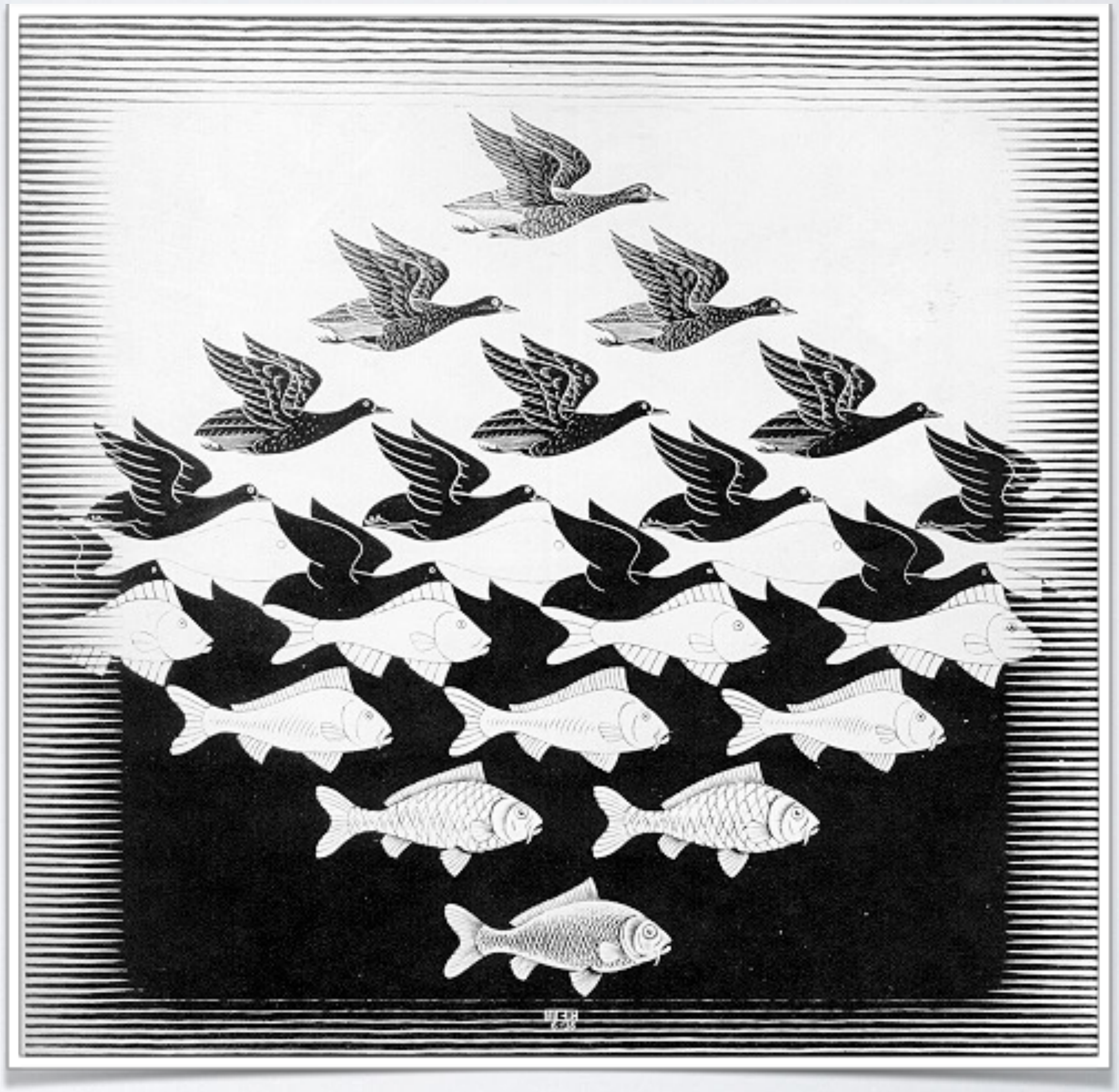


- Supersymmetry
  - extends spacetime: new “fermionic” dimensions
- T-duality
  - equivalence between compact dimension of size R & one of size 1/R
  - Lesson: distances are not absolute



Dualities in string theory

- Strong coupling: expect uncontrollably large fluctuations…
- But in fact equivalent to a distinct weakly-coupled description!
    - Often the variables used cease to be descriptive as we change a parameter (e.g. coupling)
    - Most "properties" look different on the two sides



Dualities in string theory

- Strong coupling: expect uncontrollably large fluctuations…
- But in fact equivalent to a distinct weakly-coupled description!

- D(irichlet)-branes carry the (RR) charges required by duality [Polchinski]
  - Weak coupling: topological defects on which open strings can end
  - Strong coupling: natural 'fundamental' constituents of the theory
- Dp-branes extended in p+1 dimensions
- Sufficiently many D-branes backreact on spacetime
  - → extremal black p-brane
    (higher-dimensional generalization of extremal Reissner-Nordstrom BH)

eg. for p=3: $$ds^2 = f(r)^{-1/2}\,\eta_{\mu\nu}\,dx^\mu\,dx^\nu + f(r)^{1/2}\left(dr^2 + r^2\,d\Omega_5^2\right)$$

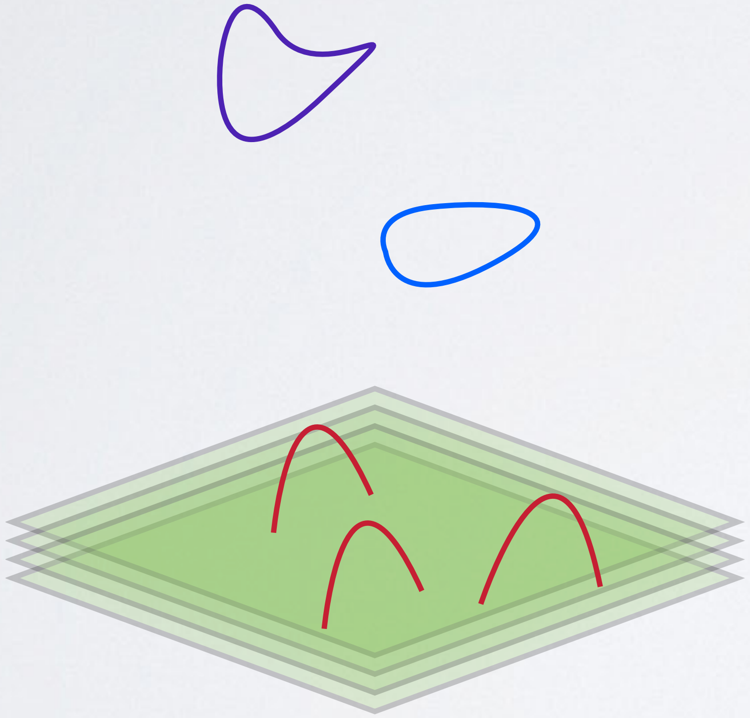
where $$f(r) = 1 + \frac{4\pi\,g_s\,N\,\ell_s^4}{r^4}$$

Maldacena's construction

(in IIB string theory)

- Consider a stack of N coincident D3-branes in 2 regimes:

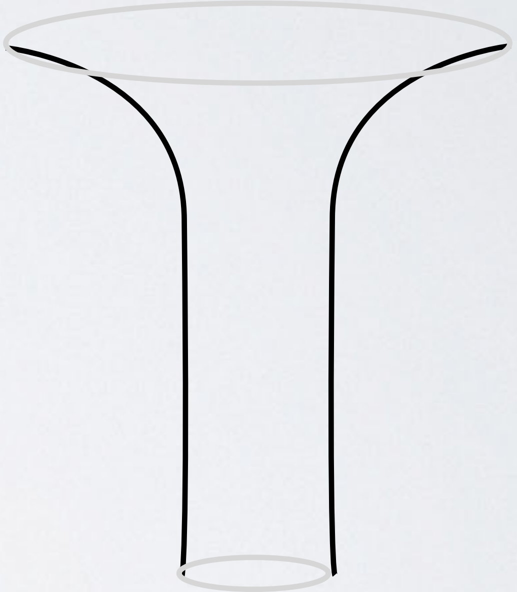
$g_s N \ll 1$ (perturbation theory valid)



$\}N$ D3-branes

closed + open string excitations
in R^{10}

$g_s N \gg 1$

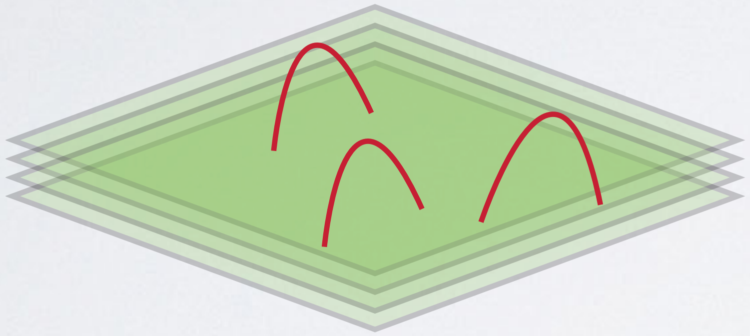


curved geometry:
extremal black 3-brane

Maldacena's construction

- take "decoupling" (low-energy) limit:

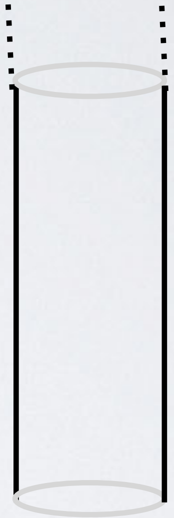
$g_s \, N \ll 1$



closed strings decouple;
we're left with low-energy
open string modes:

SU(N) gauge theory

$g_s \, N \gg 1$



Asymptotic modes decouple;
we're left with:

closed string theory on
near-horizon geometry:
AdS$_5$ x S^5

Maldacena's conjecture

$g_s N \ll 1$

SU(N) gauge theory

$g_s N \gg 1$

closed string theory on $AdS_5 \times S^5$

- ✴ but gauge theory is defined for all coupling, so identify the two descriptions:

→ AdS/CFT correspondence:

Four-dimensional $\mathcal{N} = 4$ SU(N) SYM gauge theory
is fully equivalent to
IIB string theory with $AdS_5 \times S^5$ boundary conditions.

AdS/CFT correspondence

String theory ($\ni$ gravity) $\iff$ gauge theory (CFT)

"in bulk" asymp. AdS × K *"on boundary"*

Specific example (previous case):

IIB string theory on AdS$_5$ × S^5:

g_s , $\dfrac{\ell}{\ell_s}$

string coupling

AdS size

string length

$\mathcal{N} = 4$ SU(N) SYM:

N , $\lambda \equiv g_{\rm YM}^2 N$

rank of gauge group

't Hooft coupling

where $g_s \sim \dfrac{\lambda}{N}$ and $\dfrac{\ell}{\ell_S} \sim \lambda^{1/4}$

- large λ ⇒ small stringy corrections
- large N ⇒ small quantum corrections
- Hence $N \gg \lambda \gg 1$ ⇒ classical gravity on AdS$_5$ × S^5

AdS$_5$ × S^5 geometry

AdS$_5$ × S^5

maximally symmetric
negatively curved spacetime

$\Lambda < 0$

maximally symmetric
positively curved space

$\Lambda > 0$

- can obtain via embedding into 1-higher dimensional space(time)s

5D surface

$$-X_{-1}^2 - X_0^2 + X_1^2 + \cdots + X_4^2 = -\ell^2$$

on $\mathbb{R}^{4,2}$:

$$ds^2 = -dX_{-1}^2 - dX_0^2 + dX_1^2 + \cdots + dX_4^2$$

5D surface

$$X_1^2 + \cdots + X_6^2 = \ell^2$$

on $\mathbb{R}^6$:

$$ds^2 = dX_1^2 + \cdots + dX_6^2$$

$\rightsquigarrow$ $SO(4,2) \times SO(6)$ symmetry

- full solution ($\Lambda = 0$) supported by 5-form field strength

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Lecture 1

Lecture 2

AdS/CFT correspondence

String theory ($\ni$ gravity) $\Longleftrightarrow$ gauge theory (CFT)

"in bulk" asymp. AdS *"on boundary"*

Key aspects:

- ⁎ *Holographic*: gauge theory lives in fewer dimensions.
- ⁎ Strong/weak coupling duality.
- ⁎ Gravitational theory maps to non-gravitational one!

Invaluable tool to:

- ~ Study strongly coupled field theory (hard, but describes many systems) by working with higher-dimensional gravity on AdS (easy).
- ~ Study quantum gravity in AdS (hard, but needed to understand spacetime) by using the field theory (easy for certain things)

Pre-requisite: Understand the AdS/CFT 'dictionary'...

Recall: $AdS_5 \times S^5$ geometry

- In Maldacena's construction, arose as near-horizon geometry of extremal black 3-brane (= backreaction of stack of coincident D3-branes)

$$ds^2 = f(r)^{-1/2}\,\eta_{\mu\nu}\,dx^\mu\,dx^\nu + f(r)^{1/2}\left(dr^2 + r^2\,d\Omega_5^2\right)$$

where $f(r) = 1 + \frac{4\pi\,g_s\,N\,\ell_s^4}{r^4} \equiv 1 + \frac{\ell^4}{r^4}$

  - Event horizon is at $r = 0$
  - Near horizon geometry: $$ds^2 = \underbrace{\frac{r^2}{\ell^2}\,\eta_{\mu\nu}\,dx^\mu\,dx^\nu + \frac{\ell^2}{r^2}\,dr^2}_{AdS_5} \times \underbrace{+\,\ell^2\,d\Omega_5^2}_{S^5}$$

Geometry of AdS$_5$

Poincare AdS

$$ds^2 = \frac{\ell^2}{z^2}\,\big(\underbrace{-dt^2 + dx_i\,dx^i}_{\text{4-d Poincare symmetry}} + dz^2\big)$$

scaling symmetry

Global AdS

$$ds^2 = -\left(\frac{\rho^2}{\ell^2}+1\right)\underset{\nearrow}{d\tau^2} + \frac{d\rho^2}{\left(\frac{\rho^2}{\ell^2}+1\right)} + \rho^2\,\underset{\nearrow}{d\Omega_3^2}$$

static spherically symmetric

Geometry of AdS$_3$

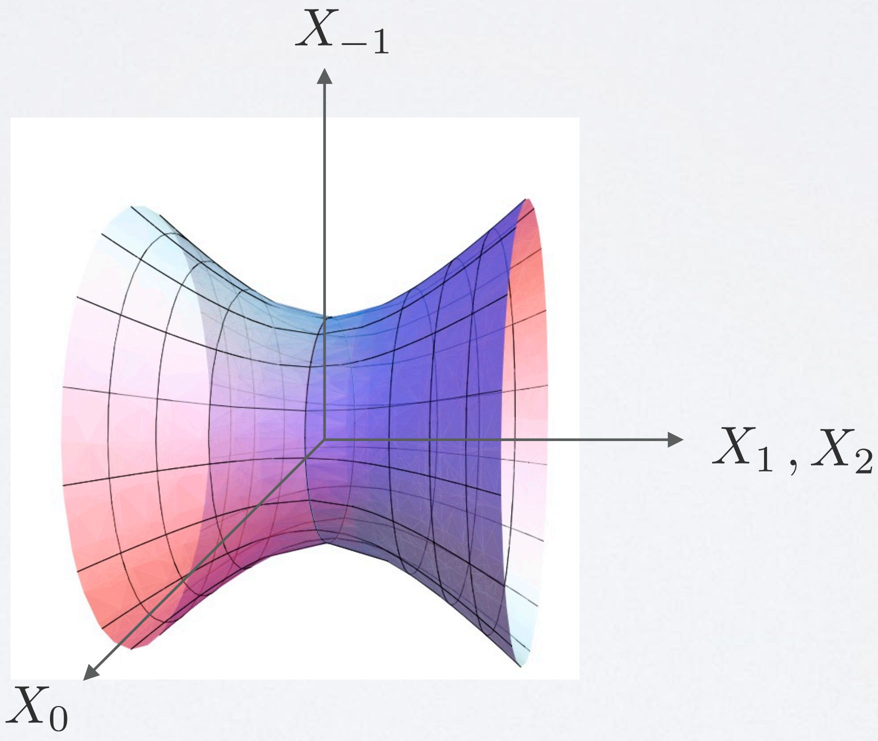
Poincare AdS$_3$

$$ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + dx^2 + dz^2\right)$$

Global AdS$_3$

$$ds^2 = -\left(\frac{\rho^2}{\ell^2} + 1\right) d\tau^2 + \frac{d\rho^2}{\left(\frac{\rho^2}{\ell^2} + 1\right)} + \rho^2\, d\varphi^2$$

view as embedded in R^4 $\quad -X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 = -\ell^2$



Geometry of AdS3

Poincare AdS3

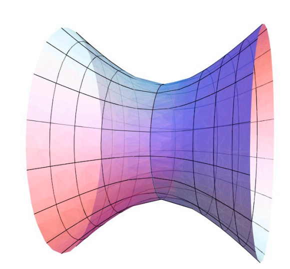
$$ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + dx^2 + dz^2\right)$$

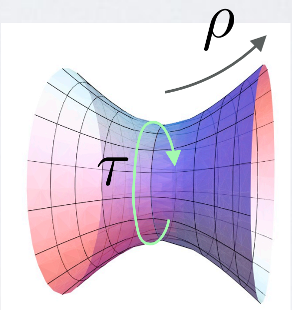
Global AdS3

$$ds^2 = -\left(\frac{\rho^2}{\ell^2}+1\right) d\tau^2 + \frac{d\rho^2}{\left(\frac{\rho^2}{\ell^2}+1\right)} + \rho^2\, d\varphi^2$$

view as embedded in R^6

$$-X_{-1}^2 - X_0^2 + X_1^2 + X_2^2 = -\ell^2$$





$$X_{-1} = \ell\, t/z$$
$$X_0 = \frac{\ell^2 - t^2 + x^2 + z^2}{2\, z}$$
$$X_1 = \frac{-\ell^2 - t^2 + x^2 + z^2}{2\, z}$$
$$X_2 = \ell\, x/z$$

$t, x \in (-\infty, \infty)\ , \quad z \in (0, \infty)$

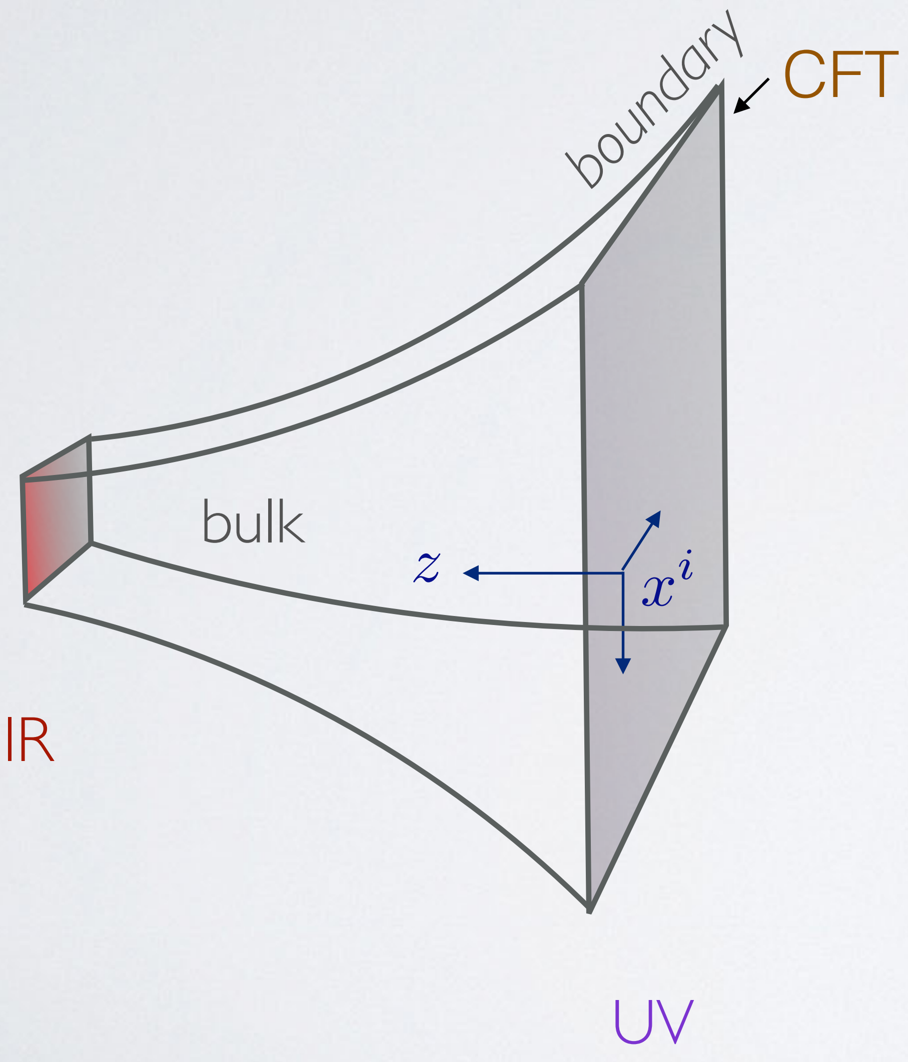
$$X_{-1} = \sqrt{\rho^2 + \ell^2}\, \sin\left(\frac{\tau}{\ell}\right)$$
$$X_0 = \sqrt{\rho^2 + \ell^2}\, \cos\left(\frac{\tau}{\ell}\right)$$
$$X_1 = \rho\, \sin\varphi$$
$$X_2 = \rho\, \cos\varphi$$

$\tau \in (-\infty, \infty)\ , \quad \rho \in [0, \infty)\ , \quad \varphi \in [0, 2\pi)$

Geometry of AdS

Poincare AdS:

$$ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + dx_i\, dx^i + dz^2\right)$$



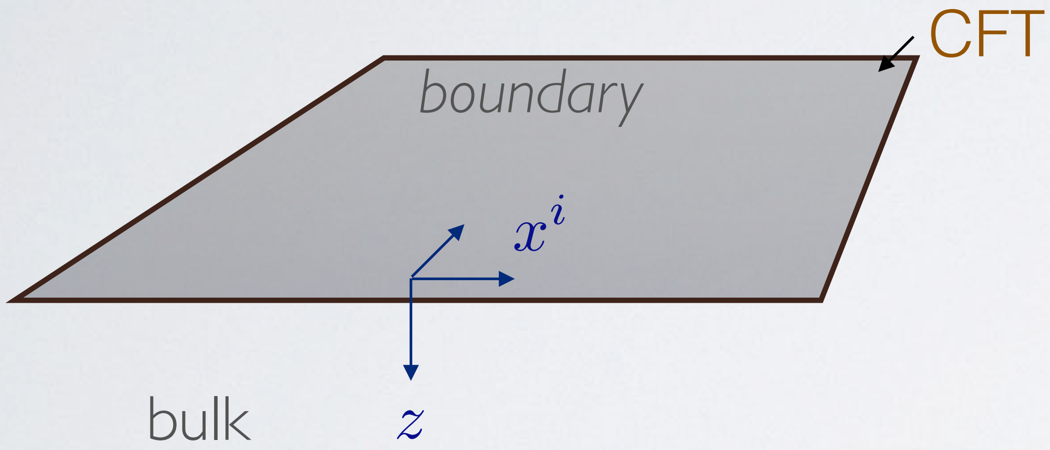
Geometry of AdS

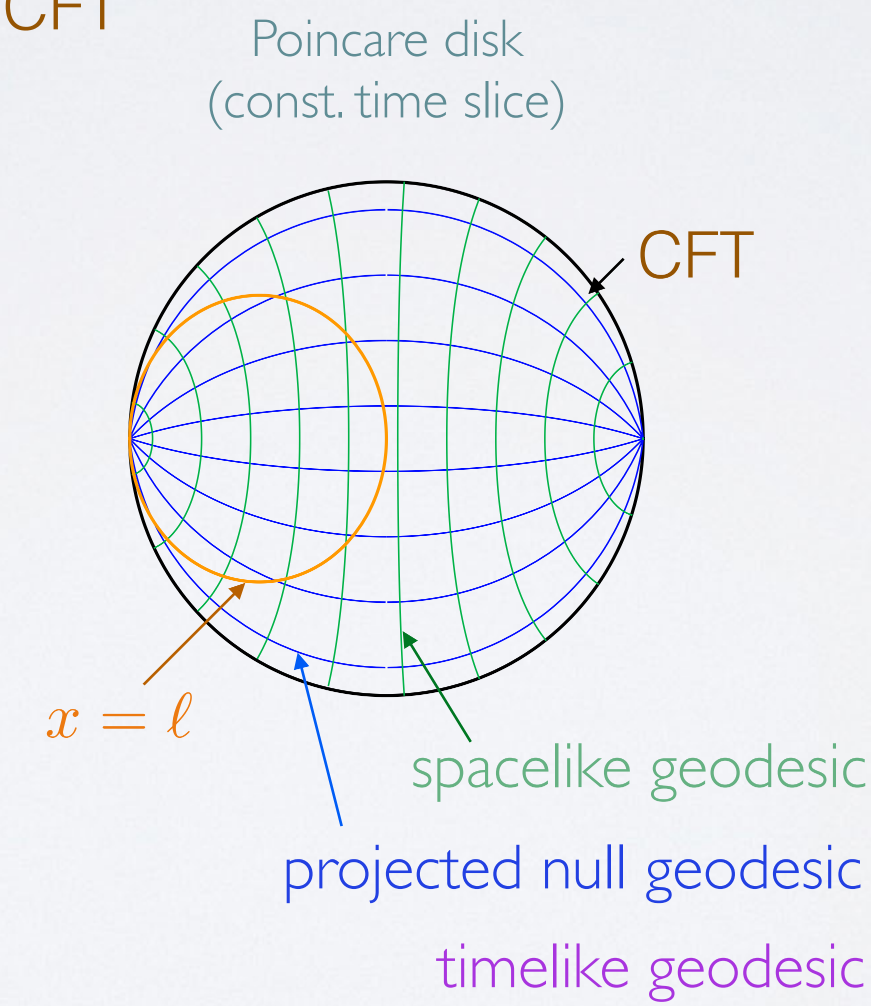
Poincare AdS:

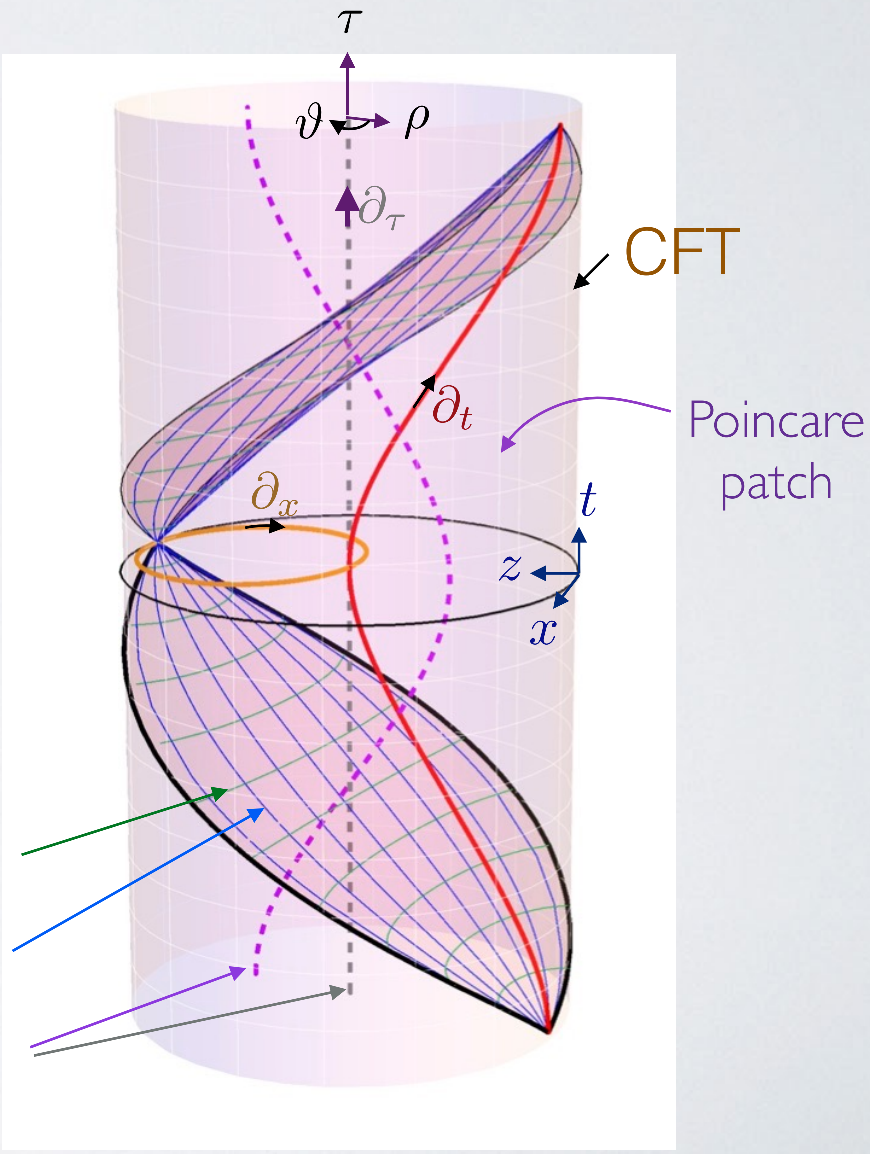
$$ds^2 = \frac{\ell^2}{z^2}\left(-dt^2 + dx_i\, dx^i + dz^2\right)$$

Global AdS:

$$ds^2 = -\left(\frac{\rho^2}{\ell^2}+1\right) d\tau^2 + \frac{d\rho^2}{\left(\frac{\rho^2}{\ell^2}+1\right)} + \rho^2\, d\Omega_3^2$$







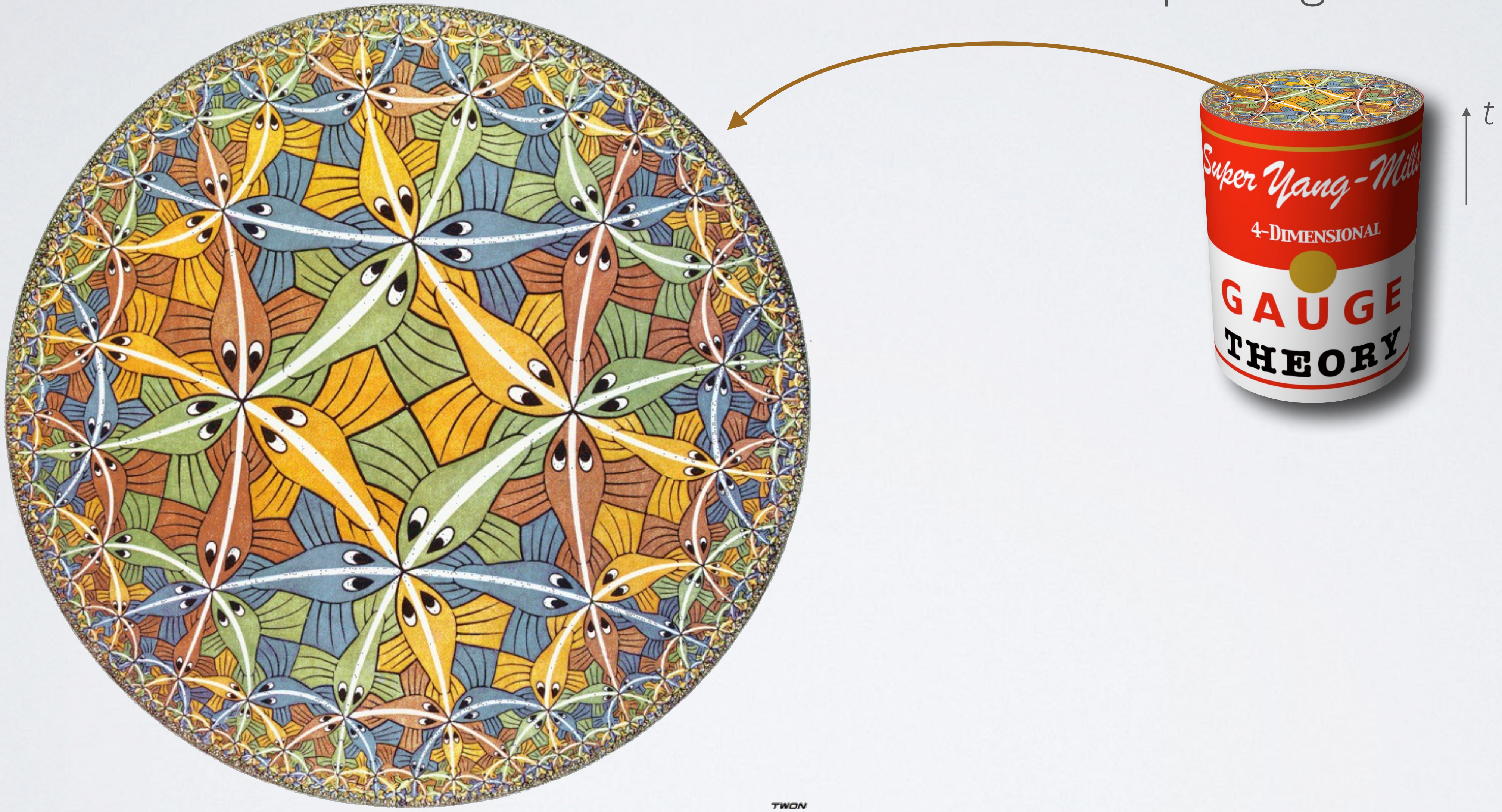
OUTLINE

- Motivation: path to QG
- Black holes & holographic principle
- Preview of gauge/gravity correspondence
- Historical detour: string theory
- Anti de Sitter geometry
- Elements of gauge/gravity dictionary
  - Scale/radius duality (“UV/IR correspondence”)
  - States / geometries
  - Fluid/gravity correspondence
  - CFT probes of bulk geometry…

Radial direction

What boundary quantity encodes the extra bulk direction?

- hints from spatial geometry of AdS:



Radial direction

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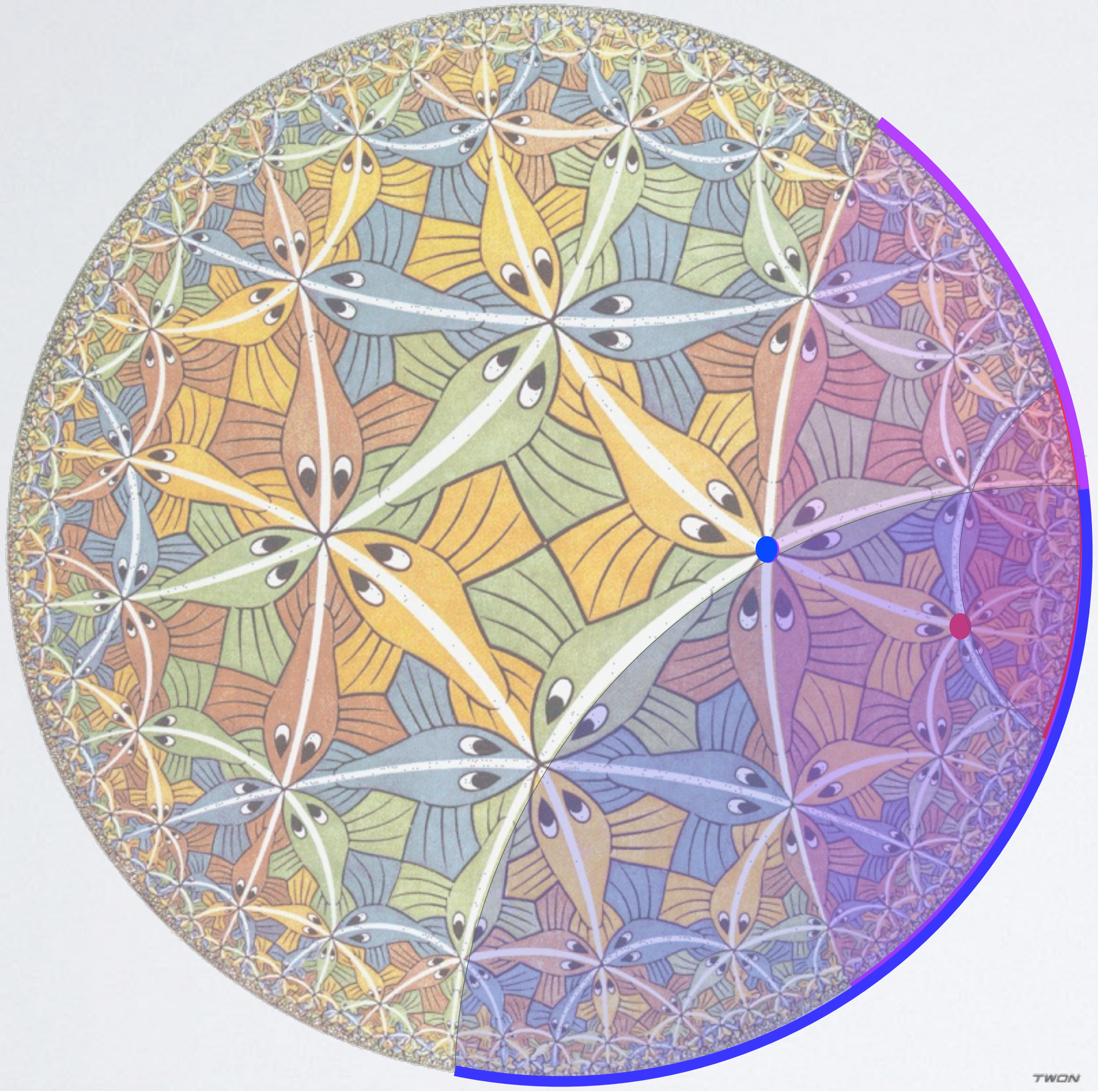


The radial bulk direction comes from a scale size on boundary:

- points near the boundary ⟷ small arcs
- points further in the bulk ⟷ larger arcs
- same point represented by multiple arcs

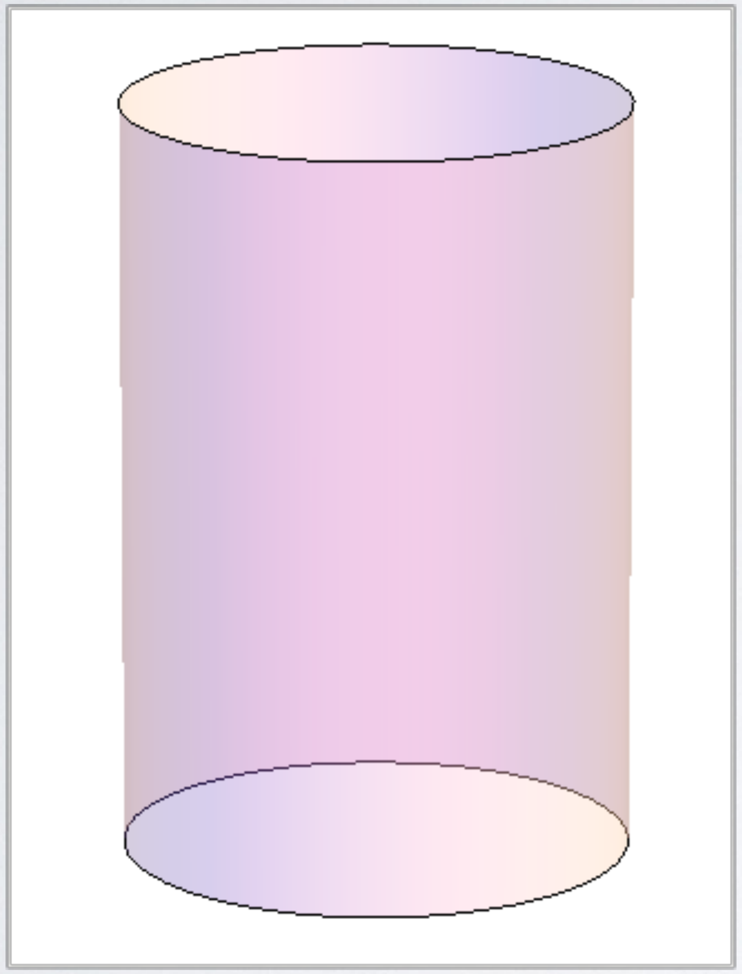
- Provides useful intuition:

e.g. bulk particle falling due to gravity ⟷ boundary excitation spreading outward
… falling into black hole ⟷ … thermalizing

Bulk geometries and CFT states

different bulk geometries ⇝ different states in CFT

(asymptotically AdS)



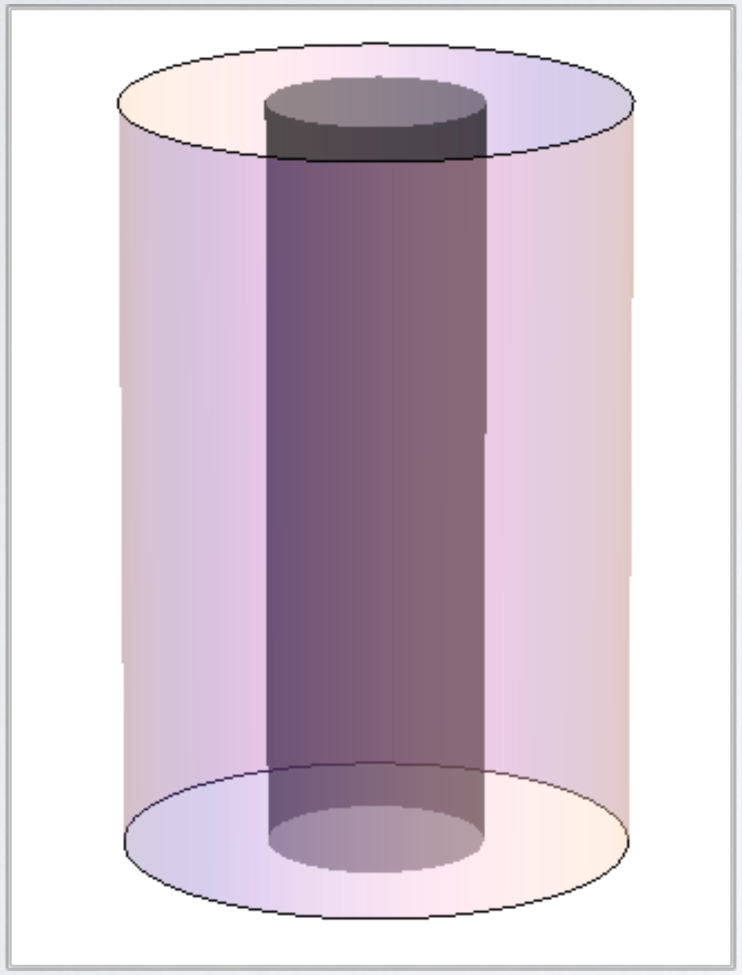
- Pure AdS ⟷ vacuum state in CFT

Finite-mass deformations
of the bulk geometry result in
non-zero boundary stress tensor

Bulk geometries and CFT states

different bulk geometries ⤳ different states in CFT

(asymptotically AdS)

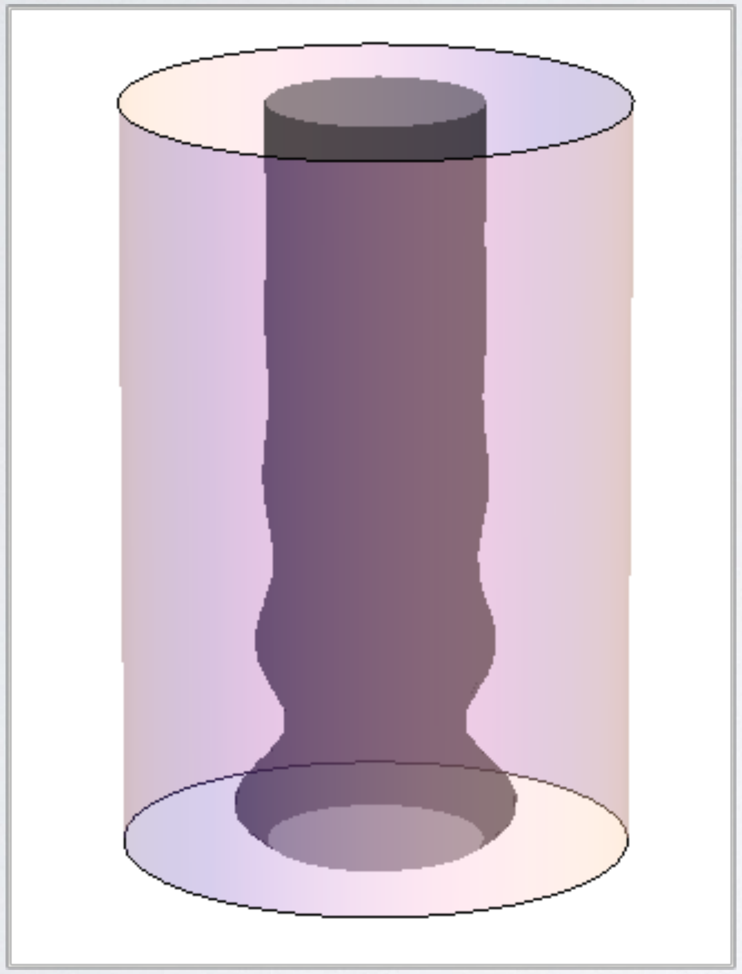


- Pure AdS ⇜⇝ vacuum state in CFT
- Black hole ⇜⇝ thermal state in CFT

(large BH ⇜⇝ high temperature)

Bulk geometries and CFT states

evolving bulk geometries $\longleftrightarrow$ corresponding dynamics



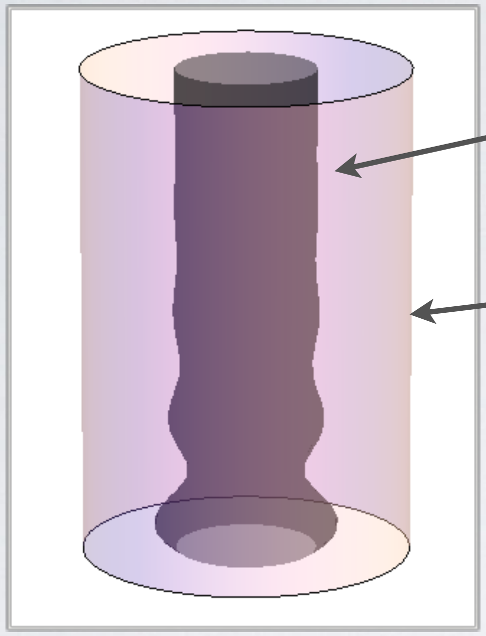
- Pure AdS $\leftrightsquigarrow$ vacuum state in CFT
- Black hole $\leftrightsquigarrow$ thermal state in CFT
- quasinormal modes of perturbed black hole $\leftrightsquigarrow$ approach to thermal equilibrium

  [Horowitz & VH]
- horizon response properties $\leftrightsquigarrow$ CFT transport coefficients

  [Kovtun, Son, Starinets]
- at non-linear level, in hydro regime (large BHs) → fluid/gravity correspondence

  [Bhattacharyya, VH, Minwalla, Rangamani]

Description of dynamics



bulk geometry specified by $g_{ab}(r, x^\mu)$

boundary state characterized by $T_{\mu\nu}(x^\mu)$

Relation between them:
asymptotic falloff of g_{ab} induces $T_{\mu\nu}$

- Bulk dynamics is specified by Einstein's equations.

$$E_{ab} \equiv R_{ab} - \frac{1}{2} R\, g_{ab} + \Lambda\, g_{ab} = 0$$

- Boundary dynamics satisfies stress tensor conservation.

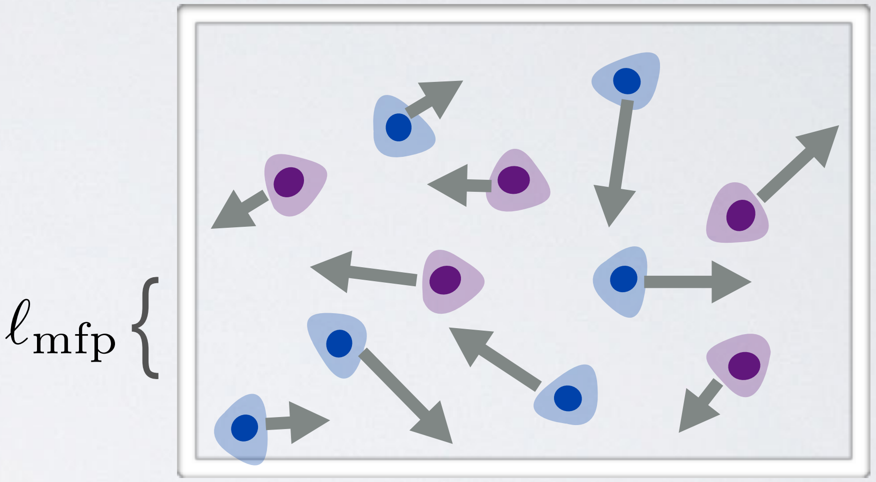
$$\nabla_\mu T^{\mu\nu} = 0$$

(Relativistic) fluid dynamics

= effective description of interacting systems which have achieved local equilibrium

- valid on length scales

$$L \gg \ell_{\rm mfp} \sim \frac{1}{T}$$



- Variables describing the system:
  - local temperature $T(x^\mu)$ and local velocity $u^i(x^\mu)$
- Key *dynamical equation*: conservation of the stress tensor $T^{\mu\nu}$

$$\nabla_\mu T^{\mu\nu} = 0$$

- supplemented by *constitutive relations*, which express $T^{\mu\nu}$ in terms of T and u^μ.

  Eg. for ideal fluid, $T^{\mu\nu} = (\rho + P)\, u^\mu\, u^\nu + P\, \eta^{\mu\nu}$

(Relativistic) fluid dynamics

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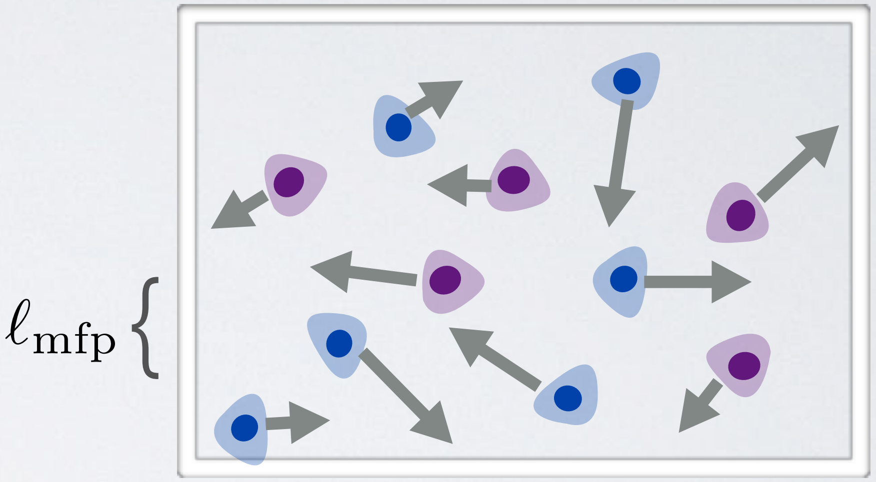

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$$\nabla_\mu T^{\mu\nu} = 0$$

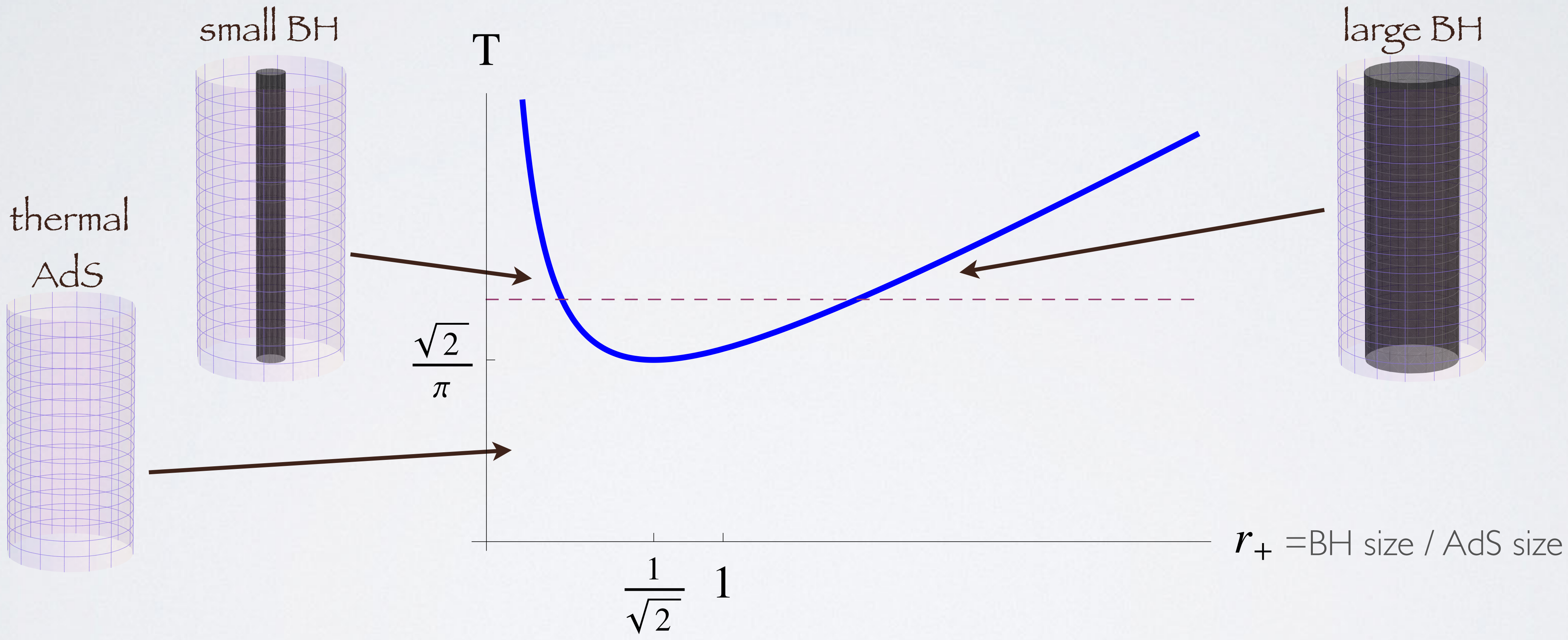
- supplemented by *constitutive relations*, which express $T^{\mu\nu}$ in terms of T and u^μ.

Eg. for dissipative fluid, $T^{\mu\nu} = (\rho + P)\, u^\mu\, u^\nu + P\, \eta^{\mu\nu} - 2\, \eta\, \sigma^{\mu\nu} + \ldots$

Fluid regime for AdS black hole?

When does the boundary stress tensor describe a fluid?

need scale of variation ≫ thermal scale $= 1/T$

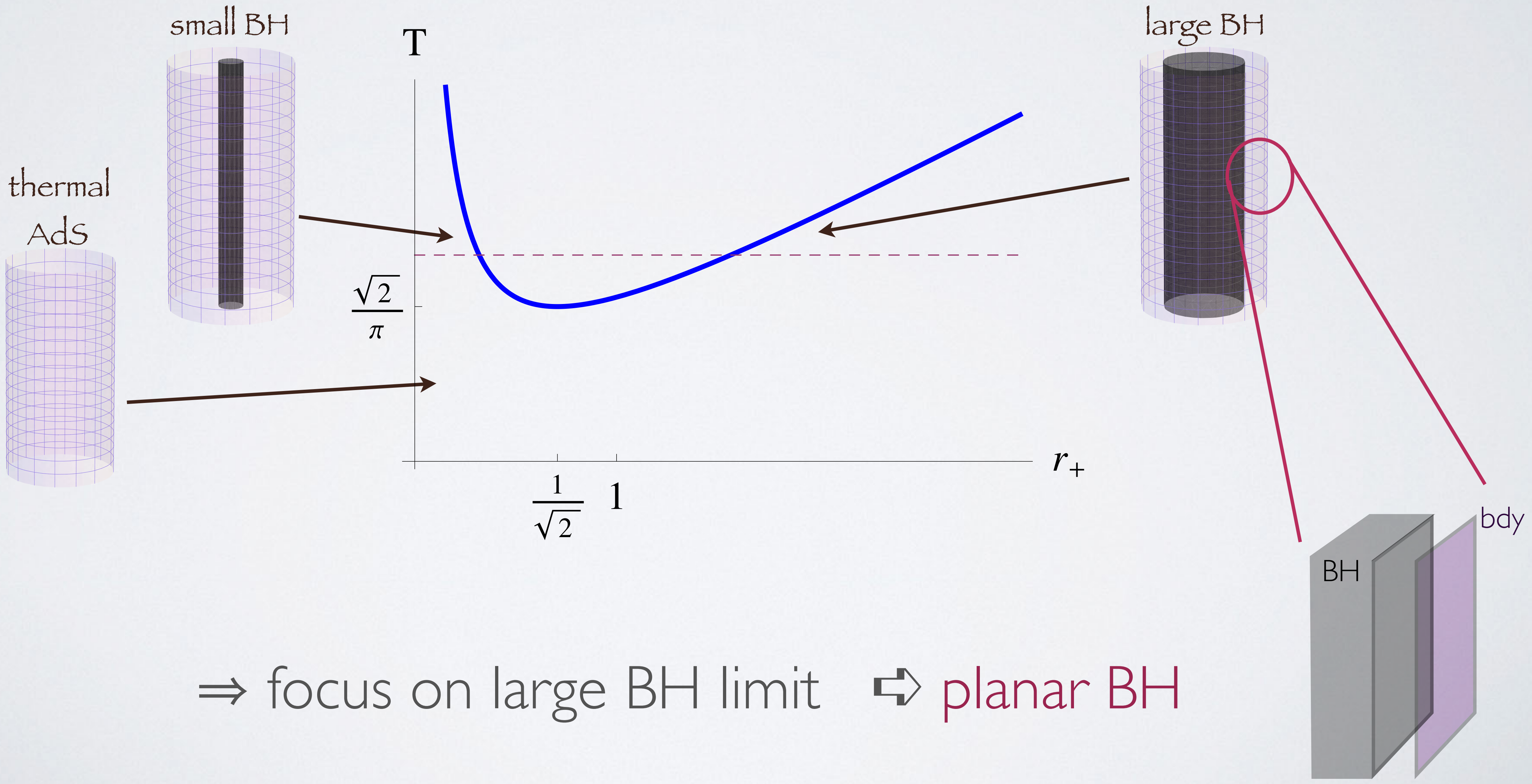


(for 5-dimensional bulk $\Rightarrow$ 4-dimensional boundary)

Fluid regime for AdS black hole?

When does the boundary stress tensor describe a fluid?

need scale of variation ≫ thermal scale



⇒ focus on large BH limit ➪ planar BH

Thermal equilibrium

Planar black hole geometry given by metric:

$$ds^2 = -2\, u_\mu\, dx^\mu dr + r^2 \left(\eta_{\mu\nu} + \frac{\pi^4 T^4}{r^4}\, u_\mu\, u_\nu \right) dx^\mu dx^\nu$$

= 4-parameter family of solutions parametrized by BH temperature T and horizon velocity u^μ

Induced boundary stress tensor:

$$T^{\mu\nu} = \pi^4\, T^4\, \left(\eta^{\mu\nu} + 4\, u^\mu\, u^\nu\right)$$

= Perfect fluid at temperature T moving with velocity u^μ
(such that $u^\mu\, u_\mu = -1$).

Note: this describes a conformal fluid ($T^\mu_{\ \mu} = 0$) with no dissipation.

Black holes with variation

Mathematically, instead of specifying 4 parameters in

$$ds^2 = -2\, u_\mu\, dx^\mu dr + r^2 \left(\eta_{\mu\nu} + \frac{\pi^4 T^4}{r^4}\, u_\mu\, u_\nu \right) dx^\mu dx^\nu$$

suppose we specified the metric by 4 functions of x^μ :

$$ds^2 = -2\, u_\mu(x)\, dx^\mu dr + r^2 \left(\eta_{\mu\nu} + \frac{\pi^4 T(x)^4}{r^4}\, u_\mu(x)\, u_\nu(x) \right) dx^\mu dx^\nu$$

This metric is still regular (for regular $u_\mu(x)$ and $T(x)$),
but does not solve Einstein's equations unless $u_\mu(x)$ and $T(x)$ are constant.

Iterative construction

However, we can use this as a starting point for an iterative construction:

Assuming slow variations

$$\frac{\partial_\mu \log T}{T} \sim \mathcal{O}\left(\epsilon\right) \ , \qquad \frac{\partial_\mu u}{T} \sim \mathcal{O}\left(\epsilon\right)$$

we can expand

$$g_{ab} = \sum_{k=0}^{\infty} \epsilon^k \, g_{ab}^{(k)} \ , \qquad T = \sum_{k=0}^{\infty} \epsilon^k \, T^{(k)} \ , \qquad u_\mu = \sum_{k=0}^{\infty} \epsilon^k \, u_\mu^{(k)}$$

and solve Einstein's equations order by order in ϵ.

Iterative construction

Einstein's equations

$$E_{ab} \equiv R_{ab} - \frac{1}{2} R\, g_{ab} + \Lambda\, g_{ab} = 0$$

separate into two sets:

- **Constraint equations:** $E_{r\mu} = 0 \qquad \Longleftrightarrow \qquad \nabla_\mu T^{\mu\nu} = 0$

  these implement stress tensor conservation

  (at one lower order)

- **Dynamical equations:** $E_{\mu\nu} = 0$ and $E_{rr} = 0$

  these allow us to solve for the metric correction

  (at the given order)

Fluid/Gravity correspondence

[Bhattacharyya, Hubeny, Minwalla, Rangamani, 2008]

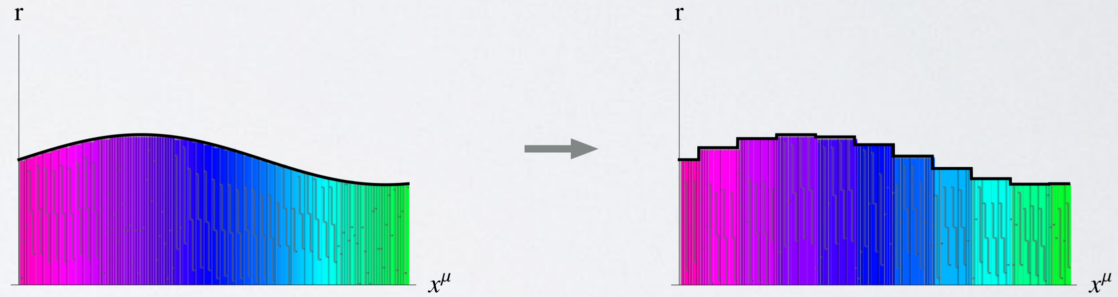
- Dynamics of bulk black hole ⇔ fluid dynamics on boundary

5-d Einstein's equations ⊃ 4-d Navier-Stokes equations

w/ negative cosmological const. (describing relativistic, conformal fluid)

- For any given fluid flow, we iteratively construct a solution of a dynamical black hole in AdS whose horizon mimics the fluid.

- Technically: expand in boundary derivatives and solve order by order
- The radial equation is fully nonlinear, and gives patched 'tubes' of different black holes



- We calculate 2nd order transport coefficients for the conformal fluid.

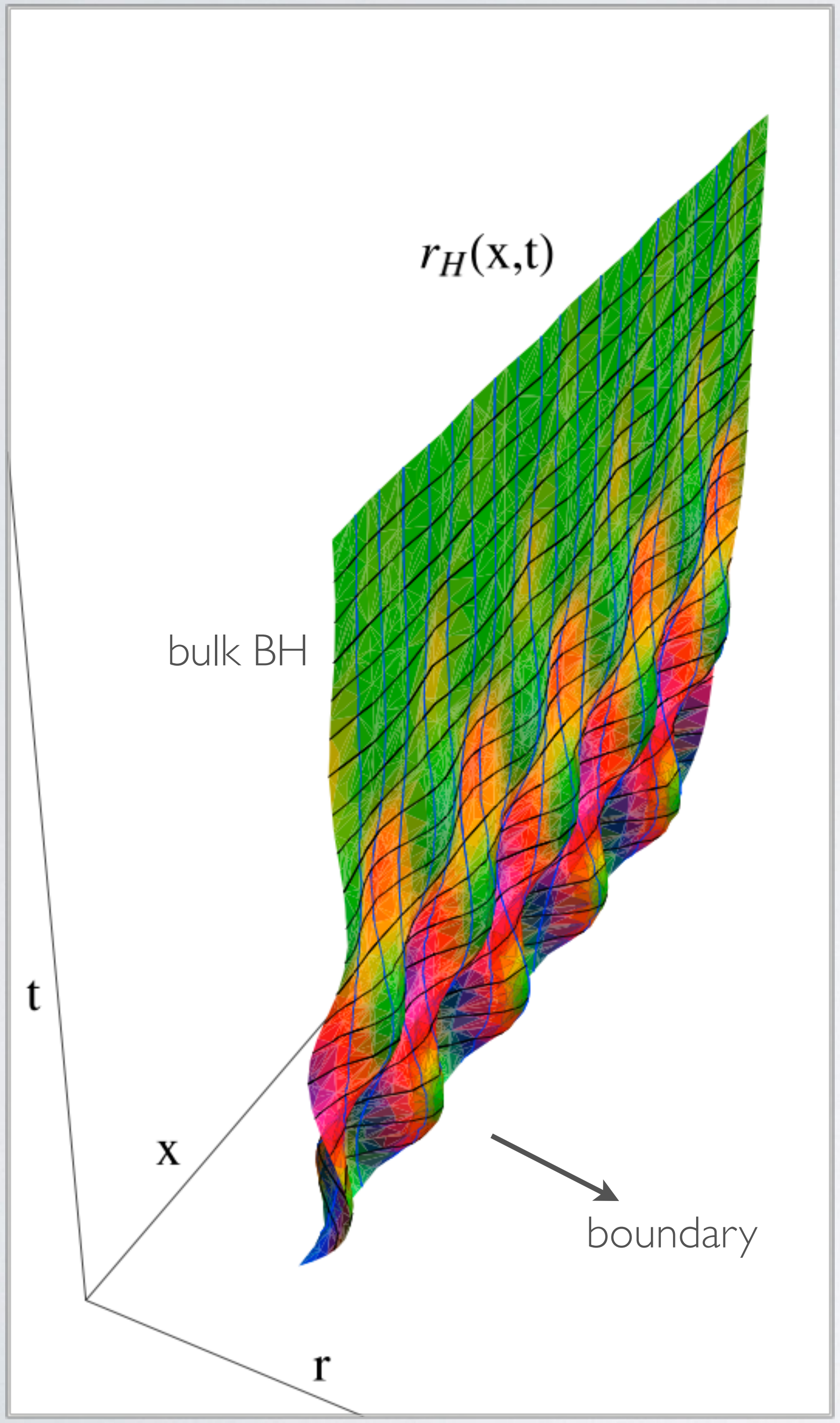
- The pull-back of the horizon area form gives a natural entropy current on the boundary, with automatically non-negative divergence.

[Bhattacharyya, Hubeny, Loganayagam, Mandal, Minwalla, Morita, Rangamani, Reall]

Key points of the construction

- The iterative construction can in principle be systematically implemented to arbitrary order in ϵ (which obtains correspondingly accurate solution).
- The resulting black hole spacetimes form a continuously-infinite set of (approximate) solutions: instead of specifying 4 parameters, we specify 4 functions of 4 variables.
- However, these solutions are given implicitly in terms of functions $T(x^\mu)$ and $u^\nu(x^\mu)$ which must solve the generalized Navier-Stokes equations.
- Any regular fluid dynamical solution corresponds to a bulk black hole with regular horizon.

Cartoon of the event horizon



- Initially non-uniform horizon evolves.
- The horizon area grows.
- No symmetries; but at late times the configuration equilibrates due to dissipation.

- The horizon area provides entropy current for the boundary fluid.
- Such entropy current automatically satisfies the 2nd law of thermodynamics.

[Bhattacharyya, VH, Loganayagam, Mandal, Minwalla, Morita, Rangamani, Reall]

Boundary stress tensor

The 1st order solution induces on the AdS boundary the stress tensor of the expected form:

$$T^{\mu\nu} = \pi^4\, T^4\, (4\, u^\mu u^\nu + \eta^{\mu\nu}) - 2\, \pi^3\, T^3\, \sigma^{\mu\nu}$$

From this we can read off the shear viscosity of the fluid; in terms of shear viscosity to entropy density ratio (calculated previously [Policastro, Son, Starinets]),

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

This saturates the famous lower bound conjectured by [Kovtun, Son, Starinets].

At 2nd order we can read off 2nd order transport coefficients.

Fluid/Gravity correspondence

[Bhattacharyya, Hubeny, Minwalla, Rangamani, 2008]

- Dynamics of bulk black hole ⇔ fluid dynamics on boundary

5-d Einstein's equations w/ negative cosmological const. ⊃ 4-d Navier-Stokes equations (describing relativistic, conformal fluid)

- For any given fluid flow, we iteratively construct a solution of a dynamical black hole in AdS whose horizon mimics the fluid.

- Generalizations:
  - charged fluids,
  - superfluids,
  - non-conformal fluids,
  - non-relativistic fluids,
  - fluids with boundary,
  - forced fluids,
  - other dimensions, ...
  - But not yet everyday liquids, e.g. non-Newtonian fluids.

- Applications:
  - black hole physics,
  - fluid dynamics,
  - nuclear physics,
  - condensed matter physics,
  - solid state physics, ...

OUTLINE

- Motivation: path to QG
- Black holes & holographic principle
- Preview of gauge/gravity correspondence
- Historical detour: string theory
- Anti de Sitter geometry

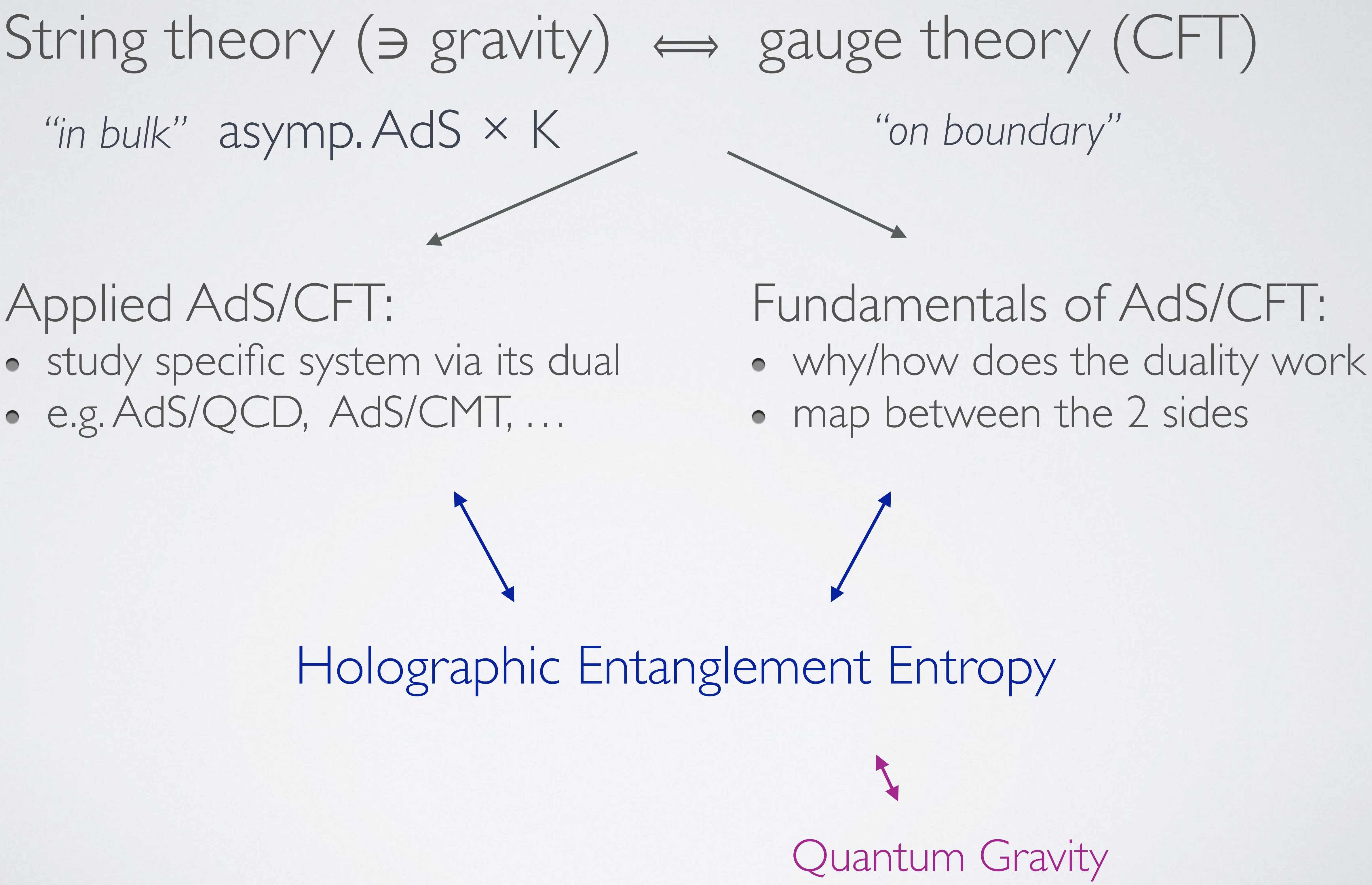
Lecture 1

- Elements of gauge/gravity dictionary
    - Scale/radius duality (“UV/IR correspondence”)
    - States / geometries
    - Fluid/gravity correspondence

Lecture 2

- CFT probes of bulk geometry… — skipped…
- Entanglement and geometry… → Lecture 3

Paths to Holographic EE



String theory ($\ni$ gravity) $\Longleftrightarrow$ gauge theory (CFT)

“in bulk” asymp. AdS × K

“on boundary”

Applied AdS/CFT:

- study specific system via its dual
- e.g. AdS/QCD, AdS/CMT, …

Fundamentals of AdS/CFT:

- why/how does the duality work
- map between the 2 sides

Holographic Entanglement Entropy

Quantum Gravity

Entanglement

- Most non-classical manifestation of quantum mechanics
  - "Best possible knowledge of a whole does not include best possible knowledge of its parts — and this is what keeps coming back to haunt us" [Schrodinger '35]
- New quantum resource for tasks which cannot be performed using classical resources [Bennet '98]
- Plays a central role in wide-ranging fields
  - quantum information (e.g. cryptography, teleportation, …)
  - quantum many body systems
  - quantum field theory
- Hints at profound connections to geometry…

Entanglement Entropy (EE)

Suppose we only have access to a subsystem A of the full system = A + B. The amount of entanglement is characterized by Entanglement Entropy S_A :

- reduced density matrix $\rho_A = \mathrm{Tr}_B \, |\psi\rangle\langle\psi|$

  (more generally, for a mixed total state, $\rho_A = \mathrm{Tr}_B \rho$)

- EE = von Neumann entropy $S_A = -\mathrm{Tr}\, \rho_A \, \log \rho_A$

Defined if we can divide a quantum system into a subsystem A and its complement B, such that the Hilbert space decomposes:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Entanglement Entropy (EE)

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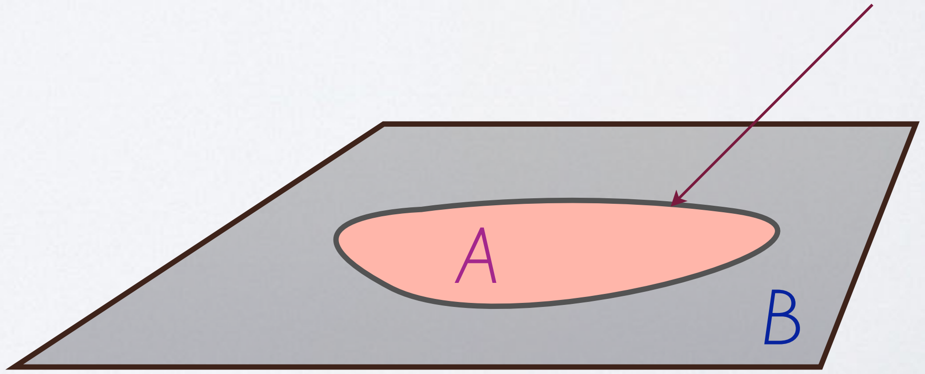
- EE = von Neumann entropy $S_A = -\mathrm{Tr}\, \rho_A \, \log \rho_A$

- e.g. in local QFT:

  A and B can be spatial regions, separated by a smooth entangling surface



The good news & the bad news

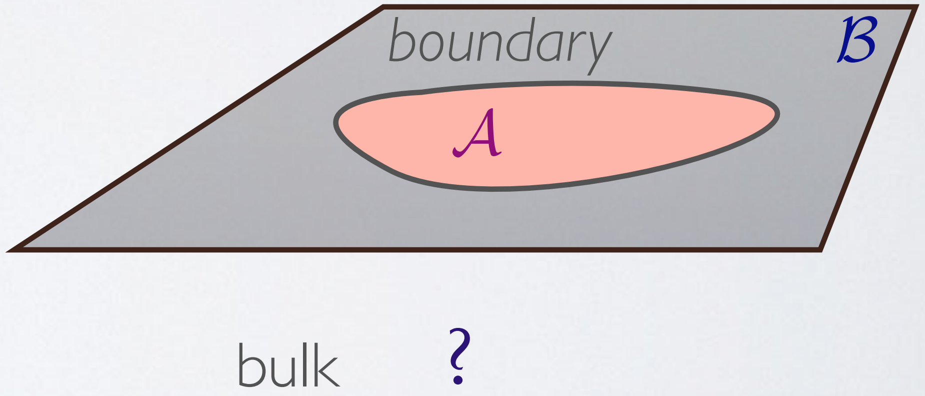
- But EE is hard to deal with…
  - non-local quantity, intricate & sensitive to environment
  - difficult to measure
  - difficult to calculate

… especially in strongly-coupled quantum systems

- AdS/CFT to the rescue?

~ Is there a natural bulk dual of EE?

(= "Holographic EE")

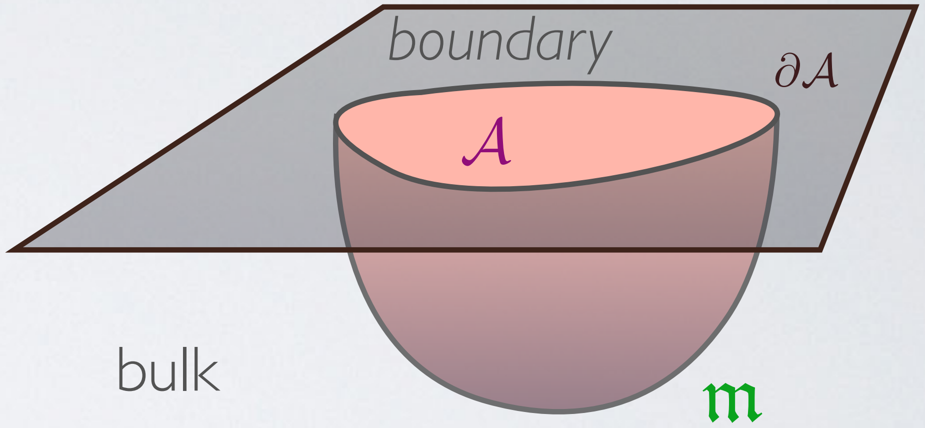


Yes! - described geometrically…

Holographic EE in static situations

Proposal [RT=Ryu & Takayanagi, '06] for *static* configurations:

In the bulk, entanglement entropy $S_{\mathcal{A}}$ for a boundary region $\mathcal{A}$ is captured by the area of a minimal co-dimension-2 bulk surface $\mathfrak{m}$ *at constant t* anchored on entangling surface $\partial\mathcal{A}$ & homologous to $\mathcal{A}$



$$S_{\mathcal{A}} = \min_{\partial\mathfrak{m}=\partial\mathcal{A}} \frac{\text{Area}(\mathfrak{m})}{4\,G_N}$$

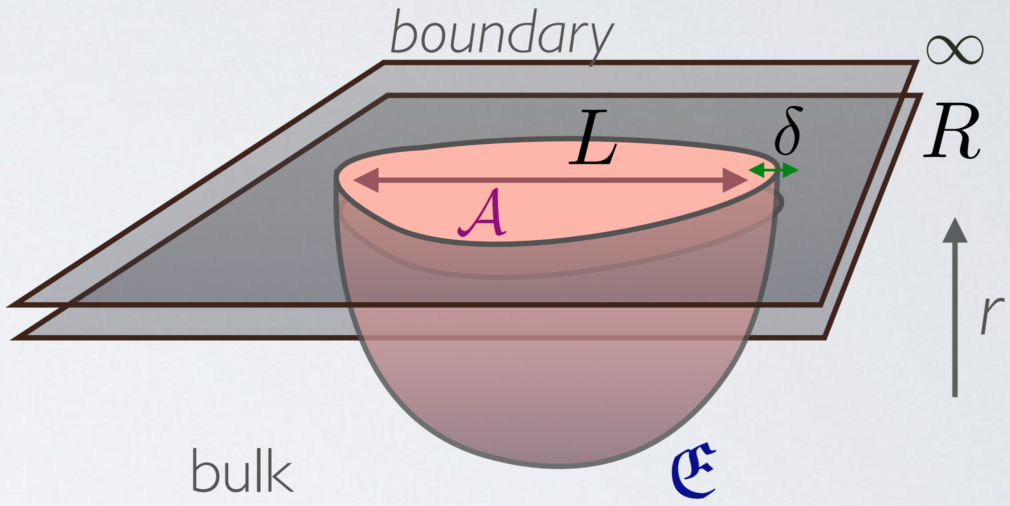
Remarks:

- Large body of evidence, culminating in [Lewkowycz & Maldacena '13]
- cf. black hole entropy…
- Minimal surface "hangs" into the bulk due to large distances near bdy.
- Note that both LHS and RHS are in fact infinite

Area-law divergence of HEE

Short-distance cutoff δ in the CFT translates to large-radius cutoff R in AdS

with $\delta = \dfrac{\ell^2}{R}$ (cf. UV/IR duality)



Bulk area reproduces the correct divergence structure:

$$S_{\mathcal{A}} = c_0 \left(\frac{L}{\delta}\right)^{d-2} + c_1 \left(\frac{L}{\delta}\right)^{d-4} + \cdots$$

cutoff-dependent coefficients

$$+ \begin{cases} c_{d-2} \log\left(\frac{L}{\delta}\right) + \cdots \quad , & d \;\; \text{even} \\ c_{d-2} + \cdots \quad , & d \;\; \text{odd} \end{cases}$$

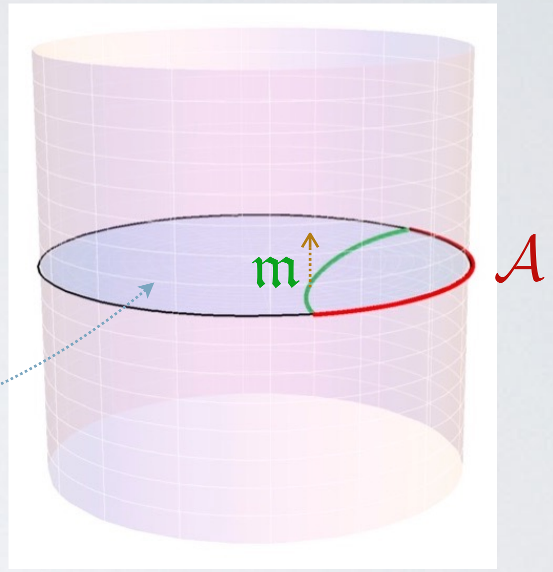
universal coefficients

We can regulate EE by e.g. background subtraction.

Covariant Holographic EE

But the RT prescription is not well-defined outside the context of static configurations:

- In Lorentzian geometry, we can decrease the area arbitrarily by timelike deformations
- In time-dependent context, no natural notion of "const. t" slice…

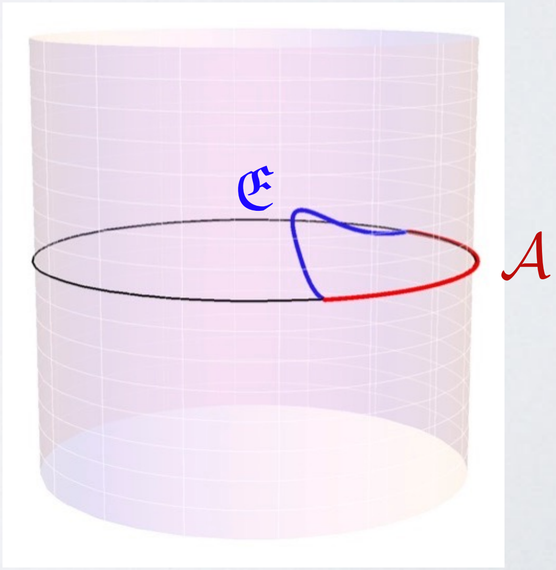


In *time-dependent* situations, RT prescription must be covariantized:

Simplest candidate: [HRT = VH, Rangamani, Takayanagi '07]

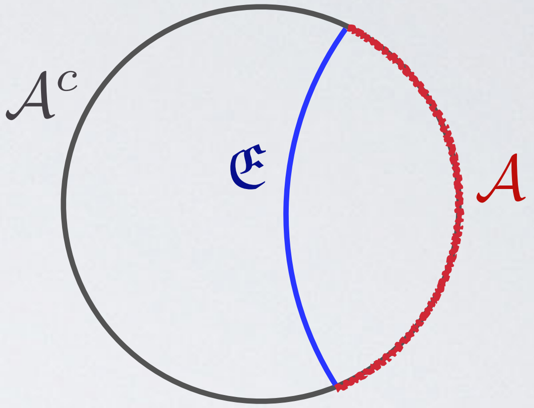
minimal surface $\mathfrak{m}$ at constant time → extremal surface $\mathfrak{E}$ in the full bulk

This gives a well-defined quantity in any (arbitrarily time-dependent asymptotically AdS) spacetime
⇒ equally robust as in CFT

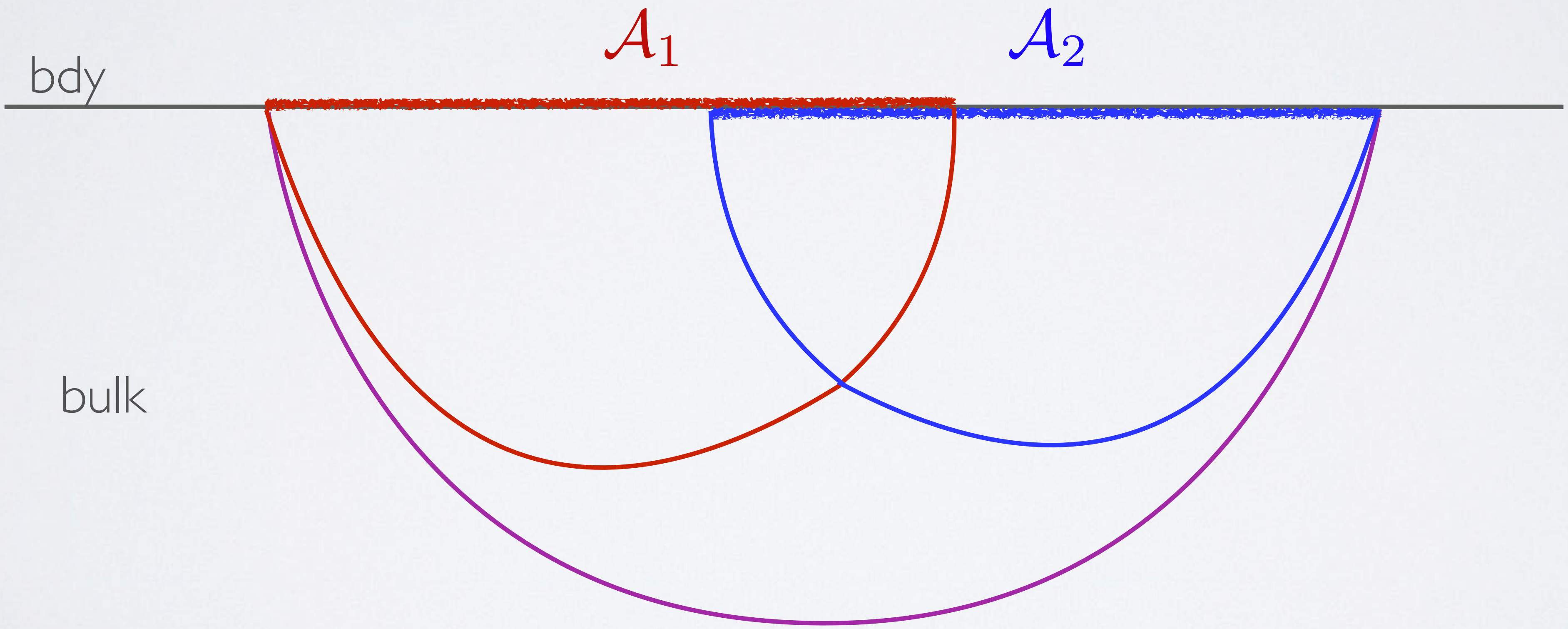


"Pf" in [Dong, Lewkowycz, Rangamani '16]

Manifest properties of EE



- For pure states $S_{\mathcal{A}} = S_{\mathcal{A}^c}$
- Positivity: $S_{\mathcal{A}} \geq 0$
- Subadditivity: $S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2}$



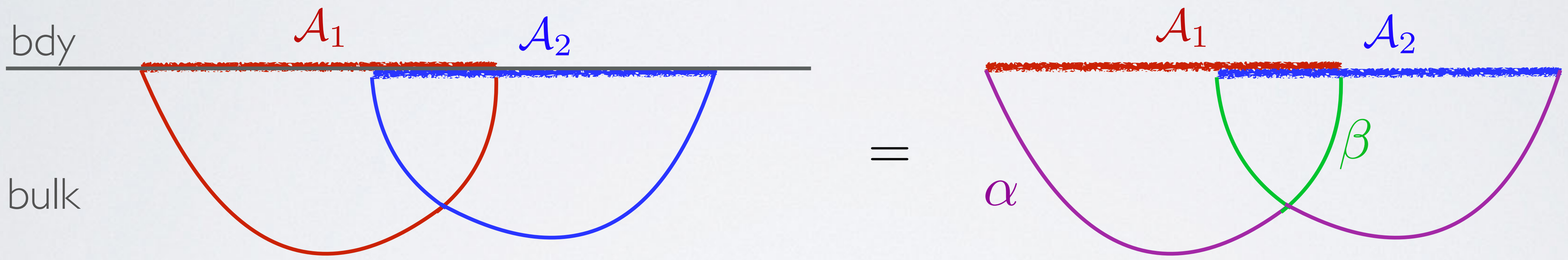
- Implies positivity of mutual information: $I(\mathcal{A}_1, \mathcal{A}_2) = S_{\mathcal{A}_1} + S_{\mathcal{A}_2} - S_{\mathcal{A}_1 \cup \mathcal{A}_2}$

Proof of Strong Subadditivity

- strong subadditivity:

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$

- proof in static configurations [Headrick & Takayanagi]



$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} = \alpha + \beta$$

Proof of Strong Subadditivity

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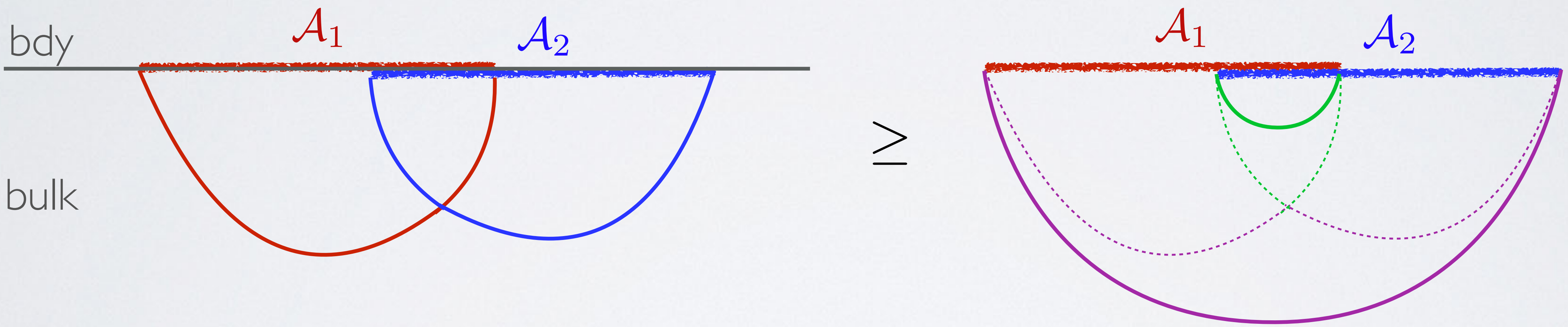


$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} = \alpha + \beta \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$

- general proof in time-dependent case uses properties of null geodesics + energy conditions [Wall]

Examples of entanglement relations

- Universal:
  - Sub-additivity (SA) $S(A)+S(B)\geq S(AB)$
  - Araki-Lieb (AL) $S(A)+S(AB)\geq S(B)$
  - Strong sub-additivity (SSA) $S(AB)+S(BC)\geq S(B)+S(ABC)$
  - Weak monotonicity (WM) $S(AB)+S(BC)\geq S(A)+S(C)$
- True in holography:
  - Monogamy of mutual information (MMI)

$$S(AB)+S(BC)+S(CA)\geq S(A)+S(B)+S(C)+S(ABC)$$

  - 5-region cyclic inequality (C5)

$$\begin{aligned}&S(ABC)+S(BCD)+S(CDE)+S(DEA)+S(EAB)\\&\qquad\geq S(AB)+S(BC)+S(CD)+S(DE)+S(EA)+S(ABCDE)\end{aligned}$$

  - k-region cyclic inequality (Ck) for k=odd is obvious...

QI interpretation

- Universal:
  - Sub-additivity (SA) $S(A) + S(B) \geq S(AB)$

    ⇨ Mutual Information $I(A:B) \equiv S(A) + S(B) - S(AB) \geq 0$
  - Strong sub-additivity (SSA) $S(AB) + S(BC) \geq S(B) + S(ABC)$

    ⇨ Conditional mutual information $I(A:C|B) \equiv I(A:BC) - I(A:B) \geq 0$
- True in holography:
  - Monogamy of mutual information (MMI)

$$S(AB) + S(BC) + S(CA) \geq S(A) + S(B) + S(C) + S(ABC)$$

⇨ Tripartite information $I3(A:B:C) \equiv I(A:B) + I(A:C) - I(A:BC) \leq 0$

→ gives interesting structure information on nature of entanglement in holography cf. [Hayden, Headrick, Maloney]

Covariant Holographic EE

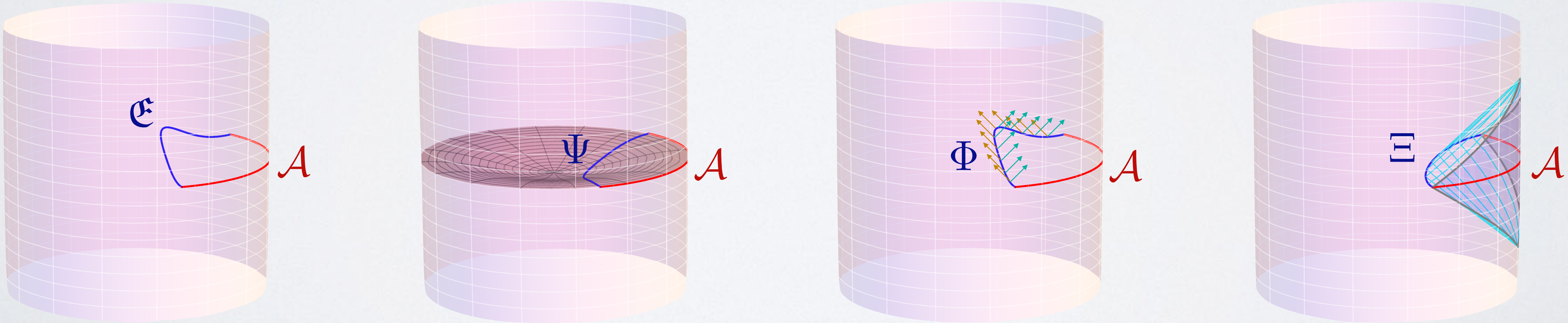
In fact, [Hubeny, Rangamani, Takayanagi '07] identified 4 natural candidates:

(all co-dim.2 surfaces ending on $\partial\mathcal{A}$, and coincident for ball regions $\mathcal{A}$ in pure AdS)

- $\mathfrak{E}$ = Extremal surface
- Ψ = Minimal-area surface on maximal-volume slice
- Φ = Surface with zero null expansions
- Ξ = Causal wedge rim

$\mathfrak{E} = \Phi$ is correct
= 'HRT prescription'

Later known as Causal Information Surface;
w/ area = causal holographic information χ [Hubeny, Rangamani '12]



Causal Wedge construction

Bulk causal region naturally corresponding to $D[\mathcal{A}]$:

- Bulk causal wedge $\blacklozenge_{\mathcal{A}}$

$$\blacklozenge_{\mathcal{A}} \equiv J^-[D[\mathcal{A}]] \cap J^+[D[\mathcal{A}]]$$

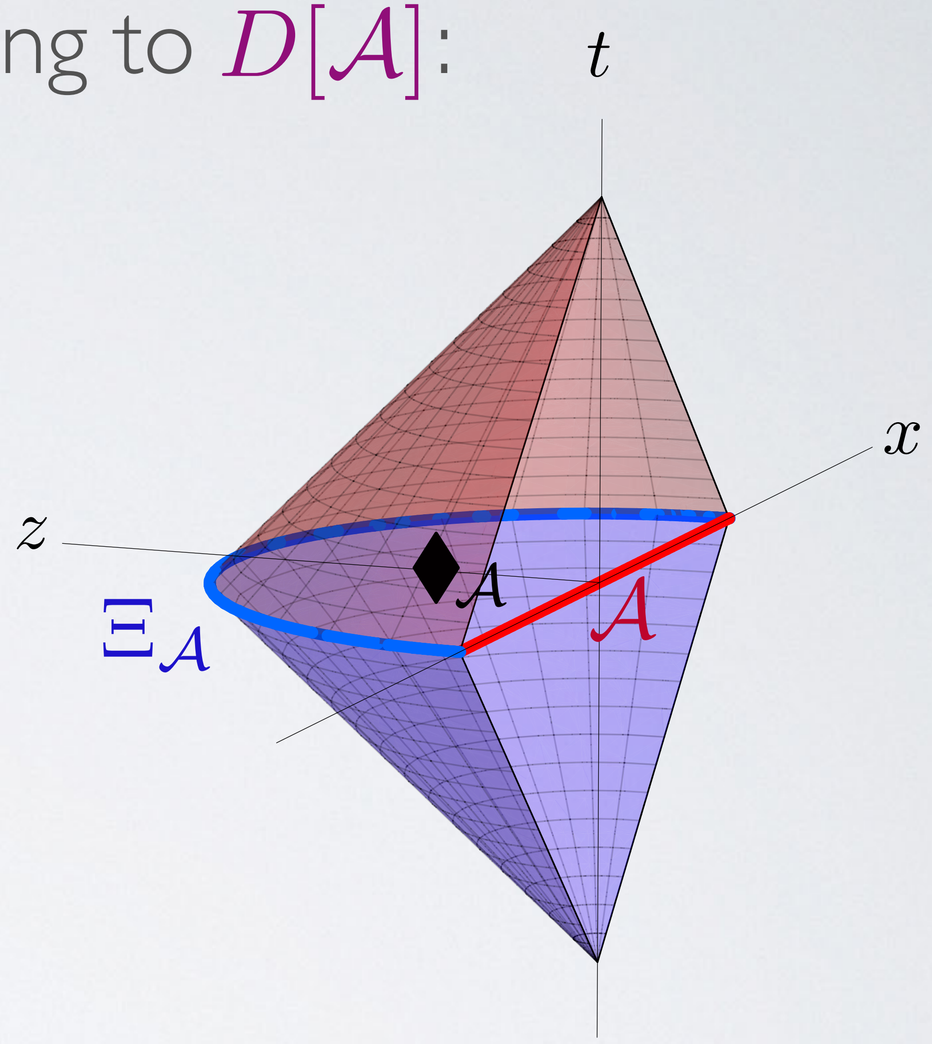
= { bulk causal curves which begin and end on $D[\mathcal{A}]$ }

- Causal information surface $\Xi_{\mathcal{A}}$

$$\Xi_{\mathcal{A}} \equiv \partial J^-[D[\mathcal{A}]] \cap \partial J^+[D[\mathcal{A}]]$$

- Causal holographic information $\chi_{\mathcal{A}}$

$$\chi_{\mathcal{A}} \equiv \frac{\text{Area}(\Xi_{\mathcal{A}})}{4\,G_N}$$



[Hubeny, Rangamani '12]

- In special cases, $\Xi_{\mathcal{A}} = \mathfrak{E}_{\mathcal{A}} \quad \Rightarrow \quad \chi = S_{\mathcal{A}}$, but in general they differ.
- Important Q: what is their interpretation within the dual CFT ?

Power of covariant constructs

- 'Natural' geometrical constructs (defined for general bulk spacetimes, independent of coordinates) provide useful candidates for dual of 'natural' quantities in CFT
- e.g. dual of $\rho_{\mathcal{A}}$? [Bousso, Leichenauer, Rosenhaus; Czech, Karczmarek, Nogueira, Van Raamsdonk;…]
- In generic Lorentzian spacetime, null congruences which define a causal set provide useful characterization of 'natural' bulk regions.

2 options:

…starting from bdy:

$D[\mathcal{A}] \rightsquigarrow$ Causal Wedge:

= future and past causally-separated from bdy region determined by $\rho_{\mathcal{A}}$

[VH & Rangamani]

…starting from bulk:

$\mathfrak{E} \rightsquigarrow$ Entanglement Wedge:

= spacelike-separated (toward $\mathcal{A}$) from $\mathfrak{E}$

[Headrick, VH, Lawrence, Rangamani]

Argued to be the correct one:

[Dong, Harlow, Wall '16] via QEC & operator algebra

[Cotler, Hayden, Salton, Swingle, Walter '17] via recovery channels

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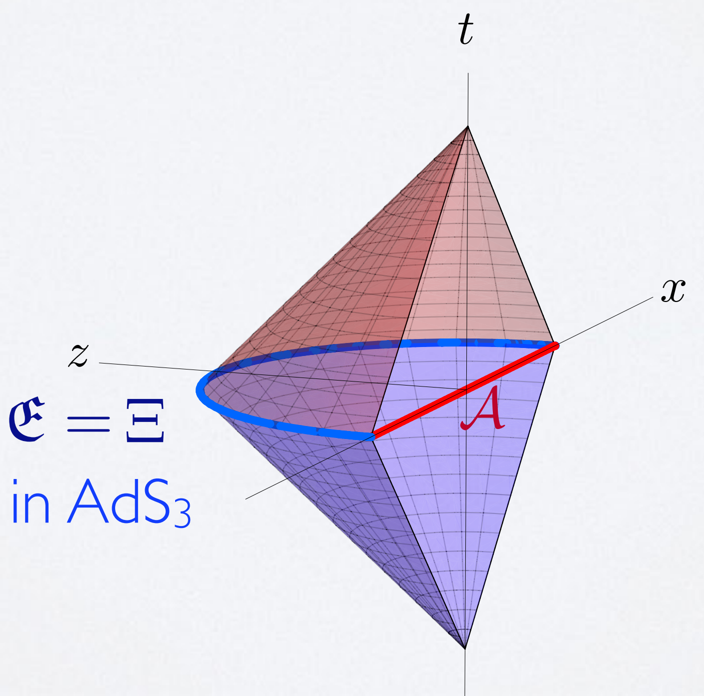
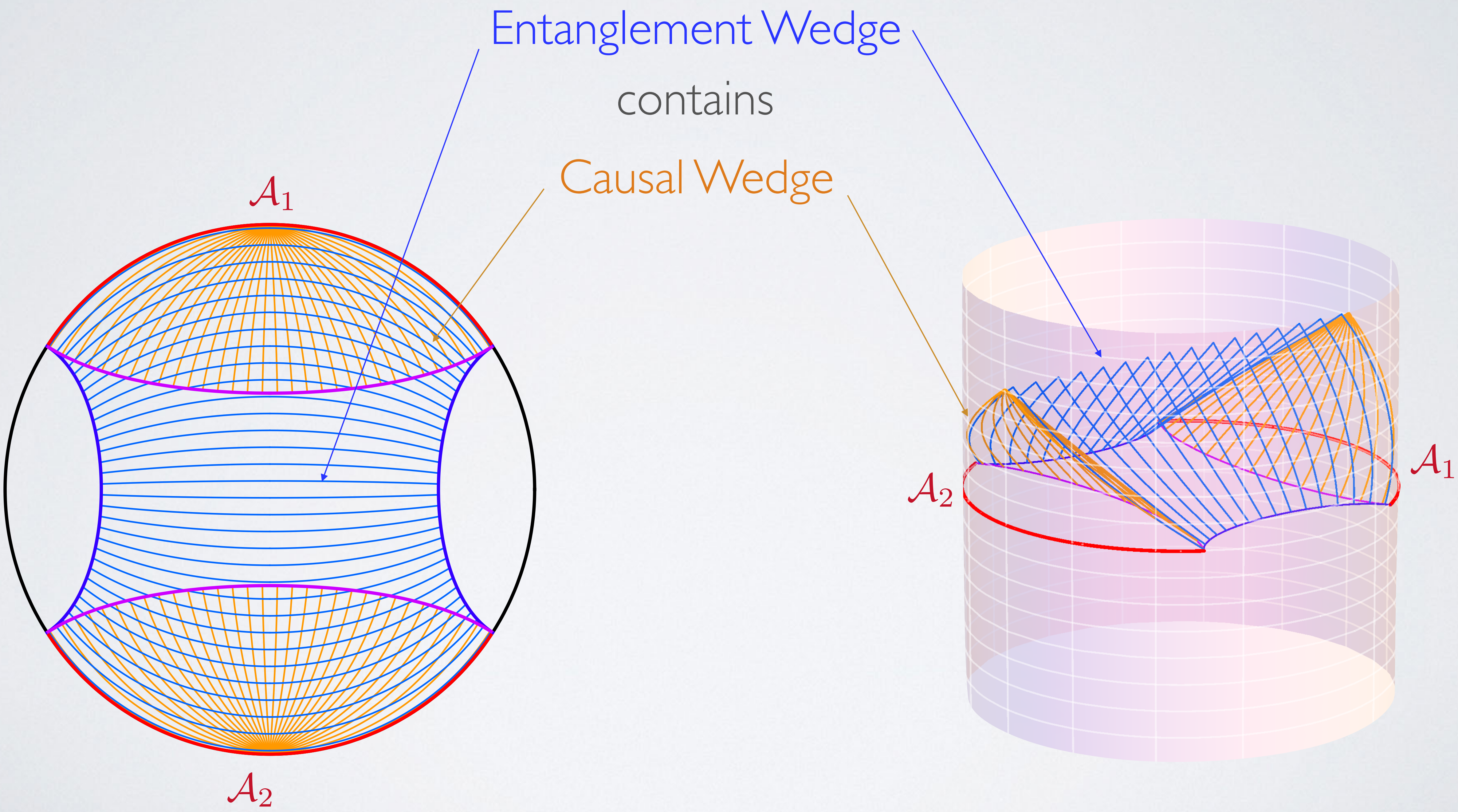


NB: in pure AdS,
& for spherical $\mathcal{A}$,
these coincide,
but not in general.

Causal wedge vs. Entanglement wedge

- Even in pure AdS₃, these can differ for composite regions $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$



Entanglement Wedge

contains

Causal Wedge

Power of covariant constructs

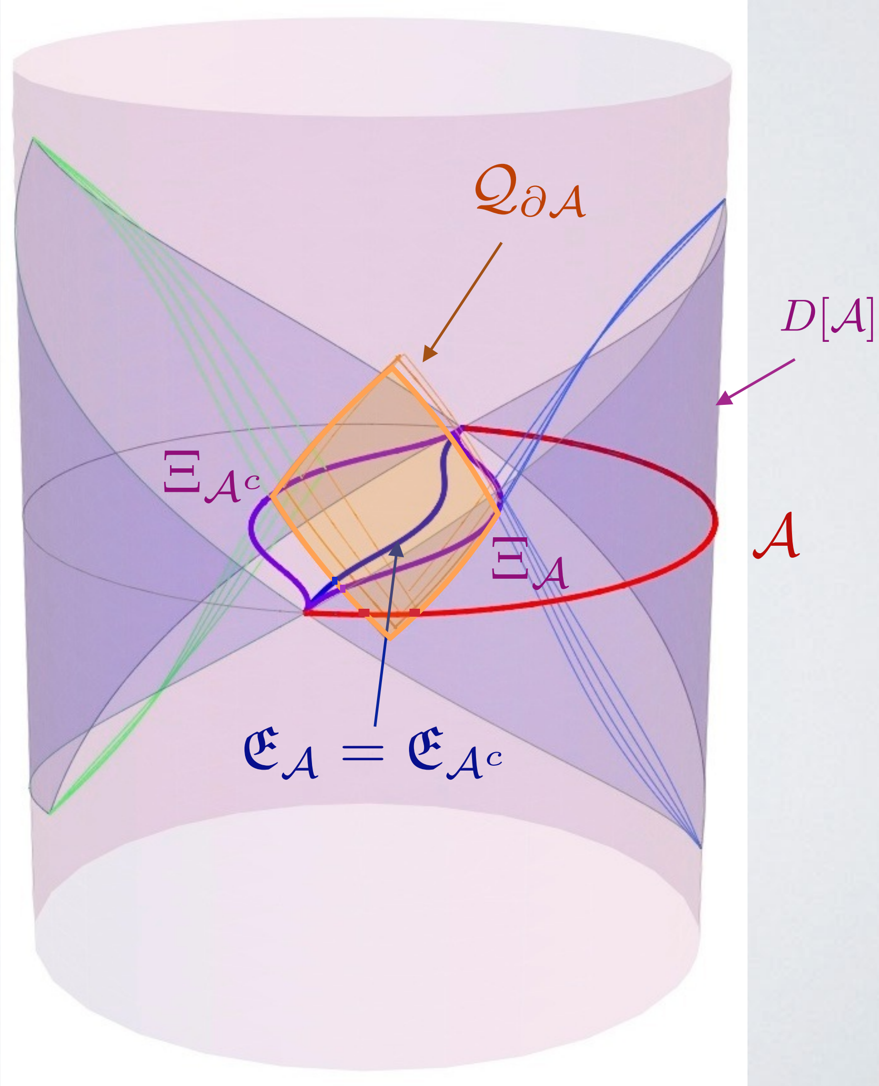
$D[\mathcal{A}]$ ⇝ Causal Wedge: $\mathfrak{E}$ ⇝ Entanglement Wedge:

…continued past Ξ: ⇝ Causal Shadow $\mathcal{Q}_{\partial\mathcal{A}}$

- We can prove the inclusion property [Headrick, VH, Lawrence, Rangamani; Wall]

CW ⊂ EW

or equivalently, $\mathfrak{E} \subset \mathcal{Q}_{\partial\mathcal{A}}$



- Consequences:
  - HRT is consistent with CFT causality (= non-trivial check of HRT)
  - Entanglement plateaux
  - Entanglement wedge can reach deep inside a black hole!

Curious features of EE:

- Extremal surfaces can have intricate behavior:
  - $\mathfrak{E}$ can have discontinuous jumps under smooth variations of $\mathcal{A}$

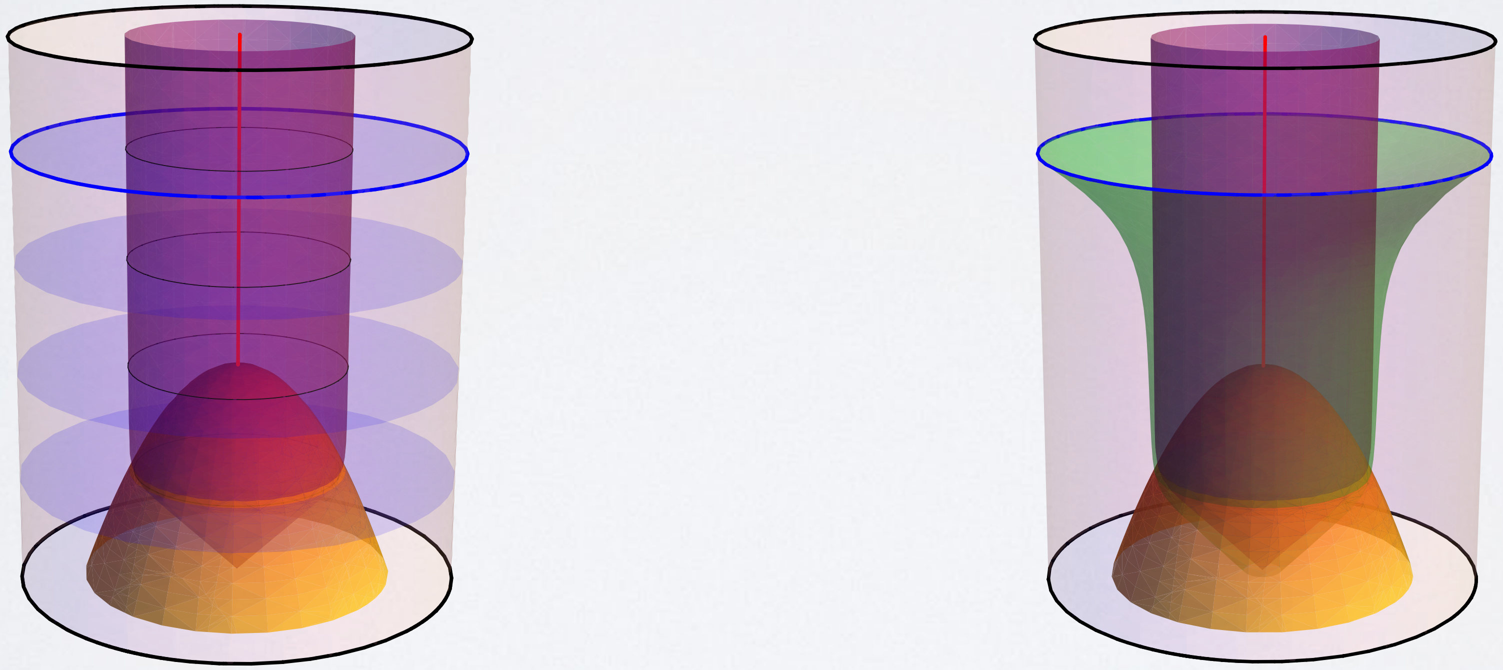
    → phase transitions in EE
  - $\mathfrak{E}$ can be topologically nontrivial even for simply-connected regions $\mathcal{A}$
- Holographic EE seems too local:
  - sharply-specified both on boundary and in bulk
  - but: → we can reconstruct the bulk metric (modulo caveats) solely from the set $\{S_{\mathcal{A}}\}$ for a suitable set of $\{\mathcal{A}\}$
- Holographic EE seems too non-local:
  - global minimization condition + homology constraint makes $S_{\mathcal{A}}$ sensitive to arbitrarily distant regions in the bulk…

EE is fine-grained observable!

Example: black hole formed from a collapse

- In contrast to the static (i.e. eternal) black hole, for a collapsed black hole, there is no non-trivial homology constraint on extremal surfaces.

[cf. Takayanagi & Ugajin]



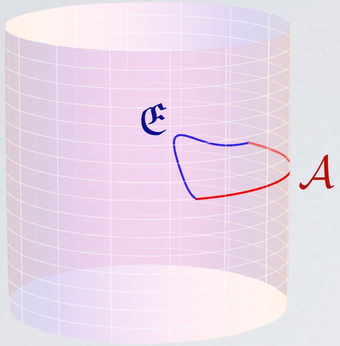
- Hence we always have $S_{\mathcal{A}} = S_{\mathcal{A}^c}$ as for a pure state.

Covariant re-formulations

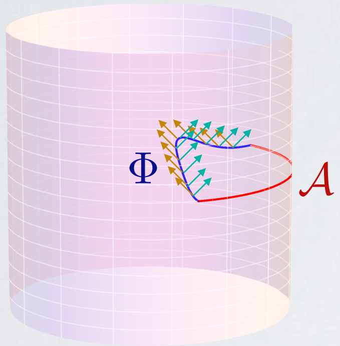
- Covariance is pre-requisite to construct being physically meaningful, but it need not be unique
  - Distinct geometrical formulations can turn out equivalent (cf. $\mathfrak{E} = \Phi$)

- This redundancy is useful
  - Each formulation can have its own advantages
  - e.g. different properties may be manifest in different formulations (cf. gauge / coordinate choice)
  - Re-formulation can reveal deeper relations (cf. ER=EPR [Maldacena, Susskind])

Covariant re-formulations of HEE



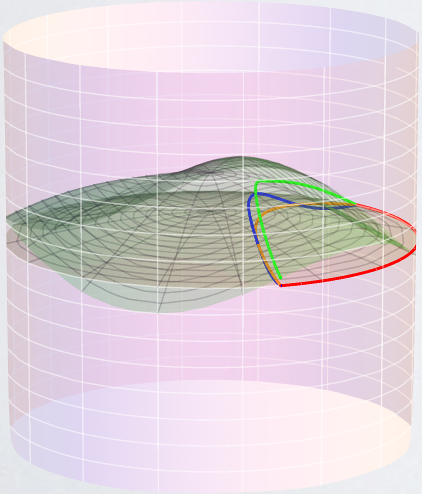
- $\mathfrak{E}$ = Extremal surface
    - (relatively) easy to find
    - minimal set of ingredients required in specification
    - need to include homology constraint as extra requirement



- Φ = Surface with zero null expansions
    - (cf. light sheet construction & covariant entropy bound [Bousso, '99]:

Bulk entropy through light sheet of surface $\sigma \leq \text{Area}(\sigma)/4$

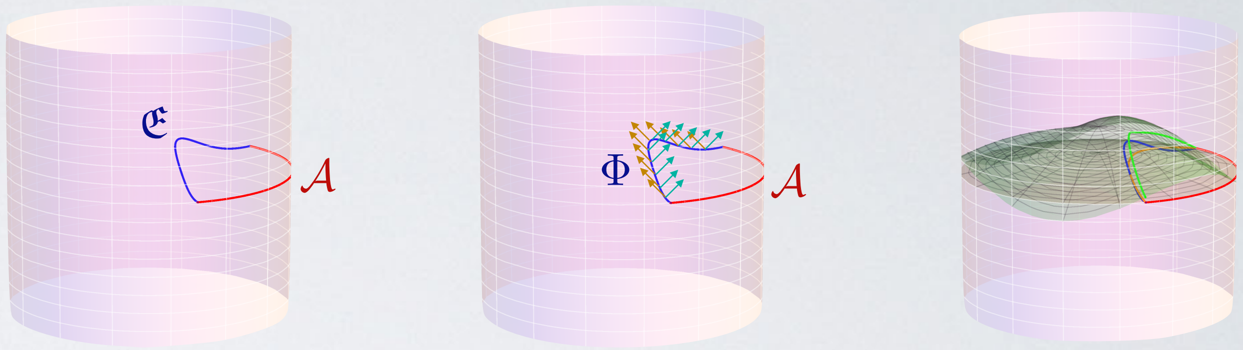
Φ = surface admitting a light sheet closest to bdy



- Maximin surface [Wall, '12]
    - maximize over minimal-area surface on a spacelike slice
    - requires the entire collection of slices & surfaces
    - implements homology constraint automatically
    - useful for proofs (e.g. SSA)

Covariant re-formulations of HEE

All of these are the same geometrical construct.



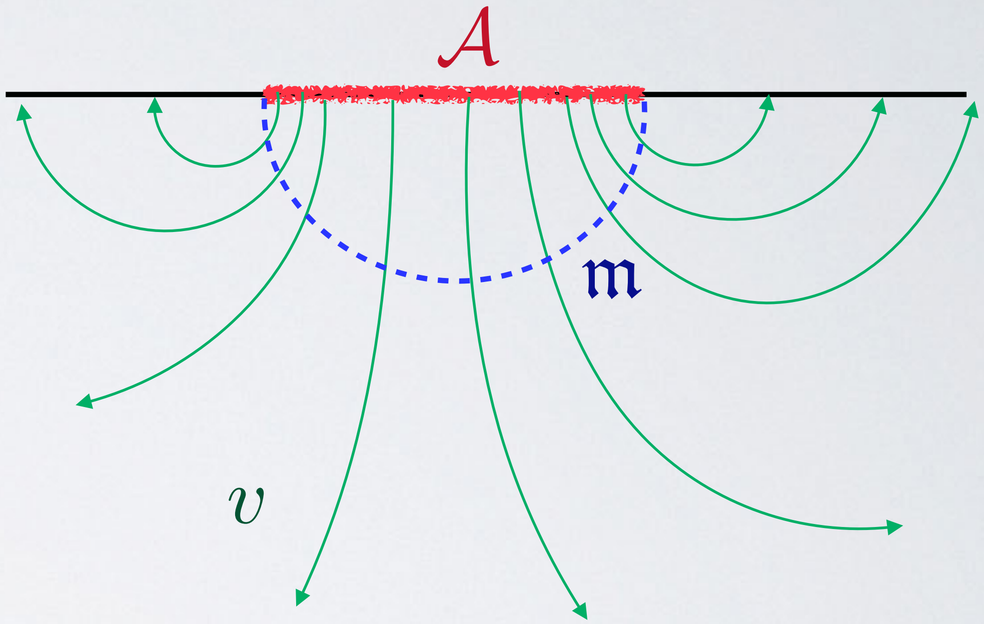
BUT it does not elucidate the relation to quantum information:

- Where does the information live?
- Mutual information I(A:B) = S(A) + S(B) - S(AB) is given by surfaces located in different spacetime regions.
- Geometric proof of SSA (S(AB) + S(BC) ≥ S(B) + S(ABC)) obscures its meaning as monotonicity under inclusion of correlations

Bit thread picture of (static) EE

- Reformulate EE in terms of flux of flow lines [Freedman & Headrick, '16]
  - let v be a vector field satisfying $\nabla \cdot v = 0$ and $|v| \leq 1$. Then EE is given by

$$S_{\mathcal{A}} = \max_{v} \int_{\mathcal{A}} v$$



  - By Max Flow - Min Cut theorem, equivalent to RT: (bottleneck for flow = minimal surface)

- Useful reformulation of holographic EE
  - flow continuous under varying region (while bottlenecks can jump discontinuously)
  - automatically implements homology constraint and global minimization of RT
  - maximal flow defined even without a regulator (when flux has UV divergence)
  - can be computed more efficiently (via linear programming methods)
  - implements QI meaning of EE and its inequalities more naturally
  - provides more intuition: think of each bit thread as connecting an EPR pair

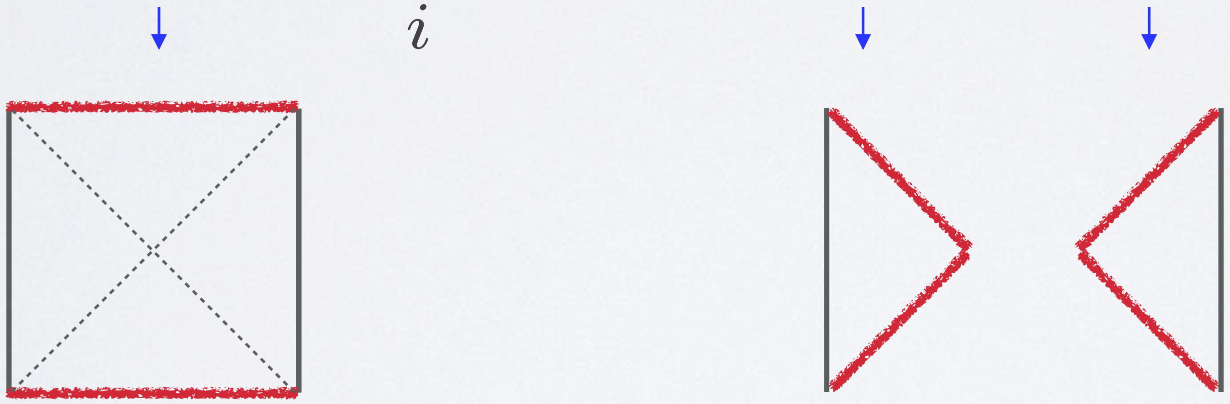
Spacetime from entanglement?

How does bulk spacetime emerge in the first place?

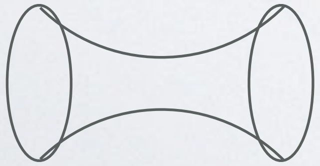
- Some connected spacetimes emerge as superpositions of disconnected spacetimes [Van Raamsdonk; Swingle]

eg. eternal AdS black hole as thermofield double:

$$|\psi\rangle = \sum_i e^{-\frac{\beta\ E_i}{2}}\ |E_i\rangle \otimes |E_i\rangle$$



- Entanglement builds bridges: ‘ER = EPR’ [Maldacena, Susskind]

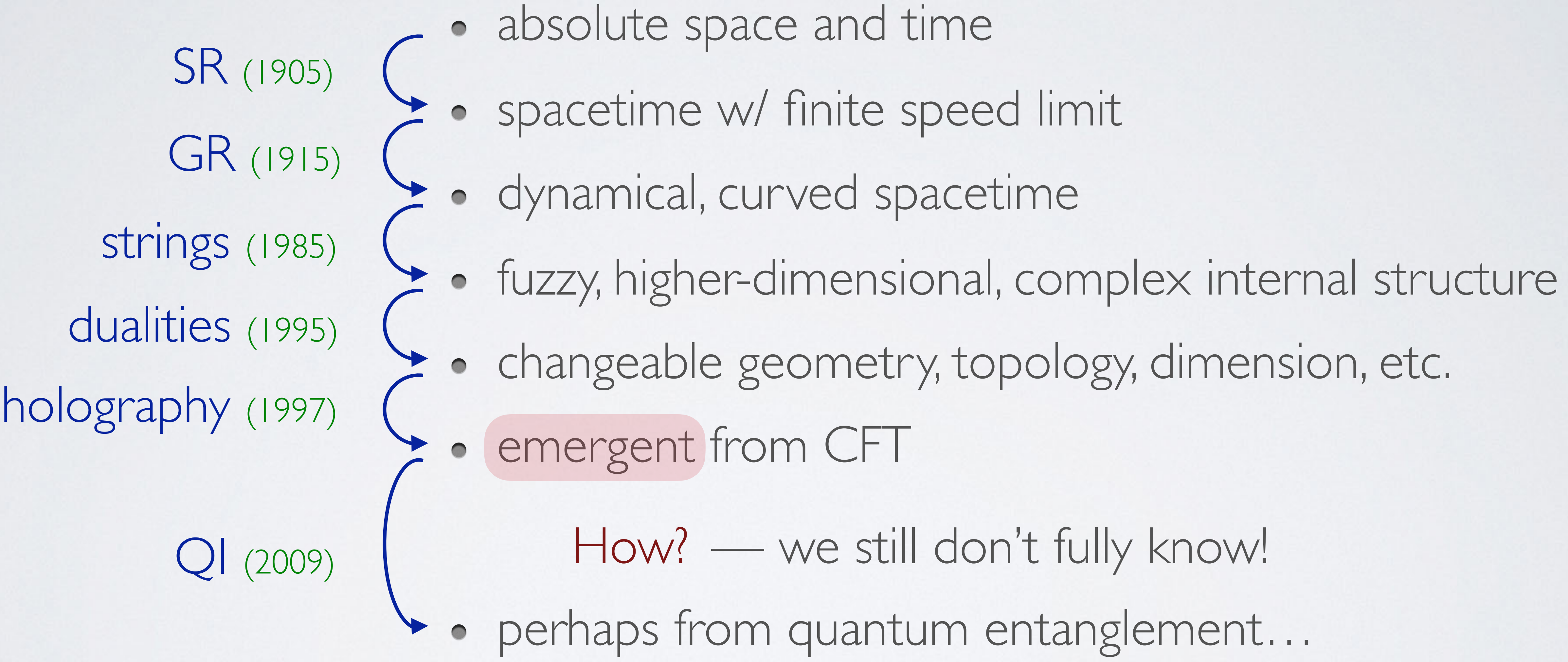


↙ Einstein-Rosen bridge

↘ Einstein-Podolsky-Rosen entanglement

Changing view of Spacetime

- Our concept of spacetime changed enormously in 111 years



- absolute space and time

SR (1905)

- spacetime w/ finite speed limit

GR (1915)

- dynamical, curved spacetime

strings (1985)

- fuzzy, higher-dimensional, complex internal structure

dualities (1995)

- changeable geometry, topology, dimension, etc.

holography (1997)

- emergent from CFT

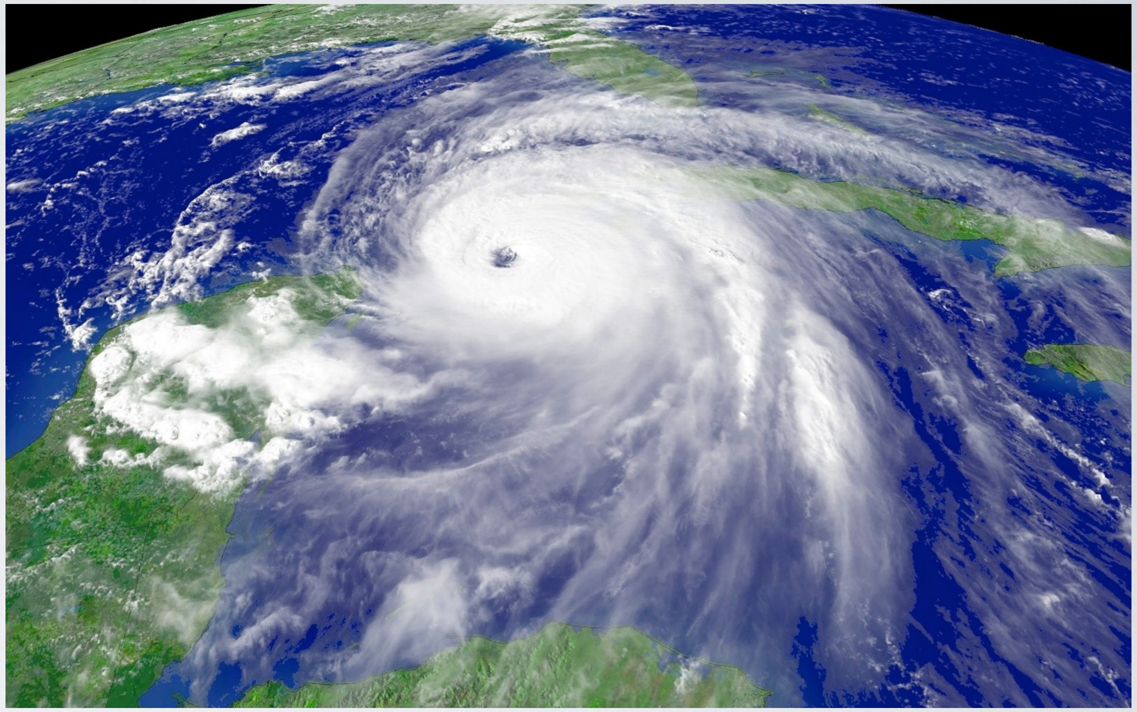
How? — we still don't fully know!

QI (2009)

- perhaps from quantum entanglement…

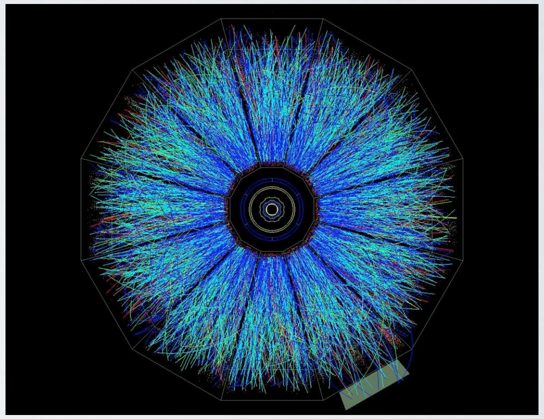
- Exciting times ahead!

Space Ref (Harvard-Smithsonian CfA)



Hurricane Ivan, NOAA

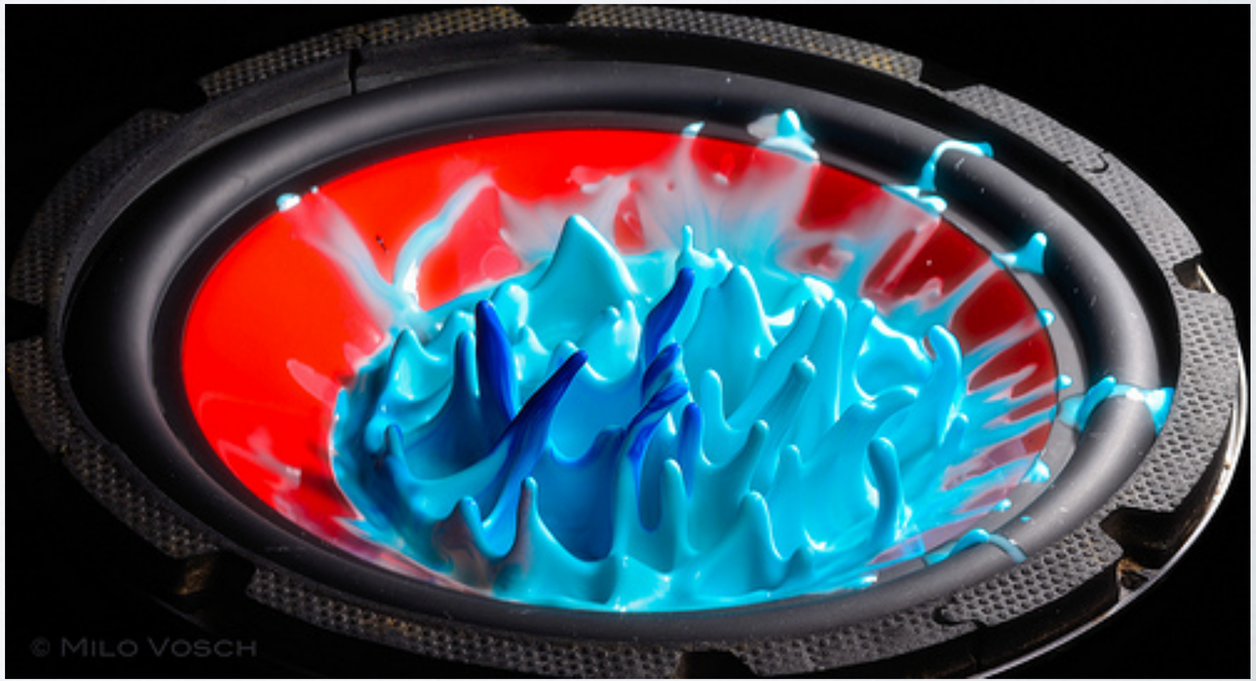
Thank you!



RHIC event, LBL

Roller Wave, by William Dalton

soihub.org/itschool



Dancing Non-Newtonian fluid by Milo Vosch